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On the φ-Normal Meromorphic Functions

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Abstract. In this paper we study a family of φ-normal meromorphic functions, and obtain some results which improve and generalize previous results in this area, especially the works of Lappan [2], Aulaskari-Rättyä [1], Xu-Qiu [7] and the recent work of Tan-Thin [6].

1. Introduction and Results

Classically, a family $\mathcal{F}$ of meromorphic functions on a domain $D \subset \mathbb{C}$ is said to be normal if every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on every compact subset of D to a meromorphic function which may be ∞ identically. See [4, 9]. In 1957, Lehto and Virtanen [3] introduced the concept of normal meromorphic functions in connection with the study of boundary behaviour of meromorphic functions. Let $\Delta = \{z;\ |z| < 1\}$ be the unit disc in $\mathbb{C}$, and let $\mathcal{M}(\Delta)$ denote the set of all meromorphic functions on Δ. A function $f \in \mathcal{M}(\Delta)$ is called normal if

$$\sup\{\ (1-|z|^2)\ f^{\#}(z);\ z \in \Delta\} < \infty$$

where $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivatives of f. The close relation between normal families and normal functions is as following. A meromorphic function f is normal if and only if the family $\mathcal{F}_f = \{f \circ \tau;\ \tau \in \mathrm{Aut}(\Delta)\}$ is normal. Since then normal meromorphic functions have been studied intensively (see [4] and [5]). For example, the well-known Lappan [2] five-point theorem says that $f \in \mathcal{M}(\Delta)$ is a normal function if $\sup\{(1-|z|^2)\ f^{\#}(z);\ z \in f^{-1}(E)\}$ is bounded for some five-point subset E of the image set $f(\Delta)$.

In 2011, R. Aulaskari and J. Rättyä [1] introduce the concept of φ-normal functions. We can state the definition as followings to cover normal functions.

Definition 1.1. *([1, 6]) An increasing function $\varphi : [0,1) \to (0,\infty)$ is called smoothly increasing if*

$$\varphi(r)(1-r) \geq 1, \quad r \in [0,1), \tag{1.1}$$

and

$$\mathcal{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|)|)}{\varphi(|a|)} \to 1 \quad as \quad |a| \to 1^- \tag{1.2}$$

uniformly on compact subsets of $\mathbb{C}$.

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Definition 1.2. *([1, 6]) For a smoothly increasing function φ, a function $f \in \mathcal{M}(\Delta)$ is called φ-normal if*

$$\|f\|_{\mathcal{N}^{\varphi}} := \sup_{z\in\Delta} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty. \tag{1.3}$$

The class of all φ-normal functions is denoted by $\mathcal{N}^{\varphi}$.

Remark 1.3. *In [1] condition (1.1) is replaced by a stricter one*

$$\varphi(r)(1-r) \to \infty \quad as \quad r \to 1^{-}. \tag{1.4}$$

Note that if φ satisfies (1.4) then we will always further assume that $\varphi(r)(1-r) \geq 1$ for all $r \in [0,1)$. This because $\varphi^{*}(r) := \varphi(r) + (1-r)^{-1}$ satisfies $\mathcal{N}^{\varphi^{*}} = \mathcal{N}^{\varphi}$.

Also in [1], Aulaskari and Rättyä obtained a version of Lappan's five-point theorem for φ-normal functions.

Theorem A. ([1, Theorem 9]) *Let φ be a smoothly increasing function and let $f \in \mathcal{M}(\Delta)$. If there exists a set E of five distinct points in $\widehat{\mathbb{C}}$ such that*

$$\sup\{\, f^{\#}(z)/\varphi(|z|);\ z \in f^{-1}(E)\} < \infty,$$

then f is φ-normal.

Recently, motivated by the extension of the spherical derivative, Y. Xu and H. L. Qiu improved Theorem A as following.

Theorem B. ([7, Theorem 2]) *Let φ be a smoothly increasing function, and let k be a positive integer. Let $f \in \mathcal{M}(\Delta)$ such that*

$$\sup\{\, f^{(i)}(z);\ z \in f^{-1}(\{0\}), i = 0, 1, \ldots, k-1\} < \infty.$$

If there exists a set E of $k+4$ distinct points in $\widehat{\mathbb{C}}$ such that

$$\sup\Big\{ \frac{1}{\varphi(|z|)^{k}} \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}};\ z \in f^{-1}(E)\Big\} < \infty,$$

then f is φ-normal.

In this paper, our first main result is as following.

Theorem 1.4. *Let φ be a smoothly increasing function, and let k be a positive integer. Let $\mathcal{F} \subset \mathcal{M}(\Delta)$ such that*

$$\sup\{\, f^{(i)}(z);\ z \in f^{-1}(\{0\}), i = 0, 1, \ldots, k-1, f \in \mathcal{F}\} < \infty. \tag{1.5}$$

If there exists a set E of $k+4$ distinct points in $\widehat{\mathbb{C}}$ such that

$$\sup\Big\{ \frac{1}{\varphi(|z|)^{k}} \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}};\ z \in f^{-1}(E), f \in \mathcal{F}\Big\} < \infty, \tag{1.6}$$

then

$$\sup\{\|f\|_{\mathcal{N}^{\varphi}};\ f \in \mathcal{F}\} < \infty. \tag{1.7}$$

Clearly, Theorem B is just Theorem 1.4 in the case of $\mathcal{F} = \{f\}$. Thus, Theorem 1.4 is an improvement of Theorems A and B. In addition, noting that the condition (1.5) holds naturally if all zeros of $f \in \mathcal{F}$ are of multiplicity at least k, we obtain the following corollary.

Corollary 1.5. *Let φ be a smoothly increasing function, and let k be a positive integer. Let $\mathcal{F} \subset \mathcal{M}(\Delta)$ such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k. If there exists a set E of $k+4$ distinct points in $\widehat{\mathbb{C}}$ such that*

$$\sup\Big\{ \frac{1}{\varphi(|z|)^{k}} \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}};\ z \in f^{-1}(E), f \in \mathcal{F}\Big\} < \infty, \tag{1.8}$$

then

$$\sup\{\|f\|_{\mathcal{N}^{\varphi}};\ f \in \mathcal{F}\} < \infty.$$

The following example shows that the existence of family $\mathcal{F}$ with property $\sup\{\|f\|_{\mathcal{N}^\varphi};\ f \in \mathcal{F}\} < \infty$.

Example 1.6. *Let $\mathcal{F} = \{f_n(z)\}_{n=1}^{\infty}$, where $f_n(z) := n(1 - z), z \in \Delta$, and let $z_n = 1 - \frac{1}{n}$. Obviously, we have $f_n^{\#}(z_n) = \frac{n}{2} \to \infty$ as $n \to \infty$. Thus,*

$$\sup_{z\in\Delta, f\in\mathcal{F}} f^{\#}(z) = \infty.$$

However, taking $E = \{0, 1, 2, 3, 4\}$, by simple calculation, we have

$$f_n(z) \in E \Rightarrow (1 - |z|) f_n^{\#}(z) \le \frac{1}{2}.$$

It follows Corollary 1.5 that

$$\sup_{z\in\Delta, f\in\mathcal{F}} (1 - |z|) f^{\#}(z) < \infty.$$

More recently, T. Van Tan and N. Van Thin [6] reduced the number "five "in Lappan's five-points theorem by bounding the spherical derivatives of meromrophic functions studied.

Theorem C. ([6, Theorem 4]) *Let φ be a smoothly increasing function and let $f \in \mathcal{M}(\Delta)$. If there exists a set E of four distinct points in $\widehat{\mathbb{C}}$ such that*

$$\sup\{\ f^{\#}(z)/\varphi(|z|);\ z \in f^{-1}(E)\} < \infty,$$

and

$$\sup\{\ (f')^{\#}(z);\ z \in f^{-1}(E\backslash\{\infty\})\} < \infty,$$

then f is φ-normal.

We also prove the following theorems generalize Theorem C.

Theorem 1.7. *Let φ be a smoothly increasing function, and let k be a positive integer. Let $\mathcal{F} \subset \mathcal{M}(\Delta)$ such that*

$$\sup\{\ f^{(i)}(z);\ z \in f^{-1}(\{0\}), i = 0, 1, \ldots, k - 1, f \in \mathcal{F}\} < \infty. \tag{1.9}$$

If there exists a set E of $[k/2] + 4$ distinct points in $\widehat{\mathbb{C}}$ such that

$$\sup\Big\{\ \frac{1}{\varphi(|z|)^k} \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}};\ z \in f^{-1}(E), f \in \mathcal{F}\Big\} < \infty \tag{1.10}$$

and

$$\sup\{\ (f^{(k)})^{\#}(z);\ z \in f^{-1}(E\backslash\{\infty\}), f \in \mathcal{F}\} < \infty < \infty, \tag{1.11}$$

then

$$\sup\{\|f\|_{\mathcal{N}^\varphi};\ f \in \mathcal{F}\} < \infty.$$

Here and in the following, $[x]$ denotes the greatest integer less than or equal to x.

As a special case, if we take $k = 1$ in Theorems 1.7,then we have:

Corollary 1.8. *Let φ be a smoothly increasing function, and let $\mathcal{F} \subset \mathcal{M}(\Delta)$. If there exists a set E of four distinct points in $\widehat{\mathbb{C}}$ such that*

$$\sup\{\ f^{\#}(z)/\varphi(|z|);\ z \in f^{-1}(E), f \in \mathcal{F}\} < \infty,$$

and

$$\sup\{\ (f')^{\#}(z);\ z \in f^{-1}(E\backslash\{\infty\}), f \in \mathcal{F}\} < \infty,$$

then

$$\sup\{\|f\|_{\mathcal{N}^\varphi};\ f \in \mathcal{F}\} < \infty.$$

2. Some Lemmas

To prove our results, we require some lemmas. We assume the standard notation of value distribution theory. For details, see [4, 5, 8].

Lemma 2.1 (Zalcman's Lemma, see [9]). *Let $\mathcal{F}$ be a family of meromorphic functions in the disk Δ. Then if $\mathcal{F}$ is not normal at a point $z_0 \in \Delta$, then there exist*

1) a real number r, $0 < r < 1$ and points z_n, $|z_n| < r, z_n \to z_0$,

2) positive numbers ϱ_n, $\varrho_n \to 0^+$,

3) functions $f_n, f_n \in \mathcal{F}$ such that

$$F_n(\xi) := f_n(z_n + \varrho_n \xi) \to F(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $F(\xi)$ is a nonconstant meromorphic function of $\mathbb{C}$.

Lemma 2.2 (First Main Theorem). *Suppose that f is meromorphic in $\mathbb{C}$ and a is any complex number. Then for $r > 0$ we have*

$$T\Big(r, \frac{1}{f-a}\Big) = T(r, f) + O(1).$$

Lemma 2.3 (Second Main Theorem). *Suppose that f is a non-constant meromorphic in $\mathbb{C}$ and a_j $(1 \le j \le q)$ are $q(\ge 3)$ distinct values in $\widehat{\mathbb{C}}$. Then*

$$(q-2)T(r, f) \le \sum_{j=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) + S(r, f).$$

Lemma 2.4. *Suppose that f is a non-constant meromorphic in $\mathbb{C}$ and k is a positive integer. Then*

$$T(r, f^{(k)}) \le T(r, f) + k\overline{N}(r, f) + S(r, f) \le (k+1)T(r, f) + S(r, f).$$

3. Proof of Theorem 1.4

Suppose, to the contrary, that assertion (1.7) fails to be valid. Then, we can find $f_n \in \mathcal{F}, z_n \in \Delta$ such that such that

$$\frac{f_n^{\#}(z_n)}{\varphi(|z_n|)} \to \infty, \quad n \to \infty. \tag{3.1}$$

By passing to a subsequence if necessary, we may assume that $z_n \to z_0$. Then $|z_0| \le 1$. We separate two cases:

Case 1. $0 \le |z_0| < 1$.

Since the function φ is increasing, the inequality

$$\frac{f_n^{\#}(z_n)}{\varphi(0)} \ge \frac{f_n^{\#}(z_n)}{\varphi(|z_n|)} \tag{3.2}$$

holds for all positive integer n. Therefore,from (3.1) and (3.2), we obtain

$$f_n^{\#}(z_n) \to \infty, \quad n \to \infty.$$

It follows from Marty's Theorem that $\{f_n\}_{n=1}^{\infty}$ is not normal at the point z_0. According to Lemma 2.1, there exist a subsequence of functions f_n (that will also be denoted by f_n), points $u_n \to z_0$, and positive numbers $\varrho_n \to 0$, such that

$$F_n(\xi) := f_n(u_n + \varrho_n \xi) \to F(\xi) \tag{3.3}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $F(\xi)$ is a nonconstant meromorphic function of $\mathbb{C}$. Consequently,

$$F_n^{(i)}(\xi) := \varrho_n^i f_n(u_n + \varrho_n \xi) \to F^{(i)}(\xi) \tag{3.4}$$

uniformly on compact subsets of $\mathbb{C}\backslash\{\text{Poles of } F\}$, $i = 1, 2, \ldots$.

Claim 1. $F(\xi) \in E \implies \frac{|F^{(k)}(\xi)|}{1+|F(\xi)|^{k+1}} = 0.$

Suppose that $F(\xi_0) = a \in E$, by Hurwitz's theorem and (3.3), there exists a sequence $\xi_n \to \xi_0$ such that $F_n(\xi_n) = f_n(u_n + \varrho_n\xi_n) = a$. By the hypothesis (1.6), there exists a constant $M > 0$ such that

$$\frac{1}{\varphi(|u_n + \varrho_n\xi_n|)^k} \frac{|f_n^{(k)}(u_n + \varrho_n\xi_n)|}{1 + |f_n(u_n + \varrho_n\xi_n)|^{k+1}} \le M. \tag{3.5}$$

for sufficiently large n. Since $u_n + \varrho_n\xi_n \to z_0$ and $|z_0| < 1$, one can take r_1, $|z_0| < r_1 < 1$. And hence, $|u_n + \varrho_n\xi_n| < r_1$ for sufficiently large n. Then, for the increasing function φ,

$$\varphi(|u_n + \varrho_n\xi_n|) \le \varphi(r_1). \tag{3.6}$$

From (3.5), (3.6) and an elementary calculation, we yield

$$\begin{aligned} \frac{|F_n^{(k)}(\xi_n)|}{1 + |F_n(\xi_n)|^{k+1}} &= \varrho_n^k \frac{|f_n^{(k)}(u_n + \varrho_n\xi_n)|}{1 + |f_n(u_n + \varrho_n\xi_n)|^{k+1}} \\ &\le \varrho_n^k M\, \varphi(|u_n + \varrho_n\xi_n|)^k \\ &\le \varrho_n{}^k M\, \varphi(r_1) \end{aligned}$$

for sufficiently large n. Then, letting $n \to \infty$ and noting (3.4) , we obtain $\frac{|F^{(k)}(\xi_0)|}{1+|F(\xi_0)|^{k+1}} = 0$. This proves the claim.

Therefore, the Claim 1 implies that if $F(\xi_0) \in E$, then ξ_0 is either the zero of $F^{(k)}(\xi)$ or the multiple pole of $F(\xi)$. On the other hand, the assumption (1.5) and Hurwitz's Theorem imply that $F^{(k)}(\xi) \not\equiv 0$. This together with Lemma 2.2 yields

$$\begin{aligned} \sum_{a_j \in E} \overline{N}\Big(r, \frac{1}{F - a_j}\Big) &\le \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_{(2}(r, F) \\ &\le T(r, F^{(k)}) + \frac{1}{2}\, N(r, F) + O(1) \\ &\le T(r, F^{(k)}) + \frac{1}{2}\, T(r, F) + O(1). \end{aligned}$$

Therefore, by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} (k+2)T(r, F) &\le \sum_{a_j \in E} \overline{N}\Big(r, \frac{1}{F - a_j}\Big) + S(r, F) \\ &\le T(r, F^{(k)}) + \frac{1}{2}\, T(r, F) + S(r, F) \\ &\le (k+1)T(r, F) + \frac{1}{2}\, T(r, F) + S(r, F) \\ &\le (k + \frac{3}{2})T(r, F) + S(r, F) \end{aligned}$$

.

So, $T(r, F) \le S(r, F)$. This is a contradiction.

Case 2. $|z_0| = 1$.

Since the function φ satisfies (1.1) and $|z_n| \to 1^-$, we see

$$\varphi(|z_n|)(1-|z_n|) \geq 1$$

for all sufficiently large n. It follows that

$$\left|z_n + \frac{z}{\varphi(|z_n|)}\right| \leq |z_n| + \frac{|z|}{\varphi(|z_n|)} < |z_n| + \frac{1}{\varphi(|z_n|)} \leq 1$$

for all $z \in \Delta$. Therefore, we have the following well-defined functions:

$$g_n(z) := f_n\Big(z_n + \frac{z}{\varphi(|z_n|)}\Big), \quad z \in \Delta. \tag{3.7}$$

Hence,

$$g_n^{\#}(0) = \frac{f_n^{\#}(z_n)}{\varphi(|z_n|)} \to \infty \quad (n \to \infty)$$

by (3.1). Hence, as in Case.1, Marty's Theorem implies that $\{g_n\}_{n=1}^{\infty}$ is not normal at the point $z = 0$. By Lemma 2.1, there exist a subsequence of functions g_n (that will also be denoted by g_n), points $v_n \to 0$, and positive numbers $\sigma_n \to 0$, such that

$$G_n(\zeta) := g_n(v_n + \sigma_n\zeta) = f_n\Big(z_n + \frac{v_n + \sigma_n\zeta}{\varphi(|z_n|)}\Big) \to G(\zeta) \tag{3.8}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\zeta)$ is a nonconstant meromorphic function of $\mathbb{C}$. Consequently,

$$G_n^{(i)}(\zeta) := \sigma_n^i\, g_n(v_n + \sigma_n\zeta) \to G^{(i)}(\zeta) \tag{3.9}$$

uniformly on compact subsets of $\mathbb{C}\backslash\{\text{Poles of } G\}$, $i = 1, 2, \ldots$.

Claim 2. $G(\zeta) \in E \implies \frac{|G^{(k)}(\zeta)|}{1+|G(\zeta)|^{k+1}} = 0.$

Suppose that $G(\zeta_0) = a \in E$, by Hurwitz's theorem and (3.8), there exists a sequence $\zeta_n \to \zeta_0$ such that $G_n(\zeta_n) = g_n(v_n + \sigma_n\zeta_n) = a$ for sufficiently large n. For brevity, we use the notation

$$\widehat{z_n} := z_n + \frac{v_n + \sigma_n\zeta_n}{\varphi(|z_n|)}.$$

Hence, $f_n(\widehat{z_n}) = a$ for all n sufficiently large.

By the hypothesis (1.6), there exists a constant $M > 0$ such that

$$\frac{1}{\varphi(|\widehat{z_n}|)^k}\frac{|f_n^{(k)}(\widehat{z_n})|}{1+|f_n(\widehat{z_n})|^{k+1}} \leq M.$$

for sufficiently large n.

Therefore, an elementary calculation gives

$$\begin{aligned}\frac{|G_n^{(k)}(\zeta_n)|}{1+|G_n(\zeta_n)|^{k+1}} &= \sigma_n^k \frac{1}{\varphi(|z_n|)^k}\frac{|f_n^{(k)}(\widehat{z_n})|}{1+|f_n(\widehat{z_n})|^{k+1}} \\ &\leq \sigma_n^k \Big(\frac{\varphi(|\widehat{z_n}|)}{\varphi(|z_n|)}\Big)^k M\end{aligned}$$

for sufficiently large n.

Noting that φ is increasing, we have

$$\frac{\varphi(|\widehat{z_n}|)}{\varphi(|z_n|)} = \frac{\varphi(|z_n + \frac{v_n+\sigma_n\zeta_n}{\varphi(|z_n|)}|)}{\varphi(|z_n|)} \to 1 \quad (n \to \infty)$$

by (1.2). Then, we obtain $\frac{|G^{(k)}(\zeta_0)|}{1+|G(\zeta_0)|^{k+1}} = 0$. Hence, Claim 2 is proved.

As the proof in Case 1, by Claim 2 and the Lemmas 2.2, 2.3 and 2.4 for the function $G(\zeta)$ and points $a_j, a_j \in E$, we may obtain a contradiction. We omit the details in order to avoid unnecessary repetition. And hence, the proof of Theorem 1.4 have been completed.

4. Proof of Theorem 1.7

With the notation used in the proof of Theorem 1.4, proceeding as in the proof of Case 1, we get that $F(\xi) \in E \implies \frac{|F^{(k)}(\xi)|}{1+|F(\xi)|^{k+1}} = 0$. Furthermore, we have

Claim 3. $F(\xi) \in E\backslash\{\infty\} \implies F^{(k+1)}(\xi) = 0$. Suppose that $F(\xi_1) = b \in E\backslash\{\infty\}$, by Hurwitz's theorem and (3.3), there exists a sequence $\xi_n^* \to \xi_1$ such that $F_n(\xi_n^*) = f_n(u_n + \varrho_n\xi_n^*) = b$. By the hypotheses (1.10) and (1.11), there exists a constant $M_1 > 0$ such that

$$\frac{1}{\varphi(|u_n + \varrho_n\xi_n^*|)^k} \frac{|f_n^{(k)}(u_n + \varrho_n\xi_n^*)|}{1 + |f_n(u_n + \varrho_n\xi_n^*)|^{k+1}} \le M_1 \tag{4.1}$$

and

$$(f_n^{(k)})^{\#}(u_n + \varrho_n\xi_n^*) \le M_1. \tag{4.2}$$

for sufficiently large n. By (4.1), we see

$$\begin{aligned} |f_n^{(k)}(u_n + \varrho_n\xi_n^*)| &\le M_1 \cdot \varphi(|u_n + \varrho_n\xi_n^*|)^k \cdot (1 + |b|^{k+1}) \\ &\le M_1 \cdot \varphi(r_1)^k \cdot (1 + |b|^{k+1}) \end{aligned}$$

where r_1 is a fixed constant number such that $|z_0| < r_1 < 1$.

This, together with (4.2) yields

$$\begin{aligned} ({F_n}^{(k)})^{\#}(\xi_n^*) &= \frac{|F_n^{(k+1)}(\xi_n^*)|}{1 + |F_n^{(k)}(\xi_n^*)|^2} \le |F_n^{(k+1)}(\xi_n^*)| \\ &= {\varrho_n}^{k+1}\, |{f_n}^{(k+1)}(u_n + \varrho_n\xi_n^*)| \\ &= {\varrho_n}^{k+1} \cdot (f_n^{(k)})^{\#}(u_n + \varrho_n\xi_n^*) \cdot (1 + |f_n^{(k)}(u_n + \varrho_n\xi_n^*)|^2) \\ &\le {\varrho_n}^{k+1} \cdot M_1 \cdot (1 + |f_n^{(k)}(u_n + \varrho_n\xi_n^*)|^2) \\ &\le {\varrho_n}^{k+1} \cdot M_1 \cdot (1 + (M_1 \cdot \varphi(r_1)^k \cdot (1 + |b|^{k+1}))^2). \end{aligned}$$

This leads to $(F^{(k)})^{\#}(\xi_1) = 0$. And hence, $F^{(k+1)}(\xi_1) = 0$. Claim 3 is obtained.

Then, by Claim 1, Claim 3 and Lemma 2.2 yields

$$\begin{aligned} \sum_{a_j\in E} \overline{N}\Big(r, \frac{1}{F - a_j}\Big) &\le \overline{N}_{(2}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_{(2}(r, F) \\ &\le \frac{1}{2}N\Big(r, \frac{1}{F^{(k)}}\Big) + \frac{1}{2}\, N(r, F) + O(1) \\ &\le \frac{1}{2}T(r, F^{(k)}) + \frac{1}{2}\, N(r, F) + O(1). \end{aligned}$$

Again, combing Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}([k/2]+2)T(r,F) &\leq \sum_{a_j\in E}\overline{N}\Big(r,\frac{1}{F-a_j}\Big)+S(r,F)\\ &\leq \frac{1}{2}T(r,F^{(k)})+\frac{1}{2}\,N(r,F)+S(r,F)\\ &\leq \frac{k+1}{2}T(r,F)+\frac{1}{2}\,T(r,F)+S(r,F)\\ &\leq (k/2+1)T(r,F)+S(r,F)\end{aligned}$$

.

So, $T(r,F) \leq S(r,F)$. This is also a contradiction.

Similarly, for the function G as in the proof of Case 2, we have

Claim 4. $G(\zeta) \in E\backslash\{\infty\} \implies G^{(k+1)}(\zeta) = 0$.

Using Claim 2, Claim 4 and value distribution theory, we also obtain a contradiction. Once again we omit the details. Therefore, $\sup\{\|f\|_{\mathcal{N}^{\varphi}};\ f \in \mathcal{F}\} < \infty$ as desired.

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