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Large Deviations for Lotka-Nagaev Estimator of a Randomly Indexed Branching Process

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Abstract. Consider a continuous time process $\{Y_t = Z_{N_t}, t \geq 0\}$, where $\{Z_n\}$ is a supercritical Galton–Watson process and $\{N_t\}$ is a renewal process which is independent of $\{Z_n\}$. Firstly, we study the asymptotic properties of the harmonic moments $\mathbb{E}(Y_t^{-r})$ of order $r > 0$ as $t \to \infty$. Then, we obtain the large deviations of the Lotka-Negaev estimator of offspring mean.

1. Introduction

Classical Galton-Watson process (GW) $\{Z_n\}$ has been naturally extended to branching process in random environments (BPRE) starting in 1970's, see [2], etc. In recent years, researchers focus on the study of large deviation results for GW and BPRE, see [1]and [3] for example.

Let $\{N_t\}$ be a Poisson process and be independent of $\{Z_n\}$. $\{Y_t = Z_{N_t}, t \geq 0\}$ is said to be a Poisson randomly indexed branching process(PRIBP). PRIBP has been firstly used to study the evolution of stock prices in [6] and its statistical investigation has been done in [5]. It was pointed out in [5] that the discrete observations $\{Y_1, Y_2, \cdots\}$ is a BPRE.

For a PRIBP with offspring distribution $\{p_i\}$, we distinguish between the Shröder case and the Böttcher case depending on whether $p_0 + p_1 > 0$ or $p_0 + p_1 = 0$.

Recently, PRIBP has been brought to attention in the following two directions.

In applied direction, a formula for the fair price of an European call option was derived in [13]. Later on, [14] obtained a formula for the fair price of an up-and-out call option.

On more theoretical side, [16] indicated that $R_t := Z_{N_t+1}Z_{N_t}^{-1}$ is a reasonable estimator of the offspring mean m, which is a naturally extension of the classical Lotka-Nagaev estimator, see [1] and [15]. They consider the supercritical PRIBP and obtained the exponential rate of decay for the large deviation probability $\mathbb{P}(|R_t - m| \geq x)$ under the conditions that the offspring distribution $\{p_i\}$ has finite exponential moments and belongs to the Shröder case. On the other hand, [11] showed that $(\lambda t)^{-1} \log Y_t$ is an estimator of $\log m$ and derived the consistency, asymptotic normality, large deviation and moderate deviation of the estimator when the PRIBP belongs to the Böttcher case. In [7], we gave the error bound in asymptotic normality. The large deviations in the Shröder case were given in [8], where the rate function $I(x)$ is deferent from the

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Böttcher case for small positive x. Similar results for branching process indexed by a renewal process were done in [9] and [10].

In this paper, we consider the rates of large deviation probability $\mathbb{P}(|R_t - m| \geq x)$ when the indexed process is a renewal process and the offspring distribution belongs to the Shröder case.

Let F be the distribution of interarrival time X of renewal process. Throughout the paper, we assume the following condition:

A1: $p_0 = 0, \ \ m = \mathbb{E}(Z_1) \in (1, \infty), \ \ \sigma^2 = \mathbb{E}(Z_1 - m)^2 \in (0, \infty), Z_0 = 1.$

A2: $F(0) = 0$, there exists $\theta_0 > 0$, $\forall \theta < \theta_0, M(\theta) := \mathbb{E}(\exp(\theta X)) < \infty$ and $M(\theta)$ is differentiable when $\theta < \theta_0$.

Our first result is the asymptotic properties of harmonic moments $\mathbb{E}(Y_t^{-r})$ of order $r > 0$ as $n \to \infty$.

Theorem 1.1. *Under condition* **A1** *and* **A2**, *for any* $r > 0$, $t^{-1} \log \mathbb{E}(Y_t^{-r}) \to A(r)$, *where*

$$A(r) = \begin{cases} -M^{-1}(p_1^{-1}), & p_1 m^r \geq 1; \\ -M^{-1}(m^r), & p_1 m^r < 1 \end{cases} \tag{1}$$

and M^{-1} *is the inverse function of* M.

Basic properties for M^{-1} are needed in following proofs. By condition A2,

(1) $M(\theta)$ is strictly increasing, then M^{-1} exists.

(2) $M(\theta)$ is differentiable when $\theta < \theta_0$, then M^{-1} is continuous and differentiable in the range of M. Furthermore, if $y = M(\theta)$, then

$$(M^{-1})'(y) = (M'(\theta))^{-1}.$$

We divided our results on large deviation probability $\mathbb{P}(|R_t - m| \geq x)$ into two parts depending on whether the offspring distribution satisfies the Cramér's condition or not.

Theorem 1.2 (Shröder case with light tails). *Assume that there exists a constant* $\alpha > 0$ *such that* $\mathbb{E}(\exp(\alpha Z_1)) < \infty$ *and* $p_1 \in (0, 1)$, *under conditions* **A1** *and* **A2**,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}(|R_t - m| \geq x) = -M^{-1}(p_1^{-1}).$$

Remark 1.3. *Cramér's condition* $\mathbb{E}(\exp(\alpha Z_1)) < \infty$ *can be weakened to* $\mathbb{E}(Z_1^{2r+\delta}) < \infty$ *for some positive constants* δ *and* r *such that* $p_1 m^r > 1$, *see [1].*

Remark 1.4. *If* $\{N_t\}$ *is a Poisson process with parameter* $\lambda > 0$, *then*

$$M(\theta) = \frac{\lambda}{\lambda - \theta}, \theta < \lambda; \ \ M^{-1}(p_1^{-1}) = \lambda(1 - p_1).$$

The following Theorem 1.5 shows that there is a " phase transition " in large deviation rates of convergence from R_t to m when the supercritical branching process indexed by a renewal process belongs to the Shröder case and the offspring distribution has Pareto type tails(Cramér's condition fails).

Theorem 1.5 (Shröder case with heavy tails). *Assume that* $p_0 = 0, p_1 \in (0, 1)$ *and there exists a constant* $r > 0$ *such that*

$$\log(P(Z_1 \geq x))/\log x \to -(r+1),$$

as $x \to \infty$. *Then*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}(|R_t - m| \geq x) = A(r),$$

where $A(r)$ *is defined in* (1).

2. Harmonic Moments

In this section, we deal with the following asymptotic properties of harmonic moments $\mathbb{E}(Y_t^{-r})$ of order $r > 0$ as $t \to \infty$. We need several lemmas to prove Theorem 1.1. Lemma 2.1 comes from [9].

Lemma 2.1. *Under condition* **A2**, *for any* $\theta \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(m^{\theta N_t}\right) = -M^{-1}(m^{-\theta}),$$

where M^{-1} *is the inverse function of* M.

Lemma 2.2. *Under condition* **A2**,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(N_t p_1^{N_t}\right) = -M^{-1}(p_1^{-1}).$$

Proof. For any $1 - p_1 > \epsilon > 0$, there exists n_0 such that for all $n \geq n_0$, one has

$$n \leq (1 + \epsilon/p_1)^n.$$

Note that

$$\begin{aligned}
\mathbb{E}\left(N_t p_1^{N_t}\right) &= \mathbb{E}\left(N_t p_1^{N_t} I\{N_t \geq n_0\}\right) + \mathbb{E}\left(N_t p_1^{N_t} I\{N_t < n_0\}\right) \\
&\leq \mathbb{E}\left((p_1 + \epsilon)^{N_t} I\{N_t \geq n_0\}\right) + \mathbb{E}\left(n_0 p_1^{N_t} I\{N_t < n_0\}\right) \\
&\leq \mathbb{E}\left((p_1 + \epsilon)^{N_t}\right) + \mathbb{E}\left(n_0 p_1^{N_t}\right),
\end{aligned}$$

where $I\{A\}$ is the indictor function of set A. According to Lemma 2.1 and Lemma 1.2.15 of [4], we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(N_t p_1^{N_t}\right) &\leq \limsup_{t\to\infty} \frac{1}{t} \log\left\{\mathbb{E}\left((p_1+\epsilon)^{N_t}\right) + \mathbb{E}\left(n_0 p_1^{N_t}\right)\right\} \\
&= \max\left\{\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left((p_1+\epsilon)^{N_t}\right), \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(n_0 p_1^{N_t}\right)\right\} \\
&= \max\{-M^{-1}((p_1+\epsilon)^{-1}),\ \ -M^{-1}(p_1^{-1})\} \\
&= -M^{-1}((p_1+\epsilon)^{-1}).
\end{aligned}$$

By condition **A2**, M^{-1} is continuous. According to the arbitrariness of ϵ, one has

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(N_t p_1^{N_t}\right) = -M^{-1}(p_1^{-1}).$$

On the other hand

$$\mathbb{E}\left(N_t p_1^{N_t}\right) \geq \mathbb{E}\left(p_1^{N_t}\right),$$

by Lemma 2.1, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(N_t p_1^{N_t}\right) = -M^{-1}(p_1^{-1}).$$

We complete the proof of Lemma 2.2. □

The following Lemma 2.3 belongs to [15], which characterizes the asymptotic properties of harmonic moments of a classical supercritical Galton–Watson process.

Lemma 2.3. *Under condition* **A1**, $A_n(r)E(Z_n^{-r}) \to C(r)$, *where*

$$A_n(r) = \begin{cases} p_1^{-n}, & \text{if } p_1 m^r > 1; \\ (np_1^n)^{-1}, & \text{if } p_1 m^r = 1; \\ (m^r)^n, & \text{if } p_1 m^r < 1 \end{cases}$$

and

$$C(r) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^\infty Q(e^{-v}) v^{r-1} dv, & \text{if } p_1 m^r > 1; \\ \frac{1}{\Gamma(r)} \int_0^m Q(\phi(v)) v^{r-1} dv, & \text{if } p_1 m^r = 1; \\ \frac{1}{\Gamma(r)} \int_0^\infty \phi(v) v^{r-1} dv, & \text{if } p_1 m^r < 1, \end{cases}$$

where $\phi(v) = \lim_n E(e^{-vZ_n/m^n})$ *and* $Q(s)$ *is the unique solution of the functional equation*

$$\begin{cases} Q(f(s)) = p_1 Q(s), \ 0 \le s < 1; \\ Q(0) = 0, \end{cases}$$

where $f(s)$ *is the generating function of the offspring distribution* $\{p_i\}$. *Furthermore,* $\{C(r), r > 0\}$ *are positive and finite.*

The proof of Theorem 1.1.

Let us see that by the total probability formula,

$$\begin{aligned} \mathbb{E}(Y_t^{-r}) &= \sum_{n=0}^{\infty} \mathbb{E}(Z_n^{-r}) \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} C(r)(A_n(r))^{-1} \mathbb{P}(N_t = n) + \sum_{n=0}^{\infty} (\mathbb{E}(Z_n^{-r}) - C(r)(A_n(r))^{-1}) \mathbb{P}(N_t = n) \\ &= I_1 + I_2, \end{aligned} \tag{2}$$

where $I_2 = \sum_{n=0}^{\infty} (\mathbb{E}(Z_n^{-r}) - C(r)(A_n(r))^{-1}) \mathbb{P}(N_t = n)$ and

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} C(r)(A_n(r))^{-1} \mathbb{P}(N_t = n) \\ &= \begin{cases} C(r)\mathbb{E}(p_1^{N_t}), & \text{if } p_1 m^r > 1; \\ C(r)\mathbb{E}(N_t p_1^{N_t}), & \text{if } p_1 m^r = 1; \\ C(r)\mathbb{E}(m^{-rN_t}), & \text{if } p_1 m^r < 1. \end{cases} \end{aligned} \tag{3}$$

According to Lemma 2.3, for any $\epsilon > 0$, there exists a constant $M = M(\epsilon, r)$ such that for any $n \ge M$,

$$\mathbb{E}(Z_n^{-r}) \in [(C(r) - \epsilon)(A_n(r))^{-1}, \ (C(r) + \epsilon)(A_n(r))^{-1}].$$

Then

$$\begin{aligned} |I_2| &\le \sum_{n=0}^{+\infty} \epsilon (A_n(r))^{-1} P(N_t = n) + \sum_{n=0}^{M} |\mathbb{E}(Z_n^{-r}) - C(r)(A_n(r))^{-1})| \mathbb{P}(N_t = n) \\ &\le \epsilon I_1 / C(r) + L(r) \mathbb{P}(N_t \le M), \end{aligned} \tag{4}$$

where

$$L(r) = \max_{1 \le n \le M} \{|\mathbb{E}(Z_n^{-r}) - C(r)(A_n(r))^{-1})|\} < \infty.$$

By (2)-(4),

$$\mathbb{E}(Y_t^{-r}) \ge (C(r) - \epsilon) \begin{cases} \mathbb{E}(p_1^{N_t}), & \text{if } p_1 m^r > 1; \\ \mathbb{E}(N_t p_1^{N_t}), & \text{if } p_1 m^r = 1; - L(r)\mathbb{P}(N_t \le M) \\ \mathbb{E}(m^{-rN_t}), & \text{if } p_1 m^r < 1 \end{cases}$$

and

$$\mathbb{E}(Y_t^{-r}) \leq (C(r)+\epsilon)\begin{cases}\mathbb{E}(p_1^{N_t}), & \text{if } p_1 m^r > 1;\\ \mathbb{E}(N_t p_1^{N_t}), & \text{if } p_1 m^r = 1; + L(r)\mathbb{P}(N_t \leq M).\\ \mathbb{E}(m^{-rN_t}), & \text{if } p_1 m^r < 1\end{cases}$$

According to the large deviations for renewal process, see [12], one has

$$\frac{1}{t}\log(\mathbb{P}(N_t \leq M)) \to -\infty.$$

Note that ϵ is arbitrary, Theorem 1.1 follows from Lemma 2.1 and Lemma 2.2. □

3. Large Deviation Probability

In this section, we deal with Theorem 1.2. The proof is dependent on the following lemma which belongs to [1].

Lemma 3.1. *Assume that $Z_0 = 1, p_0 = 0, p_1 \in (0,1)$ and there there exists a constant $\alpha > 0$ such that $\mathbb{E}(\exp(\alpha Z_1)) < \infty$, then for any $x > 0$,*

$$\lim_{n\to\infty}\frac{1}{p_1^n}\mathbb{P}\left(\left|\frac{Z_{n+1}}{Z_n} - m\right| \geq x\right) = V(x) \in (0,\infty).$$

The proof of Theorem 1.2.

Write $\psi(x) = \mathbb{P}(|Z_{n+1}/Z_n - m| \geq x)$. First, let us note that

$$\begin{aligned}\mathbb{P}(|R_t - m| \geq x) &= \sum_{n=0}^{\infty}\mathbb{P}(|Z_{n+1}/Z_n - m| \geq x)\mathbb{P}(N_t = n)\\ &= \sum_{n=0}^{\infty}V(x)p_1^n\mathbb{P}(N_t = n) + \sum_{n=0}^{\infty}(\psi(x) - V(x)p_1^n)\mathbb{P}(N_t = n)\\ &=: U_1 + U_2, \end{aligned} \tag{5}$$

where $U_1 = V(x)\mathbb{E}(p_1^{N_t})$. On the other hand, by Lemma 3.1, for any $\epsilon > 0$, there exists n_0, if $n \geq n_0$, then $\psi(x) \in ((V(x)-\epsilon)p_1^n, (V(x)+\epsilon)p_1^n)$. Thus,

$$\begin{aligned}|U_2| &\leq \sum_{n=0}^{+\infty}\epsilon p_1^n\mathbb{P}(N_t = n) + \sum_{n=0}^{n_0}|\psi(x) - V(x)p_1^n|\mathbb{P}(N_t = n)\\ &\leq \epsilon\mathbb{E}(p_1^{N_t}) + G(x)\mathbb{P}(N_t \leq n_0),\end{aligned} \tag{6}$$

where

$$G(x) = \max_{1\leq n\leq n_0}\{|\psi(x) - V(x)p_1^n|\} < \infty.$$

By (5)-(6),

$$\psi(x) \geq (V(x)-\epsilon)\mathbb{E}(p_1^{N_t}) - G(x)\mathbb{P}(N_t \leq n_0)$$

and

$$\psi(x) \geq (V(x)+\epsilon)\mathbb{E}(p_1^{N_t}) + G(x)\mathbb{P}(N_t \leq n_0)$$

According to the large deviations for renewal process, see [12], one has

$$\frac{1}{t}\log(\mathbb{P}(N_t \leq n_0)) \to -\infty.$$

Note that $0 < V(x) < \infty$ for $x \in (0,+\infty)$ and ϵ is arbitrary, Theorem 1.2 follows from Lemma 2.1. □

The proof of Theorem 1.5.

The proof is similar to that of Theorem 1.1. The only change is that Lemma 2.3 is substitute by the following lemma which belongs to [15].

Lemma 3.2. *Assume that $p_0 = 0, p_1 \in (0,1)$ and there exists a constant $r > 0$ such that*

$$\log(P(Z_1 \geq x))/\log x \to -(r+1),$$

as $x \to \infty$. Then

$$\lim_{t\to\infty} A_n(r)\mathbb{P}\left(\left|\frac{Z_{n+1}}{Z_n} - m\right| \geq a\right) = U(a) \in (0,\infty),$$

where $A_n(r)$ is defined in Lemma 2.3.

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