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NEC Rings

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Abstract. A ring R is called *NEC* if for any $a, b \in N(R)$, $ab = ba$. The class of *NEC* rings is a proper generalization of the class of *CN* rings. First, with the aid of *NEC* rings, some characterizations of *CN* rings and reduced rings are given. Next we extend many properties of *CN* rings to *NEC* rings such as we show that *NEC* rings are directly finite and left min-abel; *NEC* regular ring are strongly regular ; a ring R is *NEC* if and only if every Pierce stalk of R is *NEC*; Also we discuss some properties of *NEC* exchange rings; Finally, we give some properities of MP-invertible elements.

1. Introduction

Throughout this article, all rings considered are associated with identity, the symbols $N(R)$, $J(R)$, $U(R)$, $E(R)$, $Z(R)$, $Z_l(R)$ and $Z_r(R)$ will stand respectively for the set of all nilpotent elements, the *Jacobson* radical, the set of all invertible elements, the set of all idempotent elements, the center, the left and right singular ideal of R. And $\mathbb{Z}$ represents the set of all integers.

In [1], it is shown that if a ring R satisfies: (1) $N(R)$ is commutative, (2) for every $x \in R$ there exists an element x' in the subring $\langle x \rangle$ generated by x such that $x - x^2x' \in N(R)$, (3) for all $a \in N(R)$ and $b \in R$, $ba - ab$ commutes with b, then R is commutative.

In [2], it is shown that if R satisfies: (1) $N(R)$ is commutative, (2) for every $x \in R$ there exists an element x' in the subring $\langle x \rangle$ generated by x such that $x - x^2x' \in N(R)$, (3) for every $x, y \in R$, there exists a positive integer $n = n(x, y) \geq 1$ such that both $(xy)^n - (yx)^n$ and $(xy)^{n+l} - (yx)^{n+l}$ belong to $Z(R)$, then R is a subdirect sum of local commutative rings and nil commutative rings.

Motivated by the two theorems, we consider the class of rings satisfying the following condition:

$$ab = ba \quad a, b \in N(R)$$

A ring R is called nilpotent elements commutative (for short, *NEC*) if it satisfies the above condition. Clearly, a ring with $N(R)^2 = 0$ is always *NEC*.

Following [12], a ring R is called *CN* if $N(R) \subseteq Z(R)$. Clearly, *CN* rings are *NEC*, but the converse is not true because of the following example 2.2. Hence *NEC* rings are proper generalization of *CN* rings.

Following [22], a ring R is called *reduced* if $N(R) = 0$. And R is called $left$ $(right)$ $quasi-duo$ if every maximal left (right) ideal of R is an ideal. Recall that a ring R is said to be $directly$ $finite$ [19] if $ab = 1$ implies $ba = 1$.

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In preparation for the paper, we first state the following definitions.

An element $e \in E(R)$ is called *left minimal idempotent* if Re is a minimal left ideal of R. Write $ME_l(R)$ to denote the set of all left minimal idempotents of R. A ring R is called left min-abel [23] if either $ME_l(R) = \emptyset$ or each element e of $ME_l(R)$ is left semicentral (that is, $ae = eae$ for all $a \in R$). An element a of a ring R is called *regular* [14] if $a \in aRa$; a is said to be *strongly regular* [22] if $a \in a^2R \cap Ra^2$; and a is *unit − regular* [13] if $a = aua$ for some $u \in U(R)$. A ring R is called *regular*, *strongly regular*, *unit − regular* if every element of R is *regular*, *strongly regular* and *unit − regular*, respectively. Following [18], a ring R is called *exchange* if for every $x \in R$ there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$, and R is said to be *clean* if every element of R is a sum of a unit and an idempotent.

In section 2, we give some examples of *NEC* rings and with the aid of *NEC* rings, some characterizations of *CN* rings and reduced rings are given.

In section 3, we discuss the properties of *NEC* rings. We mainly show that *NEC* rings are directly finite and left min-abel; also give some characterizations of strongly regular rings.

In section 4, we discuss some properties of *NEC* exchange rings such as *NEC* exechange rings are clean rings and quasi-duo rings.

In section 5, we discuss some properties of Moore Penrose invertibility of *NEC* ring. Especially, we give some characterizations of *EP* elements.

2. Examples of *NEC* Rings

Definition 2.1. *A ring R is called nilpotent elements commutative (for short, NEC) if $ab = ba$ for any $a, b \in N(R)$.*

The class of *NEC* rings is rather large, and contains all commutative rings, all *CN* rings and all rings R with $N(R)^2 = 0$. However, the following example illustrates that *NEC* rings need not be *CN*.

Example 2.2. *Let F be a field and $R = T_2(F) = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is NEC because $N(R)^2 = 0$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin Z(R)$, R is not CN.*

Example 2.3. *Let $R = \mathbb{Z}_8$. Then R is NEC, while $N(R)^2 = \{0, 4\} \neq 0$. Hence there exists a NEC ring R with $N(R)^2 \neq 0$.*

Let R be a ring and $V_2(R) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R\right\}$. Then with the usual matrix addition and multiplication, $V_2(R)$ forms a ring.

Proposition 2.4. *R is a CN ring if and only if $V_2(R)$ is a NEC ring.*

Proof (⇒) Assume that $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in N(V_2(R))$, then $a_1, b_1 \in N(R) \subseteq Z(R)$, it follows that $AB = \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 \\ 0 & a_1b_1 \end{pmatrix} =$
$\begin{pmatrix} b_1a_1 & b_1a_2 + b_2a_1 \\ 0 & b_1a_1 \end{pmatrix} = BA$. Therefore $V_2(R)$ is *NEC*.

(⇐) For each $a \in N(R), b \in R$, write $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then $A, B \in N(V_2(R))$. Since $V_2(R)$ is *NEC*, $AB = BA$, that is, $\begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix}$, it follows that $ab = ba$. Hence R is *CN*. □

Let R be a ring and $R \propto R = \{(a, b) | a, b \in R\}$. Then with componentwise addition and the following multiplication:

$(a, b)(x, y) = (ax, ay + bx)$

R forms a ring and $\eta : R \propto R \longrightarrow V_2(R)$ defined by $\eta((a,b)) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a ring isomorphism. Also we have $V_2(R) \cong R[x]/(x^2)$. Hence Proposition 2.4 gives the following corollary.

Corollary 2.5. *The following conditions are equivalent for a ring R:*
(1) R is CN;
(2) $R \propto R$ is NEC;
(3) $R[x]/(x^2)$ is NEC.

Let R be a ring and set $V_3(R) = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_1 \end{pmatrix} | a_1, a_2, a_3, a_4 \in R\}$ and $SV_3(R) = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix} | a_1, a_2, a_3 \in R\}$. Then with the usual matrix addition and multiplication, $V_3(R)$ and $SV_3(R)$ form rings. Clearly, $SV_3(R)$ is a subring of $V_3(R)$. The following example illustrates $V_3(R)$ need not be NEC even if R is a division ring.

Example 2.6. *Let $R = D$ only be a division ring. Then there exist $a, b \in R$ and $ab \neq ba$. Choose $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$. Then $A, B \in N(SV_3(R))$. Since $AB = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 & ba \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $AB \neq BA$. Hence $SV_3(R)$ is not NEC. Since each subring of NEC rings is NEC, $V_3(R)$ is not NEC.*

Observing Example 2.6, we can obtain the following proposition.

Proposition 2.7. *The following conditions are equivalent for a ring R:*
(1) R is a commutative ring;
(2) $SV_3(R)$ is a commutative ring;
(3) $SV_3(R)$ is a NEC ring.

Let R be a ring and $R[x]$ the polynomial ring. Then $\sigma : R[x]/(x^3) \longrightarrow SV_3(R)$ defined by $\sigma(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{pmatrix}$ is a ring isomorphism. Hence Proposition 2.7 implies the following corollary.

Corollary 2.8. *The following conditions are equivalent for a ring R:*
(1) R is a commutative ring;
(2) $R[x]/(x^3)$ is a commutative ring;
(3) $R[x]/(x^3)$ is a NEC ring.

The following example illustrates $V_3(R)$ need not be NEC even if R is a field.

Example 2.9. *Let $R = \mathbb{Z}_3$ be a field. Choose $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(V_3(R))$. Then $AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = BA$, hence $V_3(R)$ is not NEC.*

Motivated by Example 2.2, we obtain the following theorem which gives a characterization of reduced rings.

Theorem 2.10. *R is a reduced ring if and only if the 2×2 upper triangular matrix ring $T_2(R)$ over R is a NEC ring.*

Proof ($\Longrightarrow$) Assume that R is reduced, then $N(T_2(R)) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, so $T_2(R)$ is NEC because $N(T_2(R))^2 = 0$.

($\Longleftarrow$) Assume that $a \in R$ with $a^2 = 0$. Choose $A = \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$. Then $A, B \in N(T_2(R))$. Since $T_2(R)$ is NEC, $AB = BA$, one has $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, it follows that $a = 0$. Therefore R is reduced. □

Let R be a ring and write $GT_2(R) = \{\begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix} | a_1, a_2, a_3 \in R\}$, $WGT_2(R) = \{\begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix} | a_1, a_2, a_3 \in R\}$ and $QGT_2(R) = \{\begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$

$| a_1, a_2, a_3 \in R\}$. Then by the usual matrix addition and multiplication, $GT_2(R)$, $WGT_2(R)$ and $QGT_2(R)$ form rings. Set $\rho : T_2(R) \longrightarrow GT_2(R)$ defined by $\rho(\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}) = \begin{pmatrix} a_1 & a_1 & a_2 - a_1 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}$, $\sigma : T_2(R) \longrightarrow WGT_2(R)$ defined by $\rho(\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}) = \begin{pmatrix} a_1 & 0 & a_2 - a_1 + a_3 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}$ and $\tau : WGT_2(R) \longrightarrow QGT_2(R)$ defined by $\tau(\begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_3 \end{pmatrix}) = \begin{pmatrix} a_1 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$. Then ρ, σ and τ are ring isomorphisms. Hence Theorem 2.10 implies the following corollary.

Corollary 2.11. *The following conditions are equivalent for a ring R:*
(1) R is reduced;
(2) $GT_2(R)$ is NEC;
(3) $WGT_2(R)$ is NEC;
(4) $QGT_2(R)$ is NEC.

Remark 2.12. *Example 2.9 illustrates the 3×3 upper triangular matrix ring $T_3(R)$ over a field R need not be NEC.*

Let R be a ring and write $M_2^{(0)}(R) = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} | a_{ij} \in R, i, j = 1, 2\}$. For any $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2^{(0)}(R)$, we define new multiplication as follows:

$$AB = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

Then with the usual matrix addition and the new multiplication, $M_2^{(0)}(R)$ is a ring.

Proposition 2.13. *R is a reduced ring if and only if $M_2^{(0)}(R)$ is a NEC ring.*

Proof ($\Rightarrow$) Assume $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in N(M_2^{(0)}(R))$, then $a_{11}, a_{22}, b_{11}, b_{22} \in N(R) = 0$, so $A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$, it follows that $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = BA$. Hence $M_2^{(0)}(R)$ is NEC.

($\Leftarrow$) Choose $a \in R$ with $a^2 = 0$ and $A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$. Then $A, B \in N(M_2^{(0)}(R))$. Since $M_2^{(0)}(R)$ is NEC, $AB = BA$, that is, $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, this gives $a = 0$. Hence R is reduced ring. □

Let R be a ring and write $WT_3(R) = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix} | a_i \in R, i = 1, 2, \cdots, 5\}$. Then with the usual matrix addition and multiplication, $WT_3(R)$ forms a ring.

Theorem 2.14. *R is a reduced ring if and only if $WT_3(R)$ is a NEC ring.*

Proof Assume that R is reduced, then $N(WT_3(R)) = \begin{pmatrix} 0 & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, this gives $WT_2(R)$ is NEC because $N(WT_3(R))^2 = 0$.

Conversely, assume that $WT_3(R)$ is NEC and $a \in R$ with $a^2 = 0$. Choose $A = \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, B = \begin{pmatrix} a & a & 1 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$. Then, clearly, $A, B \in N(WT_3(R))$. Since $WT_3(R)$ is NEC, $AB = BA$, this gives $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, one gets $a = 0$. Therefore R is reduced. □

Let R be a ring and write $SV_4(R) = \{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_5 \end{pmatrix} | a_1, a_2, a_3, a_4, a_5 \in R\}$. Then with the usual matrix addition and multiplication, $SV_4(R)$ forms a ring.

Theorem 2.15. *R is a commutative reduced ring if and only if $SV_4(R)$ is a NEC ring.*

Proof ($\Longrightarrow$) Assume that R is a commutative reduced ring, then $N(SV_4(R)) = \{\begin{pmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | a_2, a_3, a_4 \in R\}$. Since R is commutative, we can easily to show that $AB = BA$ for all $A, B \in N(SV_4(R))$, one gets $SV_4(R)$ is NEC.

($\Longleftarrow$) Assume that $x, y, a \in R$ with $a^2 = 0$. Choose $C = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
$D = \begin{pmatrix} 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E = \begin{pmatrix} a & 1 & 1 & a \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $C, D, E, F \in N(SV_4(R))$. Since $SV_4(R)$ is NEC, $CD = DC$ and $EF = FE$, this gives $\begin{pmatrix} 0 & 0 & xy & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & yx & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & a & a & a \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so $xy = yx$ and $a = 0$. Hence R is commutative reduced. □

3. Properties of *NEC* Rings

Let R be a ring and write $Max_l(R)$ to denote the set of all maximal left ideals of R.

Theorem 3.1. *Let R be a NEC ring, $e \in E(R)$, $a \in R$ and $M \in Max_l(R)$. Then we have*
(1) $e \in M$ or $(1-e)R \subseteq M$;
(2) $1 - ae \in M$ if and only if $1 - ea \in M$;
(3) $Ra + R(ae - 1) = R$;
(4) $Me \subseteq M$.

Proof (1) If $e \notin M$, then $Re + M = R$, so $(1-e)R \subseteq (1-e)Re + M$. Since R is NEC and $eR(1-e) \subseteq N(R)$, we have $eR(1-e)Re = (1-e)ReR(1-e) = 0$, it follows that $(1-e)Re \subseteq M$. Hence $(1-e)R \subseteq M$.

(2) If $1 - ae \in M$, then $ae \notin M$, so $e \notin M$. By (1), we have $(1-e)R \subseteq M$, so $(1-e)a, a(1-e) \in M$, this gives $1 - a = 1 - ae + ae - a = (1-ae) - a(1-e) \in M$, so $1 - ea = 1 - a + a - ea = (1-a) + (1-e)a \in M$.

Conversely, assume that $1 - ea \in M$. If $e \in M$, then $1 - e \notin M$. By (1), we have $eR \subseteq M$, it follows that $1 = (1-ea) + ea \in M$, which is a contradiction, hence $e \notin M$. By (1), we have $(1-e)R \subseteq M$, this implies that $R(1-e) \subseteq M$, one gets $1-a = 1-ae+ae-a = (1-ae)-a(1-e) \in M$ and then $1-ae = 1-a+a-ae = (1-a)+a(1-e) \in M$.

(3) If $Ra + R(ae-1) \neq R$, then there exists a maximal left ideal K of R such that $Ra + R(ae-1) \subseteq K$. Since $ae - 1 \in K$, by (2), $1 - ea \in K$. Since $a \in K$, $ea \in K$, this gives $1 \in K$, which is a contradiction. Hence $Ra + R(ae-1) = R$.

(4) If $Me \nsubseteq M$, then $Me + M = R$. Write $1 = me + n$ for some $m, n \in M$. By (3), we have $R = Rm + R(me-1) = Rm + R(-n) \subseteq M$, so $R = M$, which is a contradiction. Hence $Me \subseteq M$. □

Recall that a ring R is said to be directly finite if $ab = 1$ implies $ba = 1$.

Lemma 3.2. *Let R be a ring satisfying either $e \in M$ or $(1-e)R \subseteq M$ for each $e \in E(R)$ and $M \in Max_l(R)$. Then R is directly finite.*

Proof Assume that $ab = 1$. Write $e = ba$. Then $e \in E(R)$, $ae = a$ and $eb = b$. If $Re \neq R$, then there exists $M \in Max_l(R)$ such that $Re \subseteq M$. Since $1 - e \notin M$, by hypothesis, $eR \subseteq M$, one gets $b = eb \in M$, it follows that $1 = ab \in M$, which is a contradiction. Hence $Re = R$, this implies $ba = e = 1$. Therefore R is directly finite. □

The following corollary follows from Theorem 3.1 and Lemma 3.2.

Corollary 3.3. *NEC rings are directly finite.*

An element $e \in E(R)$ is called left minimal idempotent if Re is a minimal left ideal of R. Write $ME_l(R)$ to denote the set of all left minimal idempotents of R. A ring R is called left min-abel if either $ME_l(R) = \emptyset$ or each element e of $ME_l(R)$ is left semicentral (that is, $ae = eae$ for all $a \in R$).

Lemma 3.4. *A ring R is left min-abel if and only if $Me \subseteq M$ for each $e \in ME_l(R)$ and $M \in Max_l(R)$.*

Proof Suppose that R is left min-abel. Choose $e \in ME_l(R)$ and $M \in Max_l(R)$. If $Me \nsubseteq M$, then $Me + M = R$. Since e is left semicentral, $1-e$ is right semicentral, so $(1-e)R \subseteq (1-e)Me + (1-e)M \subseteq M$, one gets $R(1-e) = M$, so $Me = 0$, which is a contradiction. Hence $Me \subseteq M$.

Conversely, let $e \in ME_l(R)$. If $(1-e)Re \neq 0$, then there exists $a \in R$ such that $(1-e)ae \neq 0$. Write $g = e + (1-e)ae$, then $g \in ME_l(R)$, $eg = e$ and $ge = g$. Since $R(1-g) \in Max_l(R)$, $R(1-g)e \subseteq R(1-g)$ by hypothesis, it follows that $(1-g)eg = 0$, one gets $e = g$, so $(1-e)ae = 0$ which is a contradiction. Therefore $(1-e)Re = 0$, this shows that R is left min-abel. □

Theorem 3.1 and Lemma 3.4 implies the following corollary.

Corollary 3.5. *NEC rings are left min-abel.*

The following example illustrates the converses of Corollary 3.3 and Corollary 3.5 are not true.

Example 3.6. *Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in Z, a \equiv d \ (mod\ 2), b \equiv c \equiv 0 \ (mod\ 2)\}$. Then by the usual addition and multiplication of matrix, R forms a ring. It is easy to show that $E(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \subseteq Z(R)$, so R is* *left min-abel and directly finite. We claim that R is not NEC. In fact, let* $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, *we have* $A^2 = B^2 = 0$, *so* $A, B \in N(R)$. *Since* $AB = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$, *we have* $AB \neq BA$. *Therefore R is not NEC.*

A ring R is said to be n-regular [24] if every element of $N(R)$ is regular. It is well known that a ring R is strongly regular if and only if $x \in Rx^2$ for each $x \in R$.

Lemma 3.7. *Let R be a NEC ring. If* $x \in R$ *is regular, then x is strongly regular.*

Proof Since x is regular, $x = xyx$ for some $y \in R$. Set $e = xy$, then $e \in E(R)$ and $x = ex$, one gets $x(1-e) \in N(R)$. Since R is *NEC*, $x(1-e)ye = (1-e)yex(1-e) = 0$, it follows that $e = xeye$, so $x = ex = xeyex = xeyx \in x^2R$. Similarly, we can show that $x \in Rx^2$. Hence x is strongly regular. □

The following two theorems follow from Lemma 3.7.

Theorem 3.8. *The following conditions are equivalent for a ring R:*
(1) R is a strongly regular ring;
(2) R is a unit−regular ring and NEC ring;
(3) R is a regular ring and NEC ring.

Theorem 3.9. *The following conditions are equivalent for a ring R:*
(1) R is a reduced ring;
(2) R is a NEC ring and n−regular ring.

Recall that a ring R is left *NPP* [24] if for each $a \in N(R)$, Ra is projective as left R−module. And R is said to be left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for each $a \in R$ and $e \in E(R)$. Clearly, R is a left *NPP* ring if and only if for each $a \in N(R)$, $l(a) = Re$ for some $e \in E(R)$, where $l(a) = \{x \in R | xa = 0\}$.

Proposition 3.10. *Let R be a NEC left NPP ring. If R is left idempotent reflexive, then R is reduced.*

Proof Assume that $a \in R$ satisfying $a^2 = 0$. Then $l(a) = Re$ for some $e \in E(R)$ because R is left *NPP*. Hence $a = ae$ and $ea = 0$. Since R is *NEC*, $ax(1-e) = a(ex(1-e)) = ex(1-e)a = exa$ for each $x \in R$, one gets $ax(1-e) = 0$, so $aR(1-e) = 0$. Since R is left idempotent reflexive, $(1-e)Ra = 0$, it follows that $a = ea = 0$. Therefore R is reduced. □

Since semiprime rings are left idempotent reflexive, Proposition 3.10 implies the following corollary.

Corollary 3.11. *R is a reduced ring if and only if R is a semiprime NEC left NPP ring.*

Lemma 3.12. *Let R be a NEC ring and I an ideal of R. If* $I \subseteq N(R)$, *then* R/I *is NEC.*

Proof It is clear. □

Clearly, for a *NEC* ring R, $N(R)$ is only an addition subgroup of R. If $R/P(R)$ is a left *NPP* ring, then we can say more, where $P(R)$ denotes the prime radical of R.

Theorem 3.13. *Let R be a NEC ring. If* $R/P(R)$ *is left NPP, then* $N(R) = P(R)$.

Proof Since R is *NEC*, by Lemma 3.12, $R/P(R)$ is *NEC*, Since $R/P(R)$ is a semiprime left *NPP* ring, $R/P(R)$ is reduced by Corollary 3.11, so $N(R) \subseteq P(R)$. Therefore $N(R) = P(R)$. □

An ideal I of R is called reduced if $I \cap N(R) = 0$. Clearly, every ideal of reduced ring is reduced.

Proposition 3.14. *Let R be a ring and I a reduced ideal of R. If* R/I *is NEC, then so is R.*

Proof Suppose that $a, b \in N(R)$, then in $\bar{R} = R/I$, $\bar{a}, \bar{b} \in N(\bar{R})$. Since R/I is *NEC*, $ab - ba \in I$. Since $a \in N(R)$, there exists $n \geq 1$ such that $a^n = 0$. If $n = 1$, then $a = 0$, so $ab = ba$, we are done. Hence we assume that $n \geq 2$. Since $(a^{n-1}(ab-ba)a)^2 = 0$ and I is reduced, $a^{n-1}(ab-ba)a = 0$, this gives $(a^{n-1}(ab-ba))^2 = 0$, so $a^{n-1}(ab-ba) = 0$, again $(a^{n-2}(ab-ba)a)^2 = 0$ implies $a^{n-2}(ab-ba)a = 0$, further, we have $a^{n-2}(ab-ba) = 0$. Repeating this process, we can obtain that $ab - ba = 0$, this shows that R is *NEC*. □

Lemma 3.15. *Let R be a ring and I, J two ideals of R. If $R/I, R/J$ are NEC and $I \cap J = 0$, then R is NEC.*

Proof It is routine. □

Theorem 3.16. *Let R be a ring and I, J two ideals of R. If $R/I, R/J$ are NEC, then $R/(I \cap J)$ is NEC.*

Proof It is an immediate result of Lemma 3.15. □

Let R be a ring, $B(R)$ be the set of all central idempotents of R, and $S(R)$ be the nonempty set of all proper ideals of R generated by central idempotents. An ideal $P \in S(R)$ is a Pierce ideal of R if P is a maximal (with respect to inclusion) element of the set $S(R)$. The set of all Pierce ideals of R is denoted by $P(R)$. If P is a Pierce ideal of R, then the factor ring R/P is called a Pierce stalk of R.

Theorem 3.17. *The following conditions are equivalent for a ring R:*
(1) R is a NEC ring;
(2) R/S is a NEC ring for every ideal S generated by central idempotents of R;
(3) All Pierce stalks of R are NEC rings.

Proof (1) $\Longrightarrow$ (2) Assume that $x, y \in R$ such that $\bar{x}, \bar{y} \in N(R/S)$, then there exist $m, n \geq 1$ such that $x^m, y^n \in S$. Since S is generated by central idempotents of R, there exists a central idempotent $g \in S$ such that $x^m, y^n \in Rg$. Clearly $(x(1-g))^m = 0 = (y(1-g))^n$, one gets $x(1-g)y(1-g) = y(1-g)x(1-g)$ because R is *NEC*. Hence $\bar{x}\bar{y} = \bar{y}\bar{x}$, this shows that R/S is *NEC*.

(2) $\Longrightarrow$ (3) It is trivial.

(3) $\Longrightarrow$ (1) Suppose that R is not a *NEC* ring, then there exist $a, b \in N(R)$ such that $ab \neq ba$. Put $\Sigma = \{I | I$ is an ideal of R generated by central idempotents and in $\bar{R} = R/I$, $\bar{a}\bar{b} \neq \bar{b}\bar{a}\}$. Then Σ is not an empty set because $0 \in \Sigma$. One can easily show that there exists a maximal element P in Σ by Zorn's Lemma. If P is not a Pierce ideal of R, then there is a central idempotent e of R such that $P + eR$ and $P + (1-e)R$ are proper ideals of R which properly contain the ideal P. Hence $P + eR \notin \Sigma$ and $P + (1-e)R \notin \Sigma$, it follows that $ab - ba \in (P + eR) \cap (P + (1-e)R) = P$, which is a contradiction. Thus P is a Pierce ideal of R, by (3), R/P is *NEC*, which is also a contradiction because $ab - ba \notin P$. Therefore R is *NEC*. □

4. *NEC* Exchange Ring

Recall a ring is *Abelian* [4] if $E(R) \subseteq Z(R)$. It is well known that clean rings are always exchange [3]. And the converse is true when R is an Abelian ring by [26]. Example 3.6 illustrates that *NEC* ring need not be Abelian.

Theorem 4.1. *Let R be a NEC ring. If R is exchange, then R is clean.*

Proof Since R is *NEC*, $R/P(R)$ is *NEC* by Lemma 3.12. Since $R/P(R)$ is semiprime, $R/P(R)$ is Abel, this implies that $R/P(R)$ is an Abel exchange ring, so $R/P(R)$ is clean by [26]. Therefore R is clean. □

It is well known that an exchange ring with only two idempotents is local.

Lemma 4.2. *Let R be a NEC exchange ring. If P is a prime ideal of R, then R/P is local.*

Proof Since R is a *NEC* exchange ring, $R/P(R)$ is Abel. Assume that $\hat{a}$ is any idempotent of $\hat{R} = R/P$, then there exists $e \in E(R)$ such that $\hat{e} = \hat{a}$ because R is exchange. Clearly, in $\bar{R} = R/P(R)$, $\bar{e}\bar{R}(\bar{1} - \bar{e}) = \bar{0}$, so $eR(1-e) \subseteq P(R) \subseteq P$. Since P is a prime ideal of R, $e \in P$ or $1 - e \in P$, this gives $\hat{a} = \hat{0}$ or $\hat{a} = \hat{1}$. Therefore R/P is local. □

The following corollary is an immediate result of Lemma 4.2.

Corollary 4.3. *Let R be a NEC exchange ring. If P is a left (right) primitive ideal of R, then R/P is a division ring.*

Theorem 4.4. *Let R be a NEC exchange ring. Then R is a left and right quasi-duo ring.*

Proof Assume that M is any maximal left ideal of R, then R/M is a simple left $R-$module, so $P =: \{a \in R|aR \subseteq M\}$ is a left primitive ideal of R, by Corollary 4.3, R/P is a division ring. Clearly, $P \subseteq M$. If $M \neq P$, then there exists $m \in M$ such that $m \notin P$, so there exists $t \in R$ such that $1 - tm \in P$, this implies $1 = 1 - tm + tm \in M$, which is a contradiction. Hence $M = P$ is an ideal of R and so R is left quasi-duo. Similarly, we can show that R is right quasi-duo. □

A ring R is said to have right (left) square stable range one [15] if $xR + yR = R$ implies that $x^2 + yz \in U(R)$ ($x^2 + zy \in U(R)$) for some $z \in R$. A ring R is said to have idempotent stable range one (written $isr(R) = 1$) if $aR + bR = R$ implies that $a + be \in U(R)$ for some $e \in E(R)$.

Corollary 4.5. *Let R be a NEC ring with* $isr(R) = 1$*. Then R is a left and right quasi-duo ring and R has right square stable range one.*

Proof For any $a \in R$, the equation $aR + (-1)R = R$ gives $a + (-1)e \in U(R)$ for some $e \in E(R)$ because $isr(R) = 1$. Thus a is a clean element and R is a clean ring. Hence R is an exchange ring, by Theorem 4.4, R is a left and right quasi-duo ring.

Now let $xR + yR = R$. If $x^2R + yR \neq R$, then there exists a maximal right ideal M of R containing $x^2R + yR$. Since M is an ideal of R, R/M is a division ring. Clearly $xR + yR = R$ implies $xR = x^2R + xyR \subseteq M$, so $R = xR + yR \subseteq M$, which is a contradiction. Hence $x^2R + yR = R$, this leads to $x^2 + yg \in U(R)$ for some $g \in E(R)$. This shows that R has right square stable range one. □

Theorem 4.6. *Let R be a NEC exchange ring. Then R has left and right square stable range one.*

Proof Since R is a *NEC* exchange ring, R is a left and right quasi-duo ring by Theorem 4.4, so $R/J(R)$ is a left quasi-duo ring, by [25, Corollary 2.4], $R/J(R)$ is a reduced ring, hence $R/J(R)$ is an Abel exchange ring, one gets $R/J(R)$ has stable range one by [26, Theorem 6]. Therefore R has stable range one. Similar to the proof of Corollary 4.5, we can show that R has left and right square stable range one. □

Corollary 4.7. *If R is a NEC exchange ring, then* $isr(R) = 1$.

Proof Let $\bar{R} = R/J(R)$. By Theorem 4.6, R has right square stable range one and $\bar{R}$ is an Abel exchange ring. Follows from [7, Theorem 12], we have $isr(\bar{R}) = \bar{1}$. And from [7, Theorem 9], one obtains $isr(R) = 1$. □

Proposition 4.8. *Let R be a NEC exchange ring. Then the following conditions are equivalent:*

(1) there exists an $u \in U(R)$ *such that* $1 \pm u \in U(R)$*;*

(2) for any $a \in R$ *there exists* $u \in U(R)$ *such that* $a \pm u \in U(R)$.

Proof (1) $\Longrightarrow$ (2) Since R is a *NEC* exchange ring, $R/J(R)$ is an Abel exchange ring by Theorem 4.6, and by [26, Theorem 6], $R/J(R)$ is an exchange ring of bounded index. By [8, Corollary 2.4], there exists a $u \in U(R/J(R))$ such that $a \pm u \in U(R/J(R))$. Since invertible elements can be lifted modulo $J(R)$, there exists an $u \in U(R)$ such that $a \pm u \in U(R)$.

(2) $\Longrightarrow$ (1) is trivial. □

We call a ring R a left (right) $P-$exchange ring if every projective left (right) $R-$module has the exchange property. This definition is not left-right symmetric, for example, a left perfect ring which is not right perfect is a left but not a right P-exchange ring.

Theorem 4.9. *Let R be a NEC left P−exchange ring. Then* $R/J(R)$ *is a strongly regular ring.*

Proof Since R is a *NEC* left $P-$exchange ring, R is a *NEC* exchange ring, it follows that $R/J(R)$ is an Abel ring by Theorem 4.6, by [6, Corollary 2.16], $R/J(R)$ is a weakly $\pi-$regular ring. Since R is a left quasi-duo ring by Theorem 4.4, $R/J(R)$ is left quasi-duo, it follows that $R/J(R)$ is strongly regular. □

The following corollary is an immediate result of Theorem 4.9 which gives a characterization of strongly regular rings.

Corollary 4.10. *R is a strongly regular ring if and only if R is a NEC left P−exchange ring with* $J(R) = 0$.

Recall that an element a in R is uniquely clean if it has exactly one clean decomposition, and a is said to be strongly clean if it has a clean decomposition $a = e + u$ in which $eu = ue$. Following [16], we let $ucn(R)$ denote the set of uniquely clean elements and $scn(R)$ is the set of strongly clean elements. Clearly, a ring R is Abel if and only if $E(R) \subseteq ucn(R)$.

Proposition 4.11. *Let R be a NEC ring. Then $ucn(R) \subseteq scn(R)$.*

Proof Assume that $a \in ucn(R)$, then a has the uniquely clean decomposition $a = e + u$. Since R is NEC, by the proof of Theorem 3.1(1), we know that $ex(1-e)Re = 0 = eR(1-e)xe$ for each $x \in R$. Since $J(R)$ is a semiprime ideal of R, $ex(1-e) \in J(R)$ and $(1-e)xe \in J(R)$ for each $x \in R$, follows from the decomposition $a = e+u = (e+ex(1-e))+(u-ex(1-e)) = (e+(1-e)xe)+(u-(1-e)xe)$, we can see that $e+(1-e)xe = e = e+ex(1-e)$ and $u-(1-e)xe = u = u-ex(1-e)$, this gives $ex(1-e) = 0 = (1-e)xe$ for each $x \in R$. Thus $eR(1-e) = 0 = (1-e)Re$, this shows that $e \in Z(R)$ and $a \in scn(R)$. □

Theorem 4.12. *Let R be an exchange ring and I a right ideal of R, which contains no nonzero idempotents. Then R has stable range one if and only if for any regular element a of R, there exists $u \in U(R)$, such that $a - aua \in I$.*

Proof (⇒) It is evident.

(⇐) Let $a, x \in R, e \in E(R)$ such that $ax + e = 1$. If $ea = 0$, then $a = axa$, so there exists $u \in U(R)$ such that $a - aua = y \in I$. we have $1 - e = ax = (aua + y)x = auax + yx = au(1-e) + yx$, $(au-e)^2 = auau - aue - eau + e = (a-y)u - aue + e = au(1-e) - yu + e = 1 - e - yx - yu + e = 1 - y(u+x)$. Since R is an exchange ring, there exists $g^2 = g \in y(u+x)R \subseteq I$ such that $1-g \in (1-y(u+x))R$. Since I contains no nonzero idempotents, one gets $g = 0$, so $1 \in (1-y(u+x))R$. Aussme $1 = (1-y(u+x))z$ for some $z \in R$, so that $(au-e)^2z = 1$. Let $v = (au-e)z$. Then $(au-e)v = 1$; If $ea \neq 0$, let $f = ax = 1-e$, $r = fa - a$, then $rx = (fa-a)x = (axa-a)x = (ax-1)ax = -e(1-e) = 0$ and $fr = f^2a - fa = 0$. Let $a' = a + r$. Then $a'x = ax + rx = f$, $a'xa' = fa' = fa + fr = fa = r + a = a'$ and $a'x + e = ax + e = 1$, so we have $ea' = 0$. Follows from the above proof, there exists $u \in U(R), v \in R$, such that $(a'u - e)v = 1$, one gets $(au + ru - e)v = 1$. Since $fr = 0$, $r = (1-f)r = er$, we have $(au + e(ru-1))v = 1$. Hence in any case, one has $u \in U(R), v \in R$ such that $(au + es)v = 1$ for some $s \in R$, where $s = -1$ or $s = ru - 1$. Write $h = v(au + es)$. Then $h^2 = h$ and $(au + es)h = au + es$. Since $v(au + es) + 1 - h = 1$, by the above proof, there exists $w \in U(R), t, q \in R$ such that $(vw + (1-h)t)q = 1$, so $au + es = (au + es)(vw + (1-h)t)q = wq$, then $q = w^{-1}(au + es)$. Hence $(vw + (1-h)t)w^{-1}(au + es) = 1$, this implies $au + es \in U(R)$, so $a + esu^{-1} \in U(R)$. Therefore R has stable range one. □

Corollary 4.13. *[21, Proposition 5.3] An exchange ring R has stable range one if and only if for each regular element a of R, there exists $u \in U(R)$ such that $a - aua \in J(R)$.*

Corollary 4.14. *[27, Proposition 4.6] An exchange ring R has stable range one if and only if for each regular element a of R, there exists $u \in U(R)$ such that $a - aua \in Z_l(R)$.*

Corollary 4.15. *An exchange ring R has stable range one if and only if for each regular element a of R, there exists $u \in U(R)$ such that $a - aua \in Z_r(R)$.*

5. Generalized Inverses

An involution $a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^*.$$

A ring R with an involution $*$ is called $*-$ring. An element $a^\dagger$ in a $*$-ring R is called the Moore-Penrose inverse (or MP-inverse) of a, if [20]

$$aa^\dagger a = a, a^\dagger aa^\dagger = a^\dagger, aa^\dagger = (aa^\dagger)^*, a^\dagger a = (a^\dagger a)^*.$$

In this case, we call a is MP-invertible in R. The set of all MP-invertible elements of R is denoted by $R^\dagger$.

An involution $*$ of R is called proper if $x^*x = 0$ implies $x = 0$ for all $x \in R$.

Following [5], an element a of a ring R is called group invertible if there is $a^\# \in R$ such that

$$aa^\sharp a = a,\ a^\sharp aa^\sharp = a^\sharp,\ aa^\sharp = a^\sharp a.$$

Denote by $R^\sharp$ the set of all group invertible elements of R. Clearly, a ring R is strongly regular if and only if $R = R^\sharp$.

Duo to [11], an element a of a $*-$ring R is said to be *EP* if $a \in R^\sharp \cap R^\dagger$ and $a^\sharp = a^\dagger$. In [10], many characterizations of *EP* elements are given.

Noting that $a \in R^\sharp$ if and only if $a \in Ra^2 \cap a^2R$. Hence Lemma 3.7 implies that the following lemma.

Lemma 5.1. *Let R be a NEC ring. If* $a \in R^\dagger$*, then* $a \in R^\sharp$.

Theorem 5.2. *Let R be a NEC ring. If* $a \in R^\dagger$*, then* $a \in R^\sharp$ *and*

(1) $a = a^2a^\dagger a^\dagger a$;
(2) $aa^\sharp = aa^\dagger + a^\dagger a - a^\dagger a^2 a^\dagger$;
(3) $aa^\sharp = aa^\dagger a^\dagger a$;
(4) $a^\dagger = a^\dagger aa^\sharp + a^\sharp aa^\dagger - a^\sharp$;
(5) $a^\sharp a^\dagger = a^\sharp aa^\dagger a^\dagger$;
(6) $a = a^2a^\dagger + a^\dagger a^2 - a^\dagger a^3 a^\dagger$.

Proof Since R is a *NEC* ring and $a \in R^\dagger$, by Lemma 5.1, $a \in R^\sharp$, so $a^\sharp$ exists.

Write $f = aa^\dagger, g = a^\dagger a$ and $e = aa^\sharp$. Then $f = f^2, g = g^2, e = e^2$ and $a = ag = fa = ea = ae$. Noting that $a^\sharp = fa^\sharp = a^\sharp g$. Then $a(1-f), (1-g)a^\sharp \in N(R)$, this gives that $(1-g)a^\sharp a(1-f) = a(1-f)(1-g)a^\sharp$, so $a(1-f)(1-g)a^\sharp f = 0$. Noting that $a = fa$. Then $a(1-f)(1-g)a^\sharp a = 0$, which implies that

$$a(1-f)(1-g)a^\sharp = 0 \tag{5.1}$$

and

$$(1-g)a^\sharp a(1-f) = 0 \tag{5.2}$$

Equation (5.1) gives that

$$a^\sharp a = a^2a^\dagger a^\dagger aa^\sharp \tag{5.3}$$

Hence $a = (aa^\sharp)a = (a^2a^\dagger a^\dagger aa^\sharp)a = a^2a^\dagger a^\dagger a$, (1) is completed.

Noting that $a^\sharp a = a^\sharp(a^2a^\dagger a^\dagger a) = aa^\dagger a^\dagger a$. Then (3) is completed.

Equation (5.2) gives that $aa^\sharp = aa^\dagger + a^\dagger a - a^\dagger a^2 a^\dagger$, hence (2) holds.

Since $(1-e)a^\dagger(1-e) = (1-e)a^\dagger eaa^\dagger(1-e) = ((1-e)a^\dagger e)(eaa^\dagger(1-e)) = (eaa^\dagger(1-e))((1-e)a^\dagger e) = 0$, we have $a^\dagger = ea^\dagger + a^\dagger e - ea^\dagger e = a^\dagger aa^\sharp + a^\sharp aa^\dagger - a^\sharp$, which implies that (4) holds.

(5) Noting that $a^\sharp(1-f), (1-f)a^\dagger \in N(R)$ and $a^\dagger = a^\dagger f$. Then $a^\sharp(1-f)a^\dagger = (1-f)a^\dagger a^\sharp(1-f) = 0$, it follows that $a^\sharp a^\dagger = a^\sharp aa^\dagger a^\dagger$.

(6) Noting that $(a-a^2a^\dagger)^2 = 0 = (aa^\sharp - a^\dagger a)^2$. Then $(aa^\sharp - a^\dagger a)(a-a^2a^\dagger) = (a-a^2a^\dagger)(aa^\sharp - a^\dagger a)$. Since $(a-a^2a^\dagger)a = 0$, $(a-a^2a^\dagger)(aa^\sharp - a^\dagger a) = -(a-a^2a^\dagger)(a^\dagger a) = -aa^\dagger a + a^2a^\dagger a^\dagger a$, by (1), one obtains that $(a-a^2a^\dagger)(aa^\sharp - a^\dagger a) = 0$. Hence $(aa^\sharp - a^\dagger a)(a-a^2a^\dagger) = 0$, this gives that $a = a^2a^\dagger + a^\dagger a^2 - a^\dagger a^3 a^\dagger$. □

We don't know whether a is *EP* under the conditions of Theorem 5.2. However, we have the following theorem.

Theorem 5.3. *Let R be a NEC ring and* $a \in R^\dagger$*. If Ra is a minimal left ideal of R, then a is EP.*

Proof Since R is *NEC* and $a \in R^\dagger$, by Lemma 5.1, $a \in R^\sharp$. If $a^\sharp = a^\dagger aa^\sharp$, then $aR = a^\sharp R = a^\dagger aa^\sharp = a^\dagger aR = a^\dagger R$, one obtains that $(1-a^\dagger a)aR = (1-a^\dagger a)a^\dagger R = 0$, $a = a^\dagger a^2$, it follows that a is an EP element. If $a^\sharp \neq a^\dagger aa^\sharp$, then, by Theorem 5.2(4), we have $a^\dagger \neq a^\sharp aa^\dagger$, so $(1-a^\sharp a)a^\dagger \neq 0$. Noting that $a^\dagger = a^\dagger aa^\dagger$. Then $(1-a^\sharp a)a^\dagger a \neq 0$. Since Ra is a minimal left ideal of R, $Ra = R(1-a^\sharp a)a^\dagger a$. Write $a = c(1-a^\sharp a)a^\dagger a$ for some $c \in R$. Then $Ra^\dagger = Raa^\dagger = Rc(1-a^\sharp a)a^\dagger aa^\dagger = Rc(1-a^\sharp a)a^\dagger$. By Theorem 5.2(4), $Ra^\dagger = Rc(a^\dagger a - 1)a^\sharp \subseteq Ra^\sharp = Ra$. Hence $Ra = Ra^\dagger$, which implies that a is *EP*. □

Let $a \in R^\sharp \cap R^\dagger$ and write $\chi_a = \{a, a^\sharp, a^\dagger, a^*, (a^\sharp)^*, (a^\dagger)^*\}$. Then we have the following theorem.

Theorem 5.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then a is an EP element if and only if the equation*

$$a^{\dagger}axa = ax \tag{5.4}$$

has at least a solution in χ_a.

Proof The necessity is clear.

Conversely, we assume that the equation (4.1) has at least a solution in χ_a.

(1) If $x = a$ is a solution, then $a^{\dagger}a^3 = a^2$, this implies $a^{\dagger}a = aa^{\#}$. Hence a is EP.

(2) If $x = a^{\#}$ is a solution, then $a^{\dagger}aa^{\#}a = aa^{\#}$, that is, $a^{\dagger}a = aa^{\#}$, so a is EP.

(3) If $x = a^{\dagger}$ is a solution, then $a^{\dagger}aa^{\dagger}a = aa^{\dagger}$, that is $a^{\dagger}a = aa^{\dagger}$. Hence a is EP.

(4) If $x = a^*$ is a solution, then $a^{\dagger}aa^*a = aa^*$. Noting that $a^* = a^{\dagger}aa^*$. Then $a^*a = aa^*$. Since $aR = aa^*R$ and $a^*R = a^*aR$, $aR = a^*R$, this gives that $(1 - a^{\dagger}a)aR = (1 - a^{\dagger}a)a^*R = 0$. Hence $a = a^{\dagger}a^2$, which implies that a is EP.

(5) If $x = (a^{\#})^*$ is a solution, then $a^{\dagger}a(a^{\#})^*a = a(a^{\#})^*$, it follows that $(a^{\#}a^{\dagger}a)^*a = a(a^{\#})^*$. Noting that $a^{\#} = a^{\#}a^{\dagger}a$. Then $(a^{\#})^*a = a(a^{\#})^*$. Applying the involution to the last equation, we have $a^*a^{\#} = a^{\#}a^*$, this gives that $Ra^* = Raa^* = Ra^{\#}a^* = Ra^*a^{\#} \subseteq Ra^{\#} = Ra$. Noting that $a(1 - a^{\dagger}a) = 0$. Then $a^*(1 - a^{\dagger}a) = 0$, this gives that $(1 - a^{\dagger}a)a = 0$. Hence $a = a^{\dagger}a^2$, one obtains a is EP.

(6) If $x = (a^{\dagger})^*$ is a solution, then $a^{\dagger}a(a^{\dagger})^*a = a(a^{\dagger})^*$, that is, $(a^{\dagger}a^{\dagger}a)^*a = a(a^{\dagger})^*$. Applying the involution to the last equation, we have $a^{\dagger}a^* = a^*a^{\dagger}a^{\dagger}a$. Multiplying by a from the left sided, one has $(a^2a^{\dagger})^* = aa^*a^{\dagger}a^{\dagger}a$, this gives that $a^2a^{\dagger} = a^{\dagger}a(a^{\dagger})^*aa^*$. Hence $aR = a^2R = a^2a^{\dagger}R = a^{\dagger}a(a^{\dagger})^*aa^*R \subseteq a^{\dagger}R$, which implies that $(1 - a^{\dagger}a)aR = 0$. Hence a is EP. □

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