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Explicit Formulae for the Generalized Drazin Inverse of Block Matrices over a Banach Algebra ☆

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Abstract. In this paper, we give expressions for the generalized Drazin inverse of a (2,2,0) block matrix over a Banach algebra under certain circumstances, utilizing which we derive the generalized Drazin inverse of a 2×2 block matrix in a Banach algebra under weaker restrictions. Our results generalize and unify several results in the literature.

1. Introduction

The original motivation of expressions for the Drazin inverse of a (2,2,0) block matrix is from the perspective of the matrix to give the solution of the second-order differential equations (see [5–7]). It is still an open problem to find an explicit formula for the Drazin inverse of a block matrix without any restrictions upon the blocks. But there have been many formulae for the Drazin inverse of a block matrix under some restrictive assumptions (see [13, 17, 19, 26]). Especially, they can be applied to analyze the solutions of the state equations of the descriptor fractional discrete-time and the continuous-time linear systems with regular pencils [22, 23].

Let $\mathcal{A}$ be a complex unital Banach algebra. For $a \in \mathcal{A}$, let $\sigma(a)$ be the spectrum of a. We denote respectively the sets of all nilpotent and quasinilpotent elements ($\sigma(a) = 0$) of $\mathcal{A}$ by $N(\mathcal{A})$ and $QN(\mathcal{A})$. An element $a \in QN(\mathcal{A})$ if $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0$. Note that $N(\mathcal{A}) \subset QN(\mathcal{A})$. The equivalent definition of the Drazin inverse involving the condition of $N(\mathcal{A})$ and the concept of the generalized Drazin inverse in a Banach algebra were both introduced in [24].

An element a of $\mathcal{A}$ is generalized Drazin invertible in the case that there is an element $b \in \mathcal{A}$ satisfying

$$ab = ba, \quad bab = b, \quad \text{and} \quad a - a^2 b \in QN(\mathcal{A}).$$

Such b, if it exists, is unique; it is called a generalized Drazin inverse of a and will be denoted by a^d. Then the spectral idempotent a^π of a corresponding to 0 is given by $a^\pi = 1 - aa^d$. From the definition of the

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generalized Drazin inverse and from the well-known properties of the functional calculus, if a^d exists, then $(a^n)^d = (a^d)^n$, for an arbitrary integer $n \geq 1$.

The generalized Drazin inverse is a generalization of the Drazin inverse and the group inverse. Until now, some results of the generalized Drazin inverse have been developed (see [4, 18, 28]). In the rest of this section, we state some well-known and auxiliary results. We begin from classic results for the generalized Drazin inverse of triangular matrices in [9, 15], which generalize the results for the Drazin inverse of triangular matrices in [20, 30].

If $a \in \mathcal{A}$ and $p = p^2 \in \mathcal{A}$ is an idempotent, then a has the following block matrix representation relative to the idempotent p:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Lemma 1.1. *[9, 15] Let $x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ relative to an idempotent $p \in \mathcal{A}$, and let $y = \begin{bmatrix} d & 0 \\ b & a \end{bmatrix}$ relative to $1-p$. If a is generalized Drazin invertible in $p\mathcal{A}p$ and d is generalized Drazin invertible in $(1-p)\mathcal{A}(1-p)$, then x and y are generalized Drazin invertible and*

$$x^d = \begin{bmatrix} a^d & X \\ 0 & d^d \end{bmatrix}, \quad y^d = \begin{bmatrix} d^d & 0 \\ X & a^d \end{bmatrix},$$

where

$$X = a^\pi \sum_{i=0}^{\infty} a^i b (d^d)^{i+2} + \sum_{i=0}^{\infty} (a^d)^{i+2} b d^i d^\pi - a^d b d^d.$$

The following formula, proved for the case of the Drazin invertibility in [11], is called as Cline's formula. Cline's formula is generalized to the case of the generalized Drazin invertibility in [25].

Lemma 1.2. *[25, Corollary 2.6](Cline's Formula) For $a, b \in \mathcal{A}$, ab is generalized Drazin invertible if and only if so is ba. Furthermore, if ab is generalized Drazin invertible, then*

$$(ba)^d = b[(ab)^d]^2 a.$$

We state the following useful result which is proved for matrices in [21], for bounded linear operators in [16], and for elements of Banach algebra [9].

Lemma 1.3. *[9, Example 4.5] Let $a, b \in \mathcal{A}$ be generalized Drazin invertible, and let $ab = 0$. Then $a + b$ is generalized Drazin invertible, and*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

Recently, in [31–33], new formulae for the generalized Drazin inverse of block matrices and the sum in a Banach algebra were given.

Lemma 1.4. [33, Theorem 2.6] *Let $a, b \in \mathcal{A}$ be generalized Drazin invertible. If $a^\pi a^k b = a^\pi ab$ and $ba^\pi = b$, for some $k \in N$ such that $k > 1$, then $a + b$ is generalized Drazin invertible and*

$$\begin{aligned}(a+b)^d &= a^d + a^\pi \sum_{n=0}^{\infty} (b^d)^{n+1} a^n + \sum_{n=0}^{\infty} (a^d)^{n+2} b (a+b)^n b^\pi \\ &- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (a^d)^{n+2} b (a+b)^n (b^d)^{k+1} a^{k+1} - a^d b \sum_{n=0}^{\infty} (b^d)^{n+1} a^n.\end{aligned}$$

The generalized Drazin inverse of block matrices have various applications in singular differential equations and singular difference equations, Markov chains and iterative methods, and so on. We refer the reader to see [10, 27, 29, 34, 35].

In this paper, we derive new formulae for the generalized Drazin inverse of a $(2, 2, 0)$ block matrix over a Banach algebra under certain circumstances. Note that the two research routes of the generalized Drazin inverse of $(2, 2, 0)$ matrices and 2×2 matrices are both independent before. Furthermore, we apply the generalized Drazin inverse of (2,2,0) matrices to deduce the generalized Drazin inverse of 2×2 matrices over a Banach algebra under weaker restrictions, which generalizes and unifies several results of [8, 12, 14, 15, 31–33].

Throughout this paper, if the lower limit of a sum is greater than its upper limit, we always define the sum to be 0. For example, the sum $\sum_{k=0}^{-1} * = 0$. We adopt the convention that $a^0 = 1$.

2. Generalized Drazin Inverse of a (2,2,0) Operator Matrix

Let $\mathcal{A}$ be a complex unital Banach algebra. Let $\mathcal{M}_2(\mathcal{A})$ be the 2×2 matrix algebra over $\mathcal{A}$. Given an idempotent e in $\mathcal{A}$, we consider the set $\mathcal{M}_2(\mathcal{A}, e) = \begin{bmatrix} e\mathcal{A}e & e\mathcal{A}(1-e) \\ (1-e)\mathcal{A}e & (1-e)\mathcal{A}(1-e) \end{bmatrix} \subset \mathcal{M}_2(\mathcal{A})$. Then $\mathcal{M}_2(\mathcal{A}, e)$ is a unital Banach algebra with respect to the norm

$$\left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\| = \|a_{11} + a_{12} + a_{21} + a_{22}\|.$$

We deduce the relation between $\mathcal{A}$ and $\mathcal{M}_2(\mathcal{A})$ as follows.

Lemma 2.1. *Let e be an idempotent of $\mathcal{A}$. For any $a \in \mathcal{A}$, let a be generalized Drazin invertible, and*

$$\sigma(a) = \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix} \in \mathcal{M}_2(\mathcal{A}, e).$$

Then the mapping σ is an isometric Banach algebra isomorphism from $\mathcal{A}$ to $\mathcal{M}_2(\mathcal{A}, e)$ such that:

1. $(\sigma(a))^d = \sigma(a^d)$;
2. *if* $(\sigma(a))^d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, *then* $a^d = \alpha + \beta + \gamma + \delta$.

Proof. Utilizing [9, Lemma 2.1] we obtain that the mapping σ is an isometric Banach algebra isomorphism from $\mathcal{A}$ to $\mathcal{M}_2(\mathcal{A}, e)$. The rest of the proof is obvious. □

We combine the above result and Lemma 1.1 to analyze the property of the generalized Drazin invertibility under some special cases.

Lemma 2.2. *Let e be an idempotent of $\mathcal{A}$ and let a, ea (or $a(1-e)$)$\in \mathcal{A}$ be generalized Drazin invertible such that $ea(1-e) = 0$. Then $a(1-e)$ (or ea) is generalized Drazin invertible, and*

$$(ea)^d = ea^d, \quad (a(1-e))^d = a^d(1-e), \quad (ea)^n = ea^n,$$

for any positive integer n.

Proof. Since $ea(1-e) = 0$, combining Lemma 2.1 and Lemma 1.1, we have that $a(1-e)$ (or ea) is generalized Drazin invertible and $ea^d(1-e) = 0$. Then $eaea^d = eaa^d = ea^d ea$ and $ea^d eaea^d = ea^d$. Furthermore,

$$\begin{aligned} \lim_{n\to\infty} \|(ea - (ea)^2 ea^d)^n\|^{\frac{1}{n}} &= \lim_{n\to\infty} \|(eaa^\pi)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|ea^n a^\pi\|^{\frac{1}{n}} \\ &\le \lim_{n\to\infty} \|e\|^{\frac{1}{n}} \|a^n a^\pi\|^{\frac{1}{n}} = 0. \end{aligned}$$

Hence, $(ea)^d = ea^d$. Similarly, we can prove that $(a(1-e))^d = a^d(1-e)$. Using $ea(1-e) = 0$, we easily get $(ea)^n = ea^n$ for any positive integer n. $\square$

Now our first major result is demonstrated as follows.

Theorem 2.3. *Let* $x = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$, *where* a, b *and* ab^π *are generalized Drazin invertible elements of* $\mathcal{A}$. *If* $bab^\pi = 0$, *then* x *is generalized Drazin invertible, and*

$$x^d = P + Q + R + S,$$

where

$$P = \begin{bmatrix} \sum\limits_{i=0}^{\infty}(a^d)^{2i+1}b^ib^\pi & b^d - a^db^\pi ab^d - \sum\limits_{i=1}^{\infty}(a^d)^{2i+1}b^ib^\pi ab^d + \sum\limits_{i=0}^{\infty}(a^d)^{2i}b^ib^\pi \\ bb^d & -bb^dab^d \end{bmatrix},$$

$$Q = \begin{bmatrix} (a^\pi - \sum\limits_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^d & -(a^\pi - \sum\limits_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^dab^d \\ 0 & 0 \end{bmatrix},$$

$$R = \sum_{i=1}^{\infty}\begin{bmatrix} (a^\pi - \sum\limits_{j=1}^{\infty}(a^d)^{2j}b^j)b^ib^\pi a + a^\pi a^{2i}b^\pi abb^d + \sum\limits_{k=0}^{i-2}a^\pi a^{2k+2}b^{i-k-1}b^\pi a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b^d \\ bb^d & -bb^dab^d \end{bmatrix}^{2i+2},$$

$$S = \sum_{i=1}^{\infty}\begin{bmatrix} a^\pi a^{2i-1}b^\pi abb^d - \sum\limits_{j=0}^{\infty}(a^d)^{2j+1}b^{i+j}b^\pi a + \sum\limits_{k=0}^{i-2}a^\pi a^{2k+1}b^{i-k-1}b^\pi a & 0 \\ b^ib^\pi a & 0 \end{bmatrix}\begin{bmatrix} 0 & b^d \\ bb^d & -bb^dab^d \end{bmatrix}^{2i+1}.$$

Proof. We adopt the convention that $b^e = bb^d$. Let $e = \begin{bmatrix} b^e & 0 \\ 0 & 1 \end{bmatrix}$, σ as in Lemma 2.1, and $\sigma(x) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Since $bab^\pi = 0$, we have

$$\alpha = \begin{bmatrix} b^ea & b^e \\ bb^e & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} 0 & 0 \\ bb^\pi & 0 \end{bmatrix}, \ \gamma = \begin{bmatrix} b^\pi ab^e & b^\pi \\ 0 & 0 \end{bmatrix}, \ \delta = \begin{bmatrix} ab^\pi & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta \in \mathcal{A}$. Note that $(\beta+\gamma)^2 = \begin{bmatrix} bb^\pi & 0 \\ bb^\pi a & bb^\pi \end{bmatrix}$ is quasinilpotent, and then $\beta+\gamma$ is quasinilpotent too. Thus,

$$(\beta+\gamma)^d = 0.$$

Utilizing Lemma 2.2, we obtain $(ab^\pi)^d = a^db^\pi$. Further,

$$(\delta^d)^n = \begin{bmatrix} (a^d)^nb^\pi & 0 \\ 0 & 0 \end{bmatrix}$$

for any positive integer n. Combining $\beta\delta = 0, \gamma\delta = 0$ and Lemma 1.3 gives

$$\begin{aligned}(\beta+\gamma+\delta)^d &= \sum_{n=0}^{\infty}(\delta^d)^{n+1}(\beta+\gamma)^n = \delta^d + (\delta^d)^2(\beta+\gamma) + \sum_{i=1}^{\infty}(\delta^d)^{2i+1}(\beta+\gamma)^{2i} + \sum_{i=1}^{\infty}(\delta^d)^{2i+2}(\beta+\gamma)^{2i+1} \\ &= \begin{bmatrix} a^db^\pi + (a^d)^2b^\pi ab^e + \sum\limits_{i=1}^{\infty}[(a^d)^{2i+1}b^\pi b^i + (a^d)^{2i+2}b^\pi b^ia] & \sum\limits_{i=0}^{\infty}(a^d)^{2i+2}b^\pi b^i \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Then simple computations show that

$$(\beta+\gamma+\delta)^\pi = \begin{bmatrix} 1-(a^eb^\pi + a^db^\pi ab^e + \sum_{i=1}^{\infty}[(a^d)^{2i}b^\pi b^i + (a^d)^{2i+1}b^\pi b^i a]) & -\sum_{i=0}^{\infty}(a^d)^{2i+1}b^\pi b^i \\ 0 & 1 \end{bmatrix}.$$

Note that α has the group inverse

$$\alpha^\sharp = \begin{bmatrix} 0 & b^d \\ b^e & -b^eab^d \end{bmatrix},$$

and so $\alpha\alpha^\pi = 0$, where $\alpha^\pi = \begin{bmatrix} b^\pi & 0 \\ 0 & b^\pi \end{bmatrix}$. Now

$$(\beta+\gamma+\delta)^d\alpha^\pi = \begin{bmatrix} \sum_{i=0}^{\infty}(a^d)^{2i+1}b^ib^\pi & \sum_{i=0}^{\infty}(a^d)^{2i}b^ib^\pi \\ 0 & 0 \end{bmatrix}. \tag{1}$$

We consider the nth power of $\alpha^\sharp$. Utilizing Lemma 2.2 gets

$$(\alpha^\sharp)^n = \left(\begin{bmatrix} b^e & 0 \\ 0 & b^e \end{bmatrix}\begin{bmatrix} 0 & b^d \\ 1 & -ab^d \end{bmatrix}\right)^n = \begin{bmatrix} b^e & 0 \\ 0 & b^e \end{bmatrix}\begin{bmatrix} 0 & b^d \\ 1 & -ab^d \end{bmatrix}^n$$

for any positive integer n. Using $\alpha(\beta+\gamma+\delta) = 0$ and Lemma 1.3, we get

$$(\alpha+\beta+\gamma+\delta)^d = (\beta+\gamma+\delta)^d\alpha^\pi + (\beta+\gamma+\delta)^\pi \sum_{n=0}^{\infty}(\beta+\gamma+\delta)^n(\alpha^\sharp)^{n+1}. \tag{2}$$

Thus, simple computations show that

$$\sum_{n=0}^{\infty}(\beta+\gamma+\delta)^n(\alpha^\sharp)^{n+1} = \alpha^\sharp + \sum_{i=0}^{\infty}(\beta+\gamma+\delta)^{2i+1}(\alpha^\sharp)^{2i+2} + \sum_{i=1}^{\infty}(\beta+\gamma+\delta)^{2i}(\alpha^\sharp)^{2i+1}.$$

Substituting (1), the following expression for $(\beta+\gamma+\delta)^\pi\alpha^\sharp$:

$$(\beta+\gamma+\delta)^\pi\alpha^\sharp = \begin{bmatrix} 0 & b^d - a^db^\pi ab^d - \sum_{i=1}^{\infty}(a^d)^{2i+1}b^ib^\pi ab^d \\ bb^d & -bb^dab^d \end{bmatrix},$$

the following equality involving $(\beta+\gamma+\delta)^\pi \sum_{i=0}^{\infty}(\beta+\gamma+\delta)^{2i+1}(\alpha^\sharp)^{2i+2}$:

$$\begin{aligned}
&(\beta+\gamma+\delta)^\pi \sum_{i=0}^{\infty}(\beta+\gamma+\delta)^{2i+1}(\alpha^\sharp)^{2i+2} \\
&= (\beta+\gamma+\delta)^\pi \sum_{i=0}^{\infty}\left(\gamma(\beta\gamma)^i + \sum_{k=0}^{i-1}\delta^{2k+2}\gamma(\beta\gamma)^{i-k-1}\right)(\alpha^\sharp)^{2i+2} \\
&= (\beta+\gamma+\delta)^\pi \left(\gamma(\alpha^\sharp)^2 + \sum_{i=1}^{\infty}\left(\gamma(\beta\gamma)^i + \sum_{k=0}^{i-2}\delta^{2k+2}\gamma(\beta\gamma)^{i-k-1} + \delta^{2i}\gamma\right)(\alpha^\sharp)^{2i+2}\right)
\end{aligned}$$

$$= \begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^d & -(a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^d ab^d \\ 0 & 0 \end{bmatrix}$$

$$+ \sum_{i=1}^{\infty}\left(\begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^i b^\pi a & (a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^i b^\pi \\ 0 & 0 \end{bmatrix}\right.$$

$$\left. + \sum_{k=0}^{i-2}\begin{bmatrix} a^\pi a^{2k+2}b^{i-k-1}b^\pi a & a^\pi a^{2k+2}b^{i-k-1}b^\pi \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a^\pi a^{2i}b^\pi abb^d & a^\pi a^{2i}b^\pi \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & b^d \\ b^e & -b^e ab^d \end{bmatrix}^{2i+2}$$

$$= \begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^d & -(a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^\pi ab^d ab^d \\ 0 & 0 \end{bmatrix}$$

$$+ \sum_{i=1}^{\infty}\left(\begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty}(a^d)^{2j}b^j)b^i b^\pi a + a^\pi a^{2i}b^\pi abb^d & 0 \\ 0 & 0 \end{bmatrix} + \sum_{k=0}^{i-2}\begin{bmatrix} a^\pi a^{2k+2}b^{i-k-1}b^\pi a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & b^d \\ b^e & -b^e ab^d \end{bmatrix}^{2i+2},$$

and the next equality containing $(\beta+\gamma+\delta)^\pi \sum_{i=1}^{\infty}(\beta+\gamma+\delta)^{2i}(\alpha^\sharp)^{2i+1}$:

$$(\beta+\gamma+\delta)^\pi \sum_{i=1}^{\infty}(\beta+\gamma+\delta)^{2i}(\alpha^\sharp)^{2i+1}$$

$$= (\beta+\gamma+\delta)^\pi \sum_{i=1}^{\infty}\left((\beta\gamma)^i + \sum_{k=0}^{i-1}\delta^{2k+1}\gamma(\beta\gamma)^{i-k-1}\right)(\alpha^\sharp)^{2i+1}$$

$$= (\beta+\gamma+\delta)^\pi \sum_{i=1}^{\infty}\left((\beta\gamma)^i + \sum_{k=0}^{i-2}\delta^{2k+1}\gamma(\beta\gamma)^{i-k-1} + \delta^{2i-1}\gamma\right)(\alpha^\sharp)^{2i+1}$$

$$= \sum_{i=1}^{\infty}\left(\begin{bmatrix} a^\pi a^{2i-1}b^\pi abb^d - \sum_{j=0}^{\infty}(a^d)^{2j+1}b^{i+j}b^\pi a & a^\pi a^{2i-1}b^\pi - \sum_{j=0}^{\infty}(a^d)^{2j+1}b^{i+j}b^\pi \\ b^i b^\pi a & b^i b^\pi \end{bmatrix}\right.$$

$$\left. + \sum_{k=0}^{i-2}\begin{bmatrix} a^\pi a^{2k+1}b^{i-k-1}b^\pi a & a^\pi a^{2k+1}b^{i-k-1}b^\pi \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & b^d \\ b^e & -b^e ab^d \end{bmatrix}^{2i+1}$$

$$= \sum_{i=1}^{\infty}\left(\begin{bmatrix} a^\pi a^{2i-1}b^\pi abb^d - \sum_{j=0}^{\infty}(a^d)^{2j+1}b^{i+j}b^\pi a & 0 \\ b^i b^\pi a & 0 \end{bmatrix} + \sum_{k=0}^{i-2}\begin{bmatrix} a^\pi a^{2k+1}b^{i-k-1}b^\pi a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & b^d \\ b^e & -b^e ab^d \end{bmatrix}^{2i+1}$$

into (2) will give the expression of x^d that we wanted. □

We next consider the interesting representation of $(x^d)^n$ for any positive integer n, which needs to be applied to prove the next main results in Section 3.

Corollary 2.4. *Let* $x = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$*, where* a, b *and* ab^π *are generalized Drazin invertible elements of* $\mathcal{A}$*. If* $bab^\pi = 0$*, then* x *is generalized Drazin invertible, and, for any positive integer* n*,*

$$(x^d)^n = (P+Q+R)P^{n-1} + \sum_{i=0}^{n-1}P^{n-(i+1)}SP^i,$$

where P*,* Q*,* R *and* S *are defined as in Theorem 2.3.*

Proof. By Theorem 2.3, $x^d = P + Q + R + S$. Since $ba^{i+1}b^\pi = b(ab^\pi)^{i+1} = 0$, for any nonnegative integer i, and $baa^db^\pi = 0$, we have $Q^2 = R^2 = S^2 = 0$, $PQ = PR = QR = RQ = QS = SQ = RS = SR = 0$ and $SPS = 0$. Then, by a routine computation, we get the expression of $(x^d)^n$ as shown in Corollary 2.4. □

Remark that we assume a, b and $b^\pi a$ are generalized Drazin invertible elements of $\mathcal{A}$. As a dual version of Theorem 2.3, x^d can be also proved to be generalized Drazin invertible and expressed when $b^\pi ab = 0$.

3. Applications to a 2 × 2 Block Matrix

In this section, we apply the generalized Drazin inverse of (2,2,0) matrices in a Banach algebra to deduce the generalized Drazin inverse of 2×2 matrices under some restrictions, which generalizes and unifies several results of [8, 12, 14, 15, 31–33].

Theorem 3.1. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $a, b, c, d \in \mathcal{A}$ *such that* a, d, bc *and* $a(bc)^\pi$ *are generalized Drazin invertible. If* $bca(bc)^\pi = 0$ *and* $bd = 0$, *then* x *is generalized Drazin invertible, and*

$$\begin{aligned} x^d &= \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (d^d)^{n+1} \end{bmatrix} \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}^n \left(1 - \begin{bmatrix} a^2 + bc & a \\ ca & c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \right) \\ &+ \begin{bmatrix} a & 1 \\ d^\pi c & 0 \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ d^\pi d^{n+1} c & 0 \end{bmatrix} u^{n+2} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, \end{aligned}$$

where, for positive integer n,

$$u^n = (P_1 + Q_1 + R_1)P_1^{n-1} + \sum_{i=0}^{n-1} P_1^{n-(i+1)} S_1 P_1^i,$$

$$P_1 = \begin{bmatrix} \sum_{i=0}^{\infty} (a^d)^{2i+1}(bc)^i(bc)^\pi & (bc)^d - a^d(bc)^\pi a(bc)^d - \sum_{i=1}^{\infty} (a^d)^{2i+1}(bc)^i(bc)^\pi a(bc)^d + \sum_{i=0}^{\infty} (a^d)^{2i}(bc)^i(bc)^\pi \\ bc(bc)^d & -bc(bc)^d a(bc)^d \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty} (a^d)^{2j}(bc)^j)(bc)^\pi a(bc)^d & -(a^\pi - \sum_{j=1}^{\infty} (a^d)^{2j}(bc)^j)(bc)^\pi a(bc)^d a(bc)^d \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned} R_1 &= \sum_{i=1}^{\infty} \begin{bmatrix} (a^\pi - \sum_{j=1}^{\infty} (a^d)^{2j}(bc)^j)(bc)^i(bc)^\pi a + a^\pi a^{2i}(bc)^\pi abc(bc)^d + \sum_{k=0}^{i-2} a^\pi a^{2k+2}(bc)^{i-k-1}(bc)^\pi a & 0 \\ 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & (bc)^d \\ bc(bc)^d & -bc(bc)^d a(bc)^d \end{bmatrix}^{2i+2}, \end{aligned}$$

$$\begin{aligned} S_1 &= \sum_{i=1}^{\infty} \begin{bmatrix} a^\pi a^{2i-1}(bc)^\pi abc(bc)^d - \sum_{j=0}^{\infty} (a^d)^{2j+1}(bc)^{i+j}(bc)^\pi a + \sum_{k=0}^{i-2} a^\pi a^{2k+1}(bc)^{i-k-1}(bc)^\pi a & 0 \\ (bc)^i(bc)^\pi a & 0 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & (bc)^d \\ bc(bc)^d & -bc(bc)^d a(bc)^d \end{bmatrix}^{2i+1}. \end{aligned}$$

Proof. Suppose that $x = y + z$, where

$$y = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \qquad \text{and} \qquad z = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}.$$

Obviously, y is generalized Drazin invertible,

$$y^d = \begin{bmatrix} 0 & 0 \\ 0 & d^d \end{bmatrix} \quad \text{and} \quad y^\pi = \begin{bmatrix} 1 & 0 \\ 0 & d^\pi \end{bmatrix}.$$

To prove that z is generalized Drazin invertible, we write $z = pq$, where

$$p = \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}.$$

Applying Theorem 2.3, we deduce that $qp = \begin{bmatrix} a & 1 \\ bc & 0 \end{bmatrix}$ is generalized Drazin invertible and

$$(qp)^d = P_1 + Q_1 + R_1 + S_1.$$

By Lemma 1.2, $z = pq$ is generalized Drazin invertible and $z^d = p[(qp)^d]^2 q$. For positive integer n, by Corollary 2.4, we get $[(qp)^d]^n = u^n$ and $(z^d)^{n+1} = p[(qp)^d]^{n+2} q$.

Because $zy = 0$, by Lemma 1.3, we have that x is generalized Drazin invertible and

$$x^d = \sum_{n=0}^{\infty} (y^d)^{n+1} z^n z^\pi + \sum_{n=0}^{\infty} y^\pi y^n (z^d)^{n+1}$$

which implies the final formula. □

The following corollary relaxes and removes some restrictions of Theorem 2.3 in [32], where Mosić consider the conditions $bd = 0$, $a(bc)^\pi = 0$, $c(bc)^\pi = 0$, and $(bc)^\pi b = 0$.

Corollary 3.2. *Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathcal{A}$ such that a and d are generalized Drazin invertible. If $a(bc)^\pi = 0$, $bd = 0$, then x is generalized Drazin invertible, and*

$$\begin{aligned} x^d &= \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (d^d)^{n+1} \end{bmatrix} \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}^n \left(1 - \begin{bmatrix} a^2 + bc & a \\ ca & c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \right) \\ &+ \begin{bmatrix} a & 1 \\ d^\pi c & 0 \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ d^\pi d^{n+1} c & 0 \end{bmatrix} u^{n+2} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, \end{aligned}$$

where, for positive integer n,

$$u^n = (P_2 + Q_2 + R_2) P_2^{n-1} + \sum_{i=0}^{n-1} P_2^{n-(i+1)} S_2 P_2^i,$$

$$P_2 = \begin{bmatrix} 0 & (bc)^d + (bc)^\pi \\ bc(bc)^d & -bc(bc)^d a (bc)^d \end{bmatrix}, \quad R_2 = \sum_{i=1}^{\infty} \begin{bmatrix} (bc)^i (bc)^\pi a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & (bc)^d \\ bc(bc)^d & -bc(bc)^d a (bc)^d \end{bmatrix}^{2i+2},$$

$$Q_2 = \begin{bmatrix} (bc)^\pi a (bc)^d & -(bc)^\pi a (bc)^d a (bc)^d \\ 0 & 0 \end{bmatrix}, \quad S_2 = \sum_{i=1}^{\infty} \begin{bmatrix} 0 & 0 \\ (bc)^i (bc)^\pi a & 0 \end{bmatrix} \begin{bmatrix} 0 & (bc)^d \\ bc(bc)^d & -bc(bc)^d a (bc)^d \end{bmatrix}^{2i+1}.$$

Proof. The result can be deduced by routine computations. □

We now analyze some other special cases of the preceding theorem. As the corollaries of Theorem 3.1, the expressions for the generalized Drazin inverse of the block matrix in a Banach algebra and the operator matrix over a Banach space under some conditions all can be shown as follows,

(1) $a = 0$ and $d = 0$ (see [31, Lemma 1.5]);

(2) $ab = 0, ca = 0$, and $d = 0$ (see [31, Corollary 2.1(ii)]);
(3) $ca = 0$ and $d = 0$ (see [31, Corollary 2.1(i)]);
(4) $bc = 0, bd = 0$, and $dc = 0$ (see [15, Theorem 5.3], [33, Theorem 3.3(ii)]);
(5) $bc = 0, bd = 0$ (see [14, Theorem 2], [33, Theorem 3.3(i)]);
(6) $bca = 0, bd = 0, dc = 0$ (see [8, Theorem 4.4]);
(7) $bca = 0, bd = 0$, and bc is nilpotent (see [8, Theorem 4.2]).

Above all, Theorem 3.1 relaxes some conditions in each item of (1)–(7) and gives a unified generalization.

We conclude a dual version of Theorem 3.1 with some remarks. Using a dual version of Theorem 2.3, we can give an expression of the generalized Drazin inverse x^d under the following condition:

$$(bc)^\pi abc = 0, \quad bd = 0,$$

which gives a unified generalization of [31, Lemma 1.5], [33, Theorem 3.3(ii)], [33, Theorem 3.3(i)], [15, Theorem 5.3], [14, Theorem 2], [12, Theorem 1], and [32, Theorem 2.5].

Utilizing the similar methods as Theorem 3.1, we can give the following corollary.

Corollary 3.3. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $a, b, c, d \in \mathcal{A}$ *such that* a, d, bc *and* $a(bc)^\pi$ *are generalized Drazin invertible. If* $bca(bc)^\pi = 0$ *and* $dc = 0$, *then* x *is generalized Drazin invertible, and*

$$\begin{aligned} x^d &= \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & bd^\pi \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} u^{n+2} \begin{bmatrix} 1 & 0 \\ 0 & bd^n d^\pi \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \left(1 - \begin{bmatrix} a^2+bc & a \\ ca & c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}\right) \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 0 \\ 0 & (d^d)^{n+1} \end{bmatrix}, \end{aligned}$$

where u *is defined as in Theorem 3.1.*

Proof. We use the same notation as in the proof of Theorem 3.1. Since $yz = 0$, using Lemma 1.3, we prove this result. □

The following corollary relaxes and removes some restrictions of Theorem 2.4 in [32], where Mosić consider the conditions $dc = 0$, $a(bc)^\pi = 0$, $c(bc)^\pi = 0$, and $(bc)^\pi b = 0$.

Corollary 3.4. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $a, b, c, d \in \mathcal{A}$ *such that* a, d, bc *and* $a(bc)^\pi$ *are generalized Drazin invertible. If* $a(bc)^\pi = 0$ *and* $dc = 0$, *then* x *is generalized Drazin invertible, and*

$$\begin{aligned} x^d &= \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & bd^\pi \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} u^{n+2} \begin{bmatrix} 1 & 0 \\ 0 & bd^n d^\pi \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \left(1 - \begin{bmatrix} a^2+bc & a \\ ca & c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}\right) \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 0 \\ 0 & (d^d)^{n+1} \end{bmatrix}, \end{aligned}$$

where u *is defined as in Corollary 3.2.*

We consider Lemma 1.4 to obtain a new representation of x^d as follows.

Corollary 3.5. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $a, b, c, d \in \mathcal{A}$ *such that* a, d, bc *and* $a(bc)^\pi$ *are generalized Drazin invertible. If*

$bca(bc)^\pi = 0$, $d^\pi d^k c = d^\pi dc$, $k \in N$, $k > 1$, and $bd^d = 0$, then x is generalized Drazin invertible, and

$$\begin{aligned} x^d &= \begin{bmatrix} 0 & 0 \\ 0 & d^d \end{bmatrix} + \begin{bmatrix} a & 1 \\ d^\pi c - d^d ca & -d^d c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} a & 1 \\ d^\pi c - d^d ca & -d^d c \end{bmatrix} u^{n+2} \begin{bmatrix} 0 & 0 \\ 0 & bd^n \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ (d^d)^{n+2} c & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \left(1 - \begin{bmatrix} a^2 + bc & a \\ ca & c \end{bmatrix} u^2 \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \right) \\ &- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \begin{bmatrix} 0 & 0 \\ (d^d)^{n+2} c & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} u^{k+2} \begin{bmatrix} 0 & 0 \\ 0 & bd^{k+1} \end{bmatrix}. \end{aligned}$$

Proof. If we use the same notation as in the proof of Theorem 3.1, notice that

$$y^\pi y^k z = \begin{bmatrix} 0 & 0 \\ d^\pi d^k c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ d^\pi dc & 0 \end{bmatrix} = y^\pi yz$$

and

$$zy^\pi = \begin{bmatrix} a & bd^\pi \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = z.$$

Using Lemma 1.4, we get that x is generalized Drazin invertible and

$$\begin{aligned} x^d &= y^d + y^\pi \sum_{n=0}^{\infty} (z^d)^{n+1} y^n + \sum_{n=0}^{\infty} (y^d)^{n+2} z x^n z^\pi \\ &- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (y^d)^{n+2} z x^n (z^d)^{k+1} y^{k+1} - y^d z \sum_{n=0}^{\infty} (z^d)^{n+1} y^n. \end{aligned}$$

By elementary computations, we finish the proof. □

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