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Biprojectivity and Biflatness of Generalized Module Extension Banach Algebras

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Abstract. We investigate biprojectivity and biflatness of generalized module extension Banach algebra $A \bowtie B$, in which A and B are Banach algebras and B is an algebraic Banach $A-$bimodule, with multiplication: $(a,b)\cdot(a^{'},b^{'}) = (aa^{'}, ab^{'} + ba^{'} + bb^{'})$.

1. Introduction

Let A and B be Banach algebras and let B be a Banach $A-$bimodule. Then, we will say that B is an *algebraic* Banach $A-$bimodule if for all $a \in A$ and $b, b^{'} \in B$

$$a(bb^{'}) = (ab)b^{'} \ , \ \ (bb^{'})a = b(b^{'}a) \ , \ \ (ba)b^{'} = b(ab^{'}) \ .$$

The Cartesian product $A \times B$ with the multiplication

$$(a,b)\cdot(a^{'},b^{'}) = (aa^{'}, ab^{'} + ba^{'} + bb^{'}) \ ,$$

and with the norm $\|(a,b)\| = \|a\| + \|b\|$, becomes a Banach algebra, which is called the ″*generalized module extension Banach algebra*‵‵, and it is denoted by $A \bowtie B$. Also $A \cong A \times \{0\}$ is a closed subalgebra, while $B \cong \{0\} \times B$ is a closed ideal of $A \bowtie B$, and $A \bowtie B/B \cong A$. The authors in [11] have studied some properties of this kind of algebra, such as bounded approximate identity, spectrum, topological centers and $n-$weak amenability. This algebra can be a generalization of the following known algebras:

(a) Let $A \times B$ be the direct product of two Banach algebras A and B, with multiplication

$$(a,b)\cdot(a^{'},b^{'}) = (aa^{'}, bb^{'}) \ .$$

If we define the $A-$bimodule actions on B by $ab = ba = 0$, for $a \in A$ and $b \in B$, then $A \times B = A \bowtie B$.

(b) Let $A \oplus X$ be the module extension Banach algebra, in which X is a Banach $A-$bimodule, with multiplication

$$(a,x)\cdot(a^{'},x^{'}) = (aa^{'}, ax^{'} + xa^{'}) \ .$$

If we define the multiplication on X by $xx^{'} = 0$, then $A \oplus X = A \bowtie X$.

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(c) Let $A \times_\theta B$ be the $\theta-$Lau product of two Banach algebras A and B with $\theta \in \triangle(A)$ and the following multiplication

$$(a,b)\cdot(a',b') = (aa', \theta(a)b' + \theta(a')b + bb') \,.$$

These kinds of products have been investigated in two prior studies [7, 13]. If we define the $A-$bimodule actions on B by $ab = ba =: \theta(a)b$, for $a \in A$ and $b \in B$, then $A \times_\theta B = A \bowtie B$.

(d) Let $A \times_T B$ be the $T-$Lau product of two Banach algebras A and B with algebra homomorphism $T : A \to B$ with $\|T\| \le 1$, and with multiplication

$$(a,b)\cdot(a',b') = (aa', T(a)b' + bT(a') + bb') \,.$$

These kinds of products were introduced by Lau [7], and studied by many authors such as [2, 5, 13]. If we define the $A-$bimodule actions on B by $ab =: T(a)b$, $ba =: bT(a)$, for $a \in A$ and $b \in B$, then $A \times_T B = A \bowtie B$.

(e) Let $A \bowtie^\theta I$ be the amalgamation of A with B along I with respect to θ, in which A and B are Banach algebras, I is a closed ideal in B, $\theta : A \to B$ is a continuous Banach algebra homomorphism, and with the following multiplication

$$(a,i)(a',i') = (aa', \theta(a)i' + i\theta(a') + ii') \,,$$

for $a, a' \in A$ and $i, i' \in I$. These kinds of Banach algebras have been studied in some other studies [9, 10]. Now if we define the $A-$bimodule actions on I by $ai =: \theta(a)i$ and $ia = i\theta(a)$, for $a \in A$ and $i \in I$, then $A \bowtie^\theta I = A \bowtie I$.

Homological properties of Banach algebras have been studied by several authors. We refer to [4] as a standard reference in this field. The properties biprojectivity and biflatness have been studied: for $\theta-$Lau product $A \times_\theta B$ in [6], and for $T-$Lau product $A \times_T B$ in [1]. In this paper we will study biprojectivity and biflatness of $L = A \bowtie B$, in two separate sections 3 and 4. We will show that if $L = A \bowtie B$ is biprojective [biflat], then A is biprojective [biflat], but for biprojectivity [biflatness] of B, we need some conditions. Also it will be shown that if A and B are biprojective [biflat], then under a mild condition on B, we conclude the biprojectivity [biflatness] of $L = A \bowtie B$. In section 5 our results will be applied in some examples.

2. Preliminaries

Throughout this paper, A and B are Banach algebras, B is an algebraic Banach $A-$bimodule, and $L = A \bowtie B$ denotes the generalized module extension Banach algebra. Consider the $A-$bimodule and also $B-$bimodule actions on $L = A \bowtie B$ by

$$\begin{cases} a' \cdot (a,b) := (a',0)\cdot(a,b)\,, \\ (a,b)\cdot a' := (a,b)\cdot(0,a')\,, \end{cases} \quad \text{and} \quad \begin{cases} b' \cdot (a,b) = (0,b')\cdot(a,b)\,, \\ (a,b)\cdot b' = (a,b)\cdot(0,b')\,, \end{cases}$$

for all $(a,b) \in L$, $a' \in A$ and $b' \in B$. Following [4], we say that A is *biprojective* if there exists a bounded $A-$bimodule map $\rho_A : A \to A\widehat{\otimes}A$ such that $\pi_A o \rho_A = id_A$, in which $\pi_A : A\widehat{\otimes}A \to A$ denotes the product map with $\pi_A(a \otimes a') = aa'$. Also A is called *biflat* if there is a bounded $A-$bimodule map $\lambda_A : (A\widehat{\otimes}A)^* \to A^*$, such that $\lambda_A o \pi^*_A = id_{A^*}$. For the basic properties of biprojectivity and biflatness, see [3, 12].

Finally, the following maps will be introduced and then used in our results. Let $p_A : L = A \bowtie B \to A$ and $p_B : L = A \bowtie B \to B$ be the projections defined by $p_A\big((a,b)\big) = a$ and $p_B\big((a,b)\big) = b$ for all $(a,b) \in L$. Also let $q_A : A \to L = A \bowtie B$ and $q_B : B \to L = A \bowtie B$ be the injections, defined by $q_A(a) = (a,0)$ and $q_B(b) = (0,b)$, for all $a \in A$ and $b \in B$. Besides, suppose that B is unital with unit e_B, and define the following bounded linear maps

$$r_B : L = A \bowtie B \to B \ \text{ by } \ r_B(a,b) = ae_B + b \,,$$
$$s_A : A \to L = A \bowtie B \ \text{ by } \ s_A(a) = (a, -ae_B) \,.$$

Note that, the mappings p_A, q_A are bounded $A-$bimodule maps, and q_B is a bounded $B-$bimodule map. We have the following lemma about relations between bimodule structures for r_B and s_A.

Lemma 2.1. *Let A and B be Banach algebras, and let B be an algebraic Banach $A-$bimodule with unit e_B, such that $ae_B = e_Ba$ for all $a \in A$. Then the mappings r_B and s_A are $B-$bimodule map and $A-$bimodule map, respectively.*

Proof. Let $a, a' \in A$ and $b, b' \in B$. By using the assumptions, we have

$$\begin{aligned} r_B\Big(b' \cdot (a,b)\Big) &= r_B\Big((0,b') \cdot (a,b)\Big) \\ &= r_B(0, b'a + b'b) \\ &= b'a + b'b \\ &= b'e_Ba + b'b \\ &= b'ae_B + b'b \\ &= b' \cdot (ae_B + b) \\ &= b' \cdot r_B(a,b)\,. \end{aligned}$$

Similarly, we have $r_B\Big((a,b)\cdot b'\Big) = r_B\Big((a,b)\Big)\cdot b'$, and we conclude that r_B is a $B-$bimodule map. Also we have

$$\begin{aligned} s_A(aa') &= (aa', -aa'e_B) \\ &= (a,0)\cdot(a', -a'e_B) \\ &= a \cdot (a', -a'e_B) \\ &= a \cdot s_A(a')\,, \end{aligned}$$

and similarly, by the assumptions

$$\begin{aligned} s_A(aa') &= (aa', -aa'e_B) \\ &= (aa', -ae_Ba') \\ &= (a, -ae_B)\cdot(a',0) \\ &= (a, -ae_B)\cdot a' \\ &= s_A(a)\cdot a'\,, \end{aligned}$$

and so s_A is an $A-$bimodule map. □

3. Results on Biprojectivity

This section deals with relations between biprojectivity of $L = A \bowtie B$ and biprojectivity of A and B.

Theorem 3.1. *Let A and B be Banach algebras, and let B be an algebraic Banach $A-$bimodule.*

(i) If $L = A \bowtie B$ is biprojective, then A is biprojective.

(ii) Suppose that B has unit e_B, such that for all $a \in A$, $ae_B = e_Ba$. If $L = A \bowtie B$ is biprojective, then B is biprojective.

Proof. By the hypothesis, there exist a bounded $L-$bimodule map $\rho_L : L \to L\widehat{\otimes}L$, such that $\pi_L o \rho_L = id_L$.
(i) Define $\rho_A : A \to A\widehat{\otimes}A$ by $\rho_A =: (p_A \otimes p_A) o \rho_L o q_A$. Clearly ρ_A is bounded. Since ρ_L is $L-$bimodule map, for $a, a' \in A$ and $b \in B$ we have

$$\begin{aligned} \rho_L\Big(a' \cdot (a,b)\Big) &= \rho\Big((a',0)\cdot(a,b)\Big) \\ &= (a',0)\rho_L\Big((a,b)\Big) \\ &= a' \cdot \rho_L\Big((a,b)\Big)\,. \end{aligned}$$

Similarly, we have $\rho_L\Big((a,b)\cdot a'\Big) = \rho_L\Big((a,b)\Big)\cdot a'$. We conclude that ρ_L is $A-$bimodule map. Then ρ_A is a bounded $A-$bimodule map. Also for $(a,b)\otimes(a',b')\in L\widehat{\otimes}L$

$$\begin{aligned}\Big(\pi_A o(p_A\otimes p_A)\Big)\Big((a,b)\otimes(a',b')\Big) &= \pi_A(a\otimes a') = aa' ,\\ (p_A o\pi_L)\Big((a,b)\otimes(a',b')\Big) &= p_A\Big((a,b)\cdot(a',b')\Big) = aa' ,\end{aligned}$$

this shows the identity $\pi_A o(p_A\otimes p_A) = p_A o\pi_L$. Now one can have the following

$$\begin{aligned}\pi_A o\rho_A &= \pi_A o(p_A\otimes p_A)o\rho_L oq_A\\ &= p_A o\pi_L o\rho_L oq_A\\ &= p_A oid_L oq_A = id_A .\end{aligned}$$

This shows that A is biprojective.

(ii) Define $\rho_B =: (r_B\otimes r_B)o\rho_L oq_B$. Since ρ_L, q_B and r_B are bounded $B-$bimodule maps, then ρ_B is bounded $B-$bimodule map. Also for (a,b) and (a',b') in L we have

$$\begin{aligned}\Big(\pi_B o(r_B\otimes r_B)\Big)\Big((a,b)\otimes(a',b')\Big) &= \pi_B\Big((ae_B+b)\otimes(a'e_B+b')\Big)\\ &= (ae_B+b)\cdot(a'e_B+b')\\ &= ae_Ba'e_B+ae_Bb'+ba'e_B+bb'\\ &= aa'e_B+ab'+ba'+bb'\\ &= r_B\Big(aa',ab'+ba'+bb'\Big)\\ &= r_B\Big((a,b)\cdot(a',b')\Big)\\ &= (r_B o\pi_L)\Big((a,b)\otimes(a',b')\Big) .\end{aligned}$$

We conclude that $\pi_B o(r_B\otimes r_B) = r_B o\pi_L$. Moreover it is easy to check that $r_B oq_B = id_B$. Then

$$\begin{aligned}\pi_B o\rho_B &= \pi_B o(r_B\otimes r_B)o\rho_L oq_B\\ &= r_B o\pi_L o\rho_L oq_B\\ &= r_B oid_L oq_B\\ &= r_B oq_B\\ &= id_B ,\end{aligned}$$

and this shows the biprojectivity of B. □

Theorem 3.2. *Let A and B be Banach algebras, and let B be an algebraic Banach A−bimodule with unit e_B such that for all $a\in A$, $ae_B = e_Ba$. If A and B are biprojective, then $L = A\bowtie B$ is biprojective.*

Proof. By the hypothesis, there exist bounded $A-$bimodule map $\rho_A : A\to A\widehat{\otimes}A$, and bounded $B-$bimodule map $\rho_B : B\to B\widehat{\otimes}B$, such that $\pi_A o\rho_A = id_A$ and $\pi_B o\rho_B = id_B$. For $(a\otimes a')\in A\widehat{\otimes}A$ we have

$$\begin{aligned}\Big(\pi_L o(s_A\otimes s_A)\Big)(a\otimes a') &= \pi_L\Big((a,-ae_B)\otimes(a',-a'e_B)\Big)\\ &= (a,-ae_B)\cdot(a',-a'e_B)\\ &= (aa',-aa'e_B-ae_Ba'+ae_Ba'e_B)\\ &= (aa',-aa'e_B-aa'e_B+aa'e_B)\\ &= (aa',-aa'e_B)\\ &= s_A(aa')\\ &= (s_A o\pi_A)(a\otimes a') ,\end{aligned}$$

and we conclude that $\pi_L o(s_A \otimes s_A) = s_A o \pi_A$. Also, it is easy to check that $\pi_L o(q_B \otimes q_B) = q_B o \pi_B$. Now define $\rho_L : L \to L\widehat{\otimes}L$ by

$$\rho_L\Big((a,b)\Big) =: \Big((s_A \otimes s_A) o \rho_A o p_A\Big)(a,b) + (a,b) \cdot \Big((q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big) .$$

Clearly ρ_L is bounded, we first show that ρ_L is a left$-L-$module map. For all $(a,b), (c,d) \in L$, we have

$$\begin{aligned}
\rho_L\Big((a,b)\cdot(c,d)\Big) &= \Big((s_A \otimes s_A) o \rho_A o p_A\Big)(ac, ad+bc+bd) + \Big((a,b)\cdot(c,d)\Big)\cdot\Big((q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big) \\
&= (s_A \otimes s_A)\Big(\rho_A(ac)\Big) + \Big((a,b)\cdot(c,d)\Big)\cdot\Big((q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big) \\
&= (a,0)\cdot(s_A \otimes s_A)\Big(\rho_A(c)\Big) + \Big((a,b)\cdot(c,d)\Big)\cdot\Big((q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big) \\
&= (a,b)\cdot\Big[(s_A \otimes s_A)\Big(\rho_A(c)\Big) + (c,d)\cdot(q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big] \\
&- (0,b)\cdot\Big((s_A \otimes s_A)\Big(\rho_A(c)\Big)\Big) \\
&= (a,b)\cdot\rho_L(c,d) - (0,b)\cdot\Big((s_A \otimes s_A)\Big(\rho_A(c)\Big)\Big) ,
\end{aligned}$$

but $(0,b)\cdot(s_A \otimes s_A)\Big(\rho_A(c)\Big) = 0$, because for all $(a' \otimes a'') \in A\widehat{\otimes}A$, we can write

$$\begin{aligned}
(0,b)\cdot\Big((s_A \otimes s_A)(a' \otimes a'')\Big) &= (0,b)\cdot\Big(s_A(a') \otimes s_A(a'')\Big) \\
&= (0,b)\cdot\Big((a', -a'e_B) \otimes (a'', -a''e_B)\Big) \\
&= \Big((0,b)\cdot(a', -a'e_B)\Big) \otimes (a'', -a''e_B) \\
&= \Big(0, ba' - ba'e_B\Big) \otimes (a'', -a''e_B) \\
&= (0, ba' - be_Ba') \otimes (a'', -a''e_B)\Big) \\
&= (0,0) \otimes (a'', -a'e_B) \\
&= 0 ,
\end{aligned}$$

and we conclude that $(0,b)\cdot(s_A \otimes s_A)\Big(\rho_A(c)\Big) = 0$ for $\rho_A(c) = \sum_{i=1}^{\infty} a_i' \otimes a_i''$, in which $(a_i'), (a_i'')$ are some sequences in A with $\sum_{i=1}^{\infty} \|a_i'\|\|a_i''\| < \infty$.

Thus $\rho_L\Big((a,b)\cdot(c,d)\Big) = (a,b)\cdot\rho_L\Big((c,d)\Big)$, and so ρ_L is left$-L-$module map. To show that ρ_L is right$-L-$module map, we note that for all $(b' \otimes b'') \in B\widehat{\otimes}B$

$$\begin{aligned}
(a,b)\cdot\Big((q_B \otimes q_B)(b' \otimes b'')\Big) &= (q_B \otimes q_B)\Big((b+ae_B)\cdot(b' \otimes b'')\Big), \\
\Big((q_B \otimes q_B)(b' \otimes b'')\Big)\cdot(a,b) &= (q_B \otimes q_B)\Big((b' \otimes b'')\cdot(b+ae_B)\Big) .
\end{aligned}$$

Hence

$$\begin{aligned}
(a,b)\cdot\Big((q_B \otimes q_B)\Big(\rho_B(e_B)\Big)\Big) &= (q_B \otimes q_B)\Big((b+ae_B)\cdot\rho_B(e_B)\Big) \\
&= (q_B \otimes q_B)\Big(\rho_B(e_B)\cdot(b+ae_B)\Big) \\
&= \Big((q_B \otimes q_B)\Big(\rho(e_B)\Big)\Big)\cdot(a,b) .
\end{aligned}$$

It follows that $(q_B\otimes q_B)\Big(\rho_B(e_B)\Big)$ commutes with the members of L. Consequently,

$$\begin{aligned}
\rho_L\Big((c,d)\cdot(a,b)\Big) &= \Big((s_A\otimes s_A)o\rho_Aop_A\Big)\Big((c,d)\cdot(a,b)\Big)+\Big((c,d)\cdot(a,b)\Big)\Big((q_B\otimes q_B)\Big(\rho_B(e_B)\Big)\Big)\\
&= (s_A\otimes s_A)\Big(\rho_A(ca)\Big)+\Big((c,d)\cdot(a,b)\Big)\Big((q_B\otimes q_B)\Big(\rho_B(e_B)\Big)\Big)\\
&= \Big((s_A\otimes s_A)\Big(\rho_A(c)\Big)\Big)\cdot(a,0)+(c,d)\cdot\Big((q_B\otimes q_B)\Big(\rho_B(e_B)\Big)\Big)\cdot(a,b)\\
&= \Big[(s_A\otimes s_A)\Big(\rho_A(c)\Big)+(c,d)\cdot\Big((q_B\otimes q_B)\Big(\rho_B(e_B)\Big)\Big)\Big]\cdot(a,b)\\
&- \Big((s_A\otimes s_A)\Big(\rho_A(c)\Big)\Big)\cdot(0,b)\\
&= \rho_L\Big((c,d)\Big)\cdot(a,b)-\Big((s_A\otimes s_A)\Big(\rho_A(c)\Big)\Big)\cdot(0,b)\ ,
\end{aligned}$$

but with similar reasoning for $(0,b)\cdot\Big((s_A\otimes s_A)\Big(\rho_A(c)\Big)\Big)=0$ we have the identity $\Big((s_A\otimes s_A)\Big(\rho_A(c)\Big)\Big)\cdot(a,b)=0$. Thus $\rho_L\Big((c,d)\cdot(a,b)\Big)=\rho_L\Big((c,d)\Big)\cdot(a,b)$, and so ρ_L is a right$-L-$module map. Finally, we have

$$\begin{aligned}
\Big(\pi_Lo\rho_L\Big)(a,b) &= \Big(\pi_Lo(s_A\otimes s_A)o\rho_Aop_A\Big)(a,b)+(a,b)\cdot\pi_L\Big((q_B\otimes q_B)\Big(\rho_B(e_B)\Big)\Big)\\
&= \Big(s_Ao\pi_Ao\rho_Aop_A\Big)(a,b)+(a,b)\cdot\Big(\Big(q_Bo\pi_Bo\rho_B\Big)(e_B)\Big)\\
&= (s_Aop_A)(a,b)+(a,b)\cdot\Big(q_B(e_B)\Big)\\
&= s_A(a)+(a,b)\cdot(0,e_B)\\
&= (a,-ae_B)+(0,ae_B+b)\\
&= (a,b)\ ,
\end{aligned}$$

therefore $\pi_Lo\rho_L=id_L$, and hence $L=A\bowtie B$ is biprojective. □

4. Results on Biflatness

This section is devoted to the relations between biflatness of $L=A\bowtie B$ and biflatness of A and B.

Theorem 4.1. *Let A and B be Banach algebras, and let B be an algebraic Banach $A-$bimodule.*

(i) If $L=A\bowtie B$ is biflat, then A is biflat.
(ii) Suppose that B has unit e_B, such that for all $a\in A$, $ae_B=e_Ba$. If $L=A\bowtie B$ is biflat, then B is biflat.

Proof. By the hypothesis, there exist a bounded $L-$bimodule map $\lambda_L:(L\widehat{\otimes}L)^*\to L^*$, such that $\lambda_Lo\pi_L^*=id_{L^*}$. The following identities have been shown in the proof of theorem (3.1)

$$\begin{aligned}
\pi_Ao(p_A\otimes p_A) &= p_Ao\pi_L\ ,\\
\pi_Bo(r_B\otimes r_B) &= r_Bo\pi_L\ .
\end{aligned}$$

(i) Define $\lambda_A:(A\widehat{\otimes}A)^*\to A^*$ by $\lambda_A=:q_A^*o\lambda_Lo(p_A\otimes p_A)^*$, which is a bounded $A-$bimodule map and

$$\begin{aligned}
\lambda_Ao\pi_A^* &= q_A^*o\lambda_Lo(p_A\otimes p_A)^*o\pi_A^*\\
&= q_A^*o\lambda_Lo\Big(\pi_Ao(p_A\otimes p_A)\Big)^*\\
&= q_A^*o\lambda_Lo(p_Ao\pi_L)^*\\
&= q_A^*o\lambda_Lo\pi_L^*op_A^*\\
&= q_A^*oid_{L^*}op_A^*\\
&= (p_Aoq_A)^*\\
&= (id_A)^*\\
&= id_{A^*}\ .
\end{aligned}$$

Hence A is biflat.
(ii) Define $\lambda_B : (B\widehat{\otimes}B)^* \longrightarrow B^*$ by $\lambda_B =: q_B^* o\lambda_L o(r_B \otimes r_B)^*$. Since B is unital and $ae_B = e_B a$ $(a \in A)$, then r_B and hence λ_B are bounded $B-$bimodule maps, and we have

$$\begin{aligned}
\lambda_B o\pi_B^* &= q_B^* o\lambda_L o(r_B \otimes r_B)^* o\pi_B^* \\
&= q_B^* o\lambda_L o\Big(\pi_B o(r_B \otimes r_B)\Big)^* \\
&= q_B^* o\lambda_L o(r_B o\pi_L)^* \\
&= q_B^* o\lambda_L o\pi_L^* o r_B^* \\
&= q_B^* o id_{L^*} o r_B^* \\
&= (r_B o q_B)^* \\
&= (id_B)^* \\
&= id_{B^*}\ .
\end{aligned}$$

This proves the biflatness of B. $\square$

For the converse of theorem (4.1) we should determine the $L-$bimodule structures on $L^* = (A \bowtie B)^*$. We recall that the dual space A^* of A is a Banach $A-$bimodule by module operations

$$\langle f\cdot a, b\rangle = \langle f, ab\rangle \ \text{ and } \ \langle a\cdot f, b\rangle = \langle f, ba\rangle\ ,$$

for $a, b \in A$ and $f \in A^*$. We remark that the dual space $L^* = (A \bowtie B)^*$ can be identified with $A^* \times B^*$ by the following bounded linear map

$$\theta : A^* \times B^* \longrightarrow (A \bowtie B)^* = L^* \quad , \quad \Big(\langle\theta(f, g), (a, b)\rangle = f(a) + g(b)\Big)\ .$$

Now suppose that B has unit e_B such that for all $a \in A$, $ae_B = e_B a$. Define $\varphi : A \to B$ by $\varphi(a) = e_B a$. For $(a, b), (a', b') \in L = A \bowtie B$ and $(f, g) \in L^*$ we have

$$\begin{aligned}
\Big((f, g)\cdot(a, b)\Big)(a', b') &= (f, g)\Big((a, b)\cdot(a', b')\Big) \\
&= (f, g)(aa', ab' + ba' + bb') \\
&= f(aa') + g(ab') + g(ba') + g(bb') \\
&= (f\cdot a)(a') + g(ae_B b') + g(be_B a') + (g\cdot b)(b') \\
&= (f\cdot a)(a') + (g\cdot(ae_B))b' + \Big((g\cdot b)o\varphi\Big)(a') + (g\cdot b)(b') \\
&= \Big(f\cdot a + (g\cdot b)o\varphi\Big)(a') + \Big(g\cdot(ae_B) + g\cdot b\Big)(b') \\
&= \Big(f\cdot a + (g\cdot b)o\varphi\ , g\cdot(ae_B) + g\cdot b\Big)(a', b')\ ,
\end{aligned}$$

therefore

$$(f, g)\cdot(a, b) = \Big(f\cdot a + (g\cdot b)o\varphi\ , g\cdot(ae_B) + g\cdot b\Big),$$

and similarly

$$(a, b)\cdot(f, g) = \Big(a\cdot f + (b\cdot g)o\varphi\ , (e_B a)\cdot g + b\cdot g\Big).$$

Theorem 4.2. *Let A and B be Banach algebras, and let B be an algebraic Banach $A-$bimodule with unit e_B such that for all $a \in A$, $ae_B = e_B a$. If A and B are biflat, then $L = A \bowtie B$ is biflat.*

Proof. By the hypothesis, there exist a bounded $A-$bimodule map $\lambda_A : (A\widehat{\otimes}A)^* \to A^*$ and a bounded $B-$bimodule map $\lambda_B : (B\widehat{\otimes}B)^* \to B^*$, such that $\lambda_A o\pi_A^* = id_{A^*}$ and $\lambda_B o\pi_B^* = id_{B^*}$. Define $\lambda_L : (L\widehat{\otimes}L)^* \to L^* \cong A^* \times B^*$ by

$$\lambda_L(h) =: \Big(\big(\lambda_A o(s_A \otimes s_A)^*\big)(h) + \big(\varphi^* o\lambda_B o(q_B \otimes q_B)^*\big)(h), \big(\lambda_B o(q_B \otimes q_B)^*\big)(h)\Big) ,$$

for $h \in (L\widehat{\otimes}L)^*$ and $\varphi : A \to B$ ($\varphi(a) = ae_B$). Clearly λ_L is a bounded map. To see that λ_L is a $L-$bimodule map we need the following identities for $h \in (L\widehat{\otimes}L)^*$ and $(a, b) \in L$

(1) $(q_B \otimes q_B)^*\big(h \cdot (a, b)\big) = (q_B \otimes q_B)^*(h) \cdot (ae_B + b)$,

(2) $(q_B \otimes q_B)^*\big((a, b) \cdot h\big) = (ae_B + b) \cdot (q_B \otimes q_B)^*(h)$,

(3) $(s_A \otimes s_A)^*\big(h \cdot (a, b)\big) = (s_A \otimes s_A)^*(h) \cdot a$,

(4) $(s_A \otimes s_A)^*\big((a, b) \cdot h\big) = a \cdot (s_A \otimes s_A)^*(h)$.

To prove the equality (1), for $(b^{'} \otimes b^{''}) \in B\widehat{\otimes}B$ we can write

$$\begin{aligned}
\Big((q_B \otimes q_B)^*\big(h \cdot (a, b)\big)\Big)(b^{'} \otimes b^{''}) &= \big(h \cdot (a, b)\big)\big((q_B \otimes q_B)(b^{'} \otimes b^{''})\big) \\
&= \big(h \cdot (a, b)\big)\big((0, b^{'}) \otimes (0, b^{''})\big) \\
&= h\big((a, b) \cdot (0, b^{'}) \otimes (0, b^{''})\big) \\
&= h\big((0, ab^{'} + bb^{'}) \otimes (0, b^{''})\big) \\
&= h\big((0, ae_B b^{'} + bb^{'}) \otimes (0, b^{''})\big) \\
&= h\big((0, (ae_B + b) \cdot b^{'}) \otimes (0, b^{''})\big) \\
&= h\big((q_B \otimes q_B)\big((ae_B + b)b^{'} \otimes b^{''}\big)\big) \\
&= \big((q_B \otimes q_B)^*(h)\big)\big((ae_B + b)(b^{'} \otimes b^{''})\big) \\
&= \big((q_B \otimes q_B)^*(h) \cdot (ae_B + b)\big)(b^{'} \otimes b^{''}) .
\end{aligned}$$

This proves the identity (1). Similarly, we can prove the identity in (2). To investigate the equality (3), for $(a^{'} \otimes a^{''}) \in A\widehat{\otimes}A$ we can write

$$\begin{aligned}
\Big((s_A \otimes s_A)^*\big(h \cdot (a, b)\big)\Big)(a^{'} \otimes a^{''}) &= \big(h \cdot (a, b)\big)\big((s_A \otimes s_A)(a^{'} \otimes a^{''})\big) \\
&= \big(h \cdot (a, b)\big)\big((a^{'}, -a^{'}e_B) \otimes (a^{''}, -a^{''}e_B)\big) \\
&= h\big((a, b) \cdot (a^{'}, -a^{'}e_B) \otimes (a^{''}, -a^{''}e_B)\big) \\
&= h\big((aa^{'}, -aa^{'}e_B + ba^{'} - ba^{'}e_B) \otimes (a^{''}, -a^{''}e_B)\big) \\
&= h\big((aa^{'}, -aa^{'}e_B) \otimes (a^{''}, -a^{''}e_B)\big) \\
&= h\big((s_A \otimes s_A)(aa^{'} \otimes a^{''})\big) \\
&= \big((s_A \otimes s_A)^*(h)\big)\big(a \cdot (a^{'} \otimes a^{''})\big) \\
&= \Big(\big((s_A \otimes s_A)^*(h)\big) \cdot a\Big)(a^{'} \otimes a^{''}) ,
\end{aligned}$$

this proves the identity in (3), and similarly one can proves the identity in (4). Now, using the identities

(1-4) we have

$$
\begin{aligned}
\lambda_L\Big(h\cdot(a,b)\Big) &= \Big(\Big(\lambda_A o(s_A\otimes s_A)^*\Big)\Big(h\cdot(a,b)\Big)+\Big(\varphi^* o\lambda_B o(q_B\otimes q_B)^*\Big)\Big(h\cdot(a,b)\Big)\\
&,\quad \Big(\lambda_B o(q_B\otimes q_B)^*\Big)\Big(h\cdot(a,b)\Big)\Big)\\
&= \Big(\lambda_A\Big((s_A\otimes s_A)^*(h)\cdot a\Big)+\Big(\varphi^* o\lambda_B\Big)\Big((q_B\otimes q_B)^*(h)\cdot(ae_B+b)\Big)\\
&,\quad \lambda_B\Big((q_B\otimes q_B)^*(h)\cdot(ae_B+b)\Big)\Big)\\
&= \Big(\lambda_A\Big((s_A\otimes s_A)^*(h)\Big)\cdot a+\Big(\varphi^* o\lambda_B\Big)\Big((q_B\otimes q_B)^*(h)\cdot ae_B\Big)\\
&+\quad \Big(\varphi^* o\lambda_B\Big)\Big((q_B\otimes q_B)^*(h)\cdot b\Big)\\
&,\quad \lambda_B\Big((q_B\otimes q_B)^*(h)\cdot ae_B\Big)+\lambda_B\Big((q_B\otimes q_B)^*(h)\cdot b\Big)\Big)\\
&= \Big(\Big(\lambda_A o(s_A\otimes s_A)^*(h)\cdot a+\Big(\lambda_B\Big((q_B\otimes q_B)^*(h)\cdot ae_B\Big)\Big)o\varphi\\
&+\quad \Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot b\Big)o\varphi\\
&,\quad \Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot ae_B+\Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot b\Big)\\
&= \Big(\Big(\lambda_A o(s_A\otimes s_A)^*(h)\cdot a+\Big(\varphi^*\Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\Big)\cdot a\\
&+\quad \Big(\Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot b\Big)o\varphi\\
&,\quad \Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot ae_B+\Big(\lambda_B o(q_B\otimes q_B)^*(h)\Big)\cdot b\Big)\\
&= \Big(\Big((\lambda_A o(s_A\otimes s_A)^*\Big)(h)+\Big(\varphi^* o\lambda_B o(q_B\otimes q_B)^*\Big)(h)\\
&,\quad (\lambda_B o(q_B\otimes q_B)^*\Big)(h)\Big)\cdot(a,b)\\
&= \lambda_L(h)\cdot(a,b)\,,
\end{aligned}
$$

this shows that λ_L is right$-L-$module map, where we have used the fact that $\Big(g\cdot ae_B\Big)o\varphi=\Big(\varphi^*(g)\Big)\cdot a$, for $g\in B^*$ and $a\in A$. With similar arguments, we can obtain that λ_L is left$-L-$module map, and consequently λ_L is bounded $L-$bimodule map. Finally, by using the following identities, in proof of the theorem (3.2)

$$
\begin{aligned}
\pi_L o(s_A\otimes s_A) &= s_A o\pi_A\,,\\
\pi_L o(q_B\otimes q_B) &= q_B o\pi_B\,,
\end{aligned}
$$

and for $(f,g)\in L^*$ we have

$$
\begin{aligned}
(\lambda_L o\pi_L^*)(f,g) &= \lambda_L(\pi_L^*(f,g))\\
&= \Big((\lambda_A o(s_A\otimes s_A)^* o\pi_L^*)(f,g)+(\varphi^* o\lambda_B o(q_B\otimes q_B)^* o\pi_L^*)(f,g)\\
&,\quad (\lambda_B o(q_B\otimes q_B)^* o\pi_L^*)(f,g)\Big)\\
&= \Big((\lambda_A o\pi_A^* o s_A^*)(f,g)+(\varphi^* o\lambda_B o\pi_B^* o q_B^*)(f,g),(\lambda_B o\pi_B^* o q_B^*)(f,g)\Big)\\
&= \Big(s_A^*(f,g)+(\varphi^* o q_B^*)(f,g),q_B^*(f,g)\Big)\\
&= (f,g)\,,
\end{aligned}
$$

this proves that $\lambda_L o\pi_L^*=id_{L^*}$, and the proof is completed. □

5. Examples

This section includes some illustrative examples.

Example 5.1. Let $L = A \times_\theta B$ be the $\theta-$Lau product of Banach algebras A and B with $\theta \in \Delta(A)$. If B is unital with unit e_B such that $e_B a = a e_B$ for all $a \in A$, then $A \times_\theta B$ is biprojective [biflat] if and only if A and B are biprojective [biflat].

Example 5.2. Let $L = A \times_T B$ be the $T-$Lau product of Banach algebras A and B with algebra homomorphism $T : A \to B$ ($\|T\| \leq 1$). If B is unital with unit e_B, then for all $a \in A$ we have $e_B T(a) = T(a) e_B = T(a)$. Hence $A \times_T B$ is biprojective [biflat] if and only if A and B are biprojective [biflat].

Example 5.3. Let $L = A \bowtie^\theta I$ be the amalgamation of Banach algebras A and B along the closed ideal I in B, with respect to continuous Banach algebra homomorphism $\theta : A \to B$. If I has unit e_I such that $\theta(a)e_I = e_I\theta(a)$, for all $a \in A$, then $A \bowtie^\theta I$ is biprojective [biflat] if and only if A and I are biprojective [biflat].

Example 5.4. Let G be a locally compact group and let $L^1(G)$ and $M(G)$ be its group algebra and measure algebra, respectively. It is known that $L^1(G)$ is unital if and only if G is descrete, and $L^1(G)$ is biprojective if and only if G is compact [4, 12]. Also, $L^1(G)$ is biflat if and only if G is amenable [4]. Therefore we have the following results

i) If $L^1(G) \bowtie L^1(G)$ is biprojective, then G is compact.
ii) If $L^1(G) \bowtie L^1(G)$ is biflat, then G is amenable.
iii) If G is descrete group, then $l^1(G) \bowtie l^1(G)$ is biprojective if and only if G is finite, and $l^1(G) \bowtie l^1(G)$ is biflat if and only if G is amenable.
iv) $M(G) \bowtie M(G)$ is biprojective [biflat] if and only if $M(G)$ is biprojrctive [biflat].
v) If $M(G) \bowtie L^1(G)$ is biprojective [biflat], then $M(G)$ is biprojective [biflat].
vi) Suppose that G be descrete, and A be a Banach algebra, such that $l^1(G)$ be an algebraic Banach $A-$bimodule.
If $A \bowtie l^1(G)$ is biprojective, then $l^1(G)$ and A are biprojective, and G is finite.
If $A \bowtie l^1(G)$ is biflat, then $l^1(G)$ and A are biflat, and G is amenable.
If G is finite and A is biprojective, then $A \bowtie l^1(G)$ is biprojective.
If G is amenable and A is biflat, then $A \bowtie l^1(G)$ is biflat.
vii) If $C_0(G) \bowtie M(G)$ is biprojective [biflat], then $C_0(G)$ and $M(G)$ are biprojective [biflat].
viii) If G is finite, then $C_0(G)$ and $C_0(G) \bowtie M(G)$ are biprojective.

Example 5.5. Let A'' be the second dual of a Banach algebra A with first Arens product $\Box$. Then A'' can be an $A-$bimodule by $aF =: \widehat{a}\Box F$ and $Fa =: F\Box\widehat{a}$, for all $a \in A$ and $F \in A''$, and with natural embeding of A into A'' ($a \mapsto \hat{a}$). Also it is known that if A is Arens regular, then A'' is unital if and only if A has bounded approximate identity, [3]. By theorems (3.1) and (4.1), if $L = A \bowtie A''$ is biprojective [biflat], then A is biprojective [biflat]. Also we can apply part (ii) of theorems (3.1) and (4.1) and theorems (3.2) and (4.2) for Arens regular Banach algebras A with bounded approximate identity and for $L = A \bowtie A''$.

On the other hand, by using the results in [8], if A is Arens regular with bounded opproximate identity, then $L = A \bowtie A''$ is biflat if and only if A'' is biflat. Besides if $A \lhd A''$, then $L = A \bowtie A''$ is biprojective if and only if A'' is biprojective.
One can use this example for a c^*-algebra, which is Arens regular and has bounded approximate identity. Also, for $A = L^1(G)$, in which G is compact, we will have $L^1(G) \lhd L^1(G)''$.

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