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Energy Decay for a Degenerate Wave Equation under Fractional Derivative Controls

Abbes Benaissa[a], Chahira Aichi[a]

[a]*Laboratory of Analysis and Control of PDEs,*
Djillali Liabes University,
P. O. Box 89, Sidi Bel Abbes 22000, ALGERIA

Abstract. In this article, we consider a one-dimensional degenerate wave equation with a boundary control condition of fractional derivative type. We show that the problem is not uniformly stable by a spectrum method and we study the polynomial stability using the semigroup theory of linear operators.

1. Introduction

In this article, we are concerned with the boundary stabilization of convolution type for degenerate wave equation of the form

$$u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 \text{ in } (0,1)\times(0,\infty), \tag{1}$$

where the coefficient a is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (1) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a = \sup_{0<x\leq 1} \frac{x|a'(x)|}{a(x)}. \tag{2}$$

We distinguish the two following cases:

-The weakly degenerate case at 0. When $\mu_a \in [0,1[$, then the problem is called weakly degenerate at 0 and the natural boundary condition associated to (1) is the Dirichlet boundary condition $u(0) = 0$.

-The strongly degenerate case at 0. When $\mu_a > 1$, then the problem is called strongly degenerate at 0 and the natural boundary condition associated to (1) is the Neumann boundary condition $(au_x)(0) = 0$.

These type of conditions on diffusion coefficient and on the boundary were used before in the context of study of null controlability of degenerate parabolic equation.

Up to now, there are many works concerning the stabilization and controllability of nondegenerate wave equation with different types of dampings (see e.g. **[10]**, **[17]**, **[9]**, **[19]** and the references therein). In

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Email addresses: benaissa_abbes@yahoo.com (Abbes Benaissa), aichi.math@gmail.com (Chahira Aichi)

[17], for $a(x) = a_1 x + a_0$: the authors have established aymptotics stabilization with the following boundary damping

$$\begin{cases} (au_x)(0,t) = 0, \\ (au_x)(1,t) = -ku(1,t) - u_t(1,t),\ k > 0. \end{cases}$$

In [9], the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x)_x + \alpha u_t(x,t) + \beta y(x,t) = 0,\ 0 < x < 1,\ t > 0, \\ (a(x)u_x)(0) = k_1 u_t(0,t), t > 0, \\ (a(x)u_x)(1) = -k_2 u_t(1,t), t > 0, \end{cases}$$

where

$$\begin{cases} \alpha \geq 0, \beta > 0, k_1, k_2 \geq 0, k_1 + k_2 \neq 0, \\ a \in W^{1,\infty}(0,1), a(x) \geq a_0 \text{ for all } x \in [0,1]. \end{cases}$$

They proved the exponential decay of the solutions.

On the contrary, when the coefficient $a(x)$ is zero at some points, the equation will be degenerate and few results are known in this case, even though many problems that are relevant for applications are described by hyperbolic equations degenerating at the boundary of the space domain (see [20], [34] and [2]). In [20], for any $0 < \gamma < 1$, the null controllability of the following degenerate wave equation was considered:

$$\begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,T), \\ u(0,t) = \theta(t), u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases} \tag{PC}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. Recently, in [34] (see also [2]), the authors studied the null controllability problems of one-dimensional degenerate wave equations as in [20] but the control acts on the nondegenerate boundary. They proved that any initial value in state space is controllable. Also, an explicit expression for the controllability time is given.

In [2], Alabau has also considered the stabilization of the problem (1) together with boundary control of the form

$$u_t(1,t) + u_x(1,t) + \beta u(1,t) = 0, \tag{3}$$

where $\beta > 0$. Thanks to the dominant energy approach together with suitable elliptic estimates, she proved that (3) stabilizes exponentially the corresponding solution of the degenerate wave equation.

In this article, we are concerned with the system

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0 & \text{in } (0,1) \times (0,+\infty), \\ \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} & \text{in } (0,+\infty), \\ \beta u(1,t) + (au_x)(1,t) = -\varrho \partial_t^{\alpha,\eta} u(1,t) & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1). \end{cases} \tag{P}$$

where $\varrho > 0$ and $\beta \geq 0$.The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha < 1)$, with respect to the time variable (see [8] and [16]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds,\ \eta \geq 0.$$

The degenerate wave equation $(P)_1$ can describe the vibration problem of an elastic string. In a neighborhood of an endpoint $x = 0$ of this string, the elastic is sufficiently small or the linear density is large enough.

There are a few number of publications concerning the stabilization of distributed systems with fractional damping. In [28], Mbodje studies the energy decay of the wave equation with a boundary fractional derivative control. He used a new approach, when the original model is transformed into an augmented system, and by using energy methods, he proves strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq C/t$ if $\eta \neq 0$. Very recently in **[1]**, Benaissa and al. considered the Euler-Bernoulli beam equation with boundary damping of fractional derivative type defined by

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ u(0,t) = u_x(0,t) = 0 & \text{on } (0,+\infty), \\ u_{xx}(L,t) = 0 & \text{on } (0,+\infty), \\ u_{xxx}(L,t) - \gamma\partial_t^{\alpha,\eta}u_t(L,t) = 0 & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,L). \end{cases} \tag{PEF}$$

They proved, under the condition $\eta = 0$, by a spectral analysis, the non uniform stability. On the other hand, for $\eta > 0$, they also proved that the energy of system (PEF) decay as time goes to infinity as $t^{-1/(1-\alpha)}$.

Fractional calculus so often arise in many physical, chemical, biological, and economical phenomena (see **[4]**, **[5]**, **[6]** and **[26]**). In recent years, the control of PDEs with boundary damping of convolution type has become an active area of research because it improve the performance of the systems.

This paper as organized as follows. In section 2, we give preliminaries results and we reformulate the system (P) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, uniqueness of strong and weak solutions of the system, when we used Hille-Yosida Theorem. In section 4, we prove lack of exponential stability by spectral analysis for particular case $a(x) = x^\gamma, 0 \leq \gamma < 2$ by using Bessel functions. In section 5, we study asymptotic stability of above model and we establish a polynomial energy decay depending with parameter α for smooth solution. In the last section, we prove an optimal decay rate for the particular case $a(x) = x^\gamma$. The proof heavily relies on multiplier method, Bessel equations and Borichev-Tomilov Theorem.

2. Preliminaries Results

Let $a \in C([0,1] \cap C^1(]0,1])$ be a function satisfying the following assumptions:

$$\begin{cases} (i) & a(x) > 0 \ \forall x \in]0,1], a(0) = 0, \\ (ii) & \mu_a = \sup\limits_{0<x\leq 1} \dfrac{x|a'(x)|}{a(x)} < 2, \text{ and} \\ (iii) & a \in C^{[\mu_a]}([0,1]), \end{cases} \tag{4}$$

where $[\cdot]$ stands for the integer part.

When $\mu_a > 1$, we suppose $\beta > 0$ because if $\beta = 0$ and the feedback law only depends on velocities, we may encounter the situation where the closed-loop system is not well-posed in terms of the semigroups in the Hilbert space.

Examples: 1) Let $\varpi \in (0,2)$ be given. Define

$$a(x) = x^\varpi \quad \forall x \in [0,1].$$

satifies (4).
2) Let $\varpi \in [0,2)$ be given and let $\theta \in (0, 1 - \varpi/2)$. The function

$$a(x) = x^\varpi(1 + \cos^2(\ln x^\theta)) \quad \forall x \in [0,1]$$

satifies (4).
3) Let $\varpi \in [0,2)$ be given and let $\theta \in (0,\varpi)$. The function

$$a(x) = x^\varpi e^{(\theta-\varpi)x} \quad \forall x \in [0,1]$$

satifies (4).

Now, we introduce, as in **[14]**, **[18]** or **[2]**, the following weighted spaces:

$$H^1_{0,a}(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : \sqrt{a(x)}u_x \in L^2(0,1)/\ u(0) = 0\right\}$$
$$\text{if } \mu_a \in [0,1[,$$

$$H^1_a(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : \sqrt{a(x)}u_x \in L^2(0,1)\right\} \text{ if } \mu_a \in [1,2[.$$

It is easy to see that $H^1_a(0,1)$ when $\beta > 0$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_a(0,1)} = \int_0^1 a(x)u'(x)\overline{v'(x)}\,dx + \beta u(1)\overline{v(1)}.$$

Let us also set

$$|u|_{H^1_{0,a}(0,1)} = \left(\int_0^1 a(x)|u'(x)|^2\,dx\right)^{1/2} \quad \forall u \in H^1_a(0,1).$$

Actually, $|\cdot|_{H^1_{0,a}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,a}(0,1)$ to the norm of $H^1_a(0,1)$ when $\mu_a \in [0,1[$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 2.1. *Assume (4) with $\mu_a \in [0,1)$. Then there is a positive constant $C_* = C(a)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_*|u|^2_{1,a} \quad \forall u \in H^1_{0,a}(0,1). \tag{5}$$

Proof. Let $u \in H^1_{0,a}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left|\int_0^x u'(s)\,ds\right| \leq |u|_{1,a}\left\{\int_0^1 \frac{1}{a(s)}\,ds\right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\,dx \leq |u|^2_{1,a}\left\{\int_0^1 \frac{1}{a(s)}\,ds\right\}.$$

Next, we define

$$H^2_a(0,1) = \{u \in H^1_a(0,1) : \ au' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Now, we state two propositions that will be needed later (see **[14]**, **[18]** and **[2]**).

Proposition 2.2. *Assume (4). Then the following properties hold.*

(i) For every $u \in H^1_a(0,1)$

$$\lim_{x\to 0} xu^2(x) = 0. \tag{6}$$

(ii) For every $u \in H^2_a(0,1)$

$$\lim_{x\to 0} xa(x)u'(x)^2 = 0. \tag{7}$$

(iii) For every $u \in H^2_a(0,1)$

$$\lim_{x\to 0} xa(x)u(x)u'(x) = 0. \tag{8}$$

Proposition 2.3. $H^1_a(0,1) \hookrightarrow L^2(0,1)$ *with compact embedding.*

2.1. Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.4 (see [28]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{9}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\phi(\xi,t) + \xi^2\phi(\xi,t) + \eta\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{10}$$

$$\phi(\xi,0) = 0, \tag{11}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi \tag{12}$$

is given by

$$O = I^{1-\alpha,\eta}U. \tag{13}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau$$

Lemma 2.5 (see [1]). *If $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ then*

$$F_1(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}$$

and

$$F_2(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(\lambda+\eta+\xi^2)^2}\,d\xi = (1-\alpha)\frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-2}.$$

□

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.4, system (P) may be recast into the augmented model:

$$\begin{cases} u_{tt}(x,t) - (a(x)u_x(x,t))_x = 0, \\ \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(1,t)\mu(\xi) = 0, \qquad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \\ \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_a < 1 \\ (au_x)(0,t) = 0 & \text{if } 1 \leq \mu_a < 2 \end{cases} \\ \beta u(1,t) + (au_x)(1,t) = -\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi, \quad \zeta = \varrho(\pi)^{-1}\sin(\alpha\pi), \\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{P'}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + a(x)|u_x|^2)dx + \frac{\beta}{2}|u(1,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2\,d\xi. \tag{14}$$

Lemma 2.6. *Let (u, ϕ) be a regular solution of the problem (P'). Then, the energy functional defined by (14) satisfies*

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 \, d\xi \le 0. \tag{15}$$

Remark 2.7. *For an initial datum in $D(\mathcal{A})$ (see Theorem 3.1 below), we know that (u, ϕ) is of class C^1 in time, thus we can derive the energy $E(t)$.*

Proof of Lemma 2.6. Multiplying the first equation in (P') by $\overline{u}_t$, integrating over $(0, 1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x, t)\overline{u}_t dx - \int_0^1 (a(x)u_x(x, t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x, t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 a(x)|u_x(x, t)|^2 \, dx - \Re\Big[(a(x)u_x)(x, t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \left(|u_t(x, t)|^2 + a(x)|u_x(x, t)|^2\right) dx + \frac{\beta}{2}|u(1, t)|^2\right) + \zeta \Re \overline{u}_t(1, t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) \, d\xi = 0. \tag{16}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\zeta \int_{-\infty}^{+\infty} \phi_t(\xi, t)\overline{\phi}(\xi, t) d\xi + \zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 d\xi - \zeta u_t(1, t) \int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi, t) d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi + \zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 d\xi - \zeta \Re u_t(1, t) \int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi, t) d\xi = 0. \tag{17}$$

From (14), (16) and (17) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 \, d\xi \le 0.$$

This completes the proof of the lemma.

□

3. Global Existence

In this section, we give an existence and uniqueness result for problem (P') using the semigroup theory. Introducing the vector function $U = (u, v, \phi)^T$, where $v = u_t$, system (P') is equivalent to

$$\begin{cases} U' = \mathcal{A}U, & t > 0, \\ U(0) = (u_0, u_1, \phi_0), \end{cases} \tag{18}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ (a(x)u_x)_x \\ -(\xi^2 + \eta)\phi + v(1)\mu(\xi) \end{pmatrix}. \tag{19}$$

We introduce the following Hilbert space (the energy space):

$$\mathcal{H} = H^1_*(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty),$$

where

$$H^1_*(0,1) = \begin{cases} H^1_{0,a}(0,1) & \text{if } \mu_a \in [0,1), \\ H^1_a(0,1) & \text{if } \mu_a \in [1,2). \end{cases}$$

For $U = (u, v, \phi)^T$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi})^T$ we define the following inner product in $\mathcal{H}$

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \int_0^1 a(x) u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \phi \overline{\tilde{\phi}}\, d\xi + \beta u(1)\overline{\tilde{u}}(1).$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,\phi)^T \text{ in } \mathcal{H} : u \in H^2_a(0,1) \cap H^1_*(0,1), v \in H^1_*(0,1), \\ -(\xi^2+\eta)\phi + v(1)\mu(\xi) \in L^2(-\infty,+\infty), \\ \beta u(1) + (au_x)(1) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{20}$$

We have the following existence and uniqueness result.

Theorem 3.1 (Existence and uniqueness).

(1) If $U_0 \in D(\mathcal{A})$, then system (18) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (18) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof

We use the semigroup approach. In what follows, we prove that $\mathcal{A}$ is monotone. For any $U \in D(\mathcal{A})$ and using (18), (15) and the fact that

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{21}$$

we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi. \tag{22}$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Given $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists $U \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = F. \tag{23}$$

Equation (23) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (a(x)u_x)_x = f_2, \\ \lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{24}$$

Suppose u is found with the appropriate regularity. Then, $(24)_1$ $(24)_2$ yield

$$v = \lambda u - f_1 \in H^1_*(0,1) \tag{25}$$

and

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(1)}{\xi^2 + \eta + \lambda}. \tag{26}$$

By using (24) and (25) it can easily be shown that u satisfies

$$\lambda^2 u - (a(x)u_x)_x = f_2 + \lambda f_1. \tag{27}$$

Solving equation (27) is equivalent to finding $u \in H^2_a(0,1) \cap H^1_*(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (a(x)u_x)_x\overline{w})\,dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx, \tag{28}$$

for all $w \in H^1_*(0,1)$. By using (28), the boundary condition $(20)_3$ and (26) the function u satisfying the following equation

$$\begin{aligned}&\int_0^1 (\lambda^2 u\overline{w} + (a(x)u_x)\overline{w}_x)\,dx + \tilde{\zeta}v(1)\overline{w}(1) + \beta u(1)\overline{w}(1)\\ &\qquad = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\overline{w}(1)\end{aligned} \tag{29}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,d\xi$. Using again (25), we deduce that

$$v(1) = \lambda u(1) - f_1(1). \tag{30}$$

Inserting (30) into (29), we get

$$\begin{cases}\displaystyle\int_0^1 (\lambda^2 u\overline{w} + a(x)u_x\overline{w}_x)\,dx + (\lambda\tilde{\zeta} + \beta)u(1)\,\overline{w}(1)\\ \displaystyle= \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\overline{w}(1) + \tilde{\zeta}f_1(1)\overline{w}(1).\end{cases} \tag{31}$$

Problem (31) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{32}$$

where $\mathcal{B} : [H^1_*(0,1) \times H^1_*(0,1)] \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u,w) = \int_0^1 (\lambda^2 u\overline{w} + a(x)u_x\overline{w}_x)\,dx + (\lambda\tilde{\zeta} + \beta)u(1)\,\overline{w}(1)$$

and $\mathcal{L} : H^1_*(0,1) \to \mathbb{C}$ is the linear functional given by

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\overline{w}(1) + \tilde{\zeta}f_1(1)\overline{w}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (32) has a unique solution $u \in H^1_*(0,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_a(0,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, the result of Theorem 3.1 follows.

□

4. Lack of Exponential Stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (18). In order to state and prove our stability results, we need some lemmas.

Theorem 4.1 ([31]). *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 4.2 ([11]). *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim_{|\beta|\to\infty}} \frac{1}{\beta^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t} U_0\|^2 \le \frac{c}{t^{\frac{2}{l}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 4.3 ([3]). *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\ge 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\ge 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main result is the following.

Theorem 4.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi)^T$ the image of $(\sin x, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = -|\xi|^{\frac{2\alpha-5}{2}} \sin 1$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. So $(u, v, \phi)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$:
A) $a(x) = x^\gamma$**:** We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Here, we consider only the case $a(x) = x^\gamma, 0 \le \gamma < 2$ and in particular we treat the case $1 \le \gamma < 2$. The case $0 \le \gamma < 1$ is similar. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\gamma u_x)_x = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0. \end{cases} \tag{33}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(33)_1 - (33)_2$ for such λ, we find

$$\lambda^2 u - (x^\gamma u_x)_x = 0. \tag{34}$$

Using the boundary conditions and $(33)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\gamma u_x)_x = 0 \\ (x^\gamma u_x)(0) = 0 \\ u_x(1) + \zeta v(1) \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi + \beta u(1) \\ \qquad\qquad = u_x(1) + (\varrho\lambda(\lambda+\eta)^{\alpha-1} + \beta)u(1) = 0. \end{cases} \tag{35}$$

Assume that u is a solution of (35) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\gamma}{2}} \Psi\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

is a solution of the following boundary problem:

$$\begin{cases} y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\gamma-1}{2-\gamma})^2)\Psi(y) = 0 \\ (2-\gamma) y^{\frac{1}{2-\gamma}} \Psi'(y) - (\gamma-1) y^{\frac{\gamma-1}{2-\gamma}} \Psi(y) \to 0 \text{ as } y \to 0 \\ \left(\frac{1-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right)\Psi(\frac{2}{2-\gamma} i\lambda) + i\lambda\Psi'(\frac{2}{2-\gamma} i\lambda) = 0. \end{cases} \tag{36}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{37}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\gamma}{2}} J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\gamma}{2}} J_{-\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda x^{\frac{2-\gamma}{2}}\right),$$

where

$$\nu_\gamma = \frac{\gamma-1}{2-\gamma}$$

and J_{ν_γ} and $J_{-\nu_\gamma}$ are Bessel functions of the first kind of order ν_γ and $-\nu_\gamma$. We suppose $\nu_\gamma \notin \mathbb{N}$. So J_{ν_γ} and $J_{-\nu_\gamma}$ are linearly independent and therefore the pair $(J_{\nu_\gamma}, J_{-\nu_\gamma})$ (classical result) forms a fundamental system of solutions $(36)_1$.

Using the series expansion of J_{ν_α} and $J_{-\nu_\alpha}$, we deduce that (see **[15]**) $\Phi_+ \in H^1_*(0,1)$, while $\Phi_- \notin H^1_*(0,1)$, so

$$u(x) = c_+\Phi_+(x).$$

Moreover, $x^\gamma \Phi'_+(x) \to 0$ as $x \to 0$, hence the boundary condition in 0 is automatically satisfied.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \Re(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that Φ_+ remains bounded.

Lemma 4.5. *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*, |k|\ge N} \subset \sigma(\mathcal{A}), \tag{38}$$

where

$$\lambda_k = -\frac{2-\gamma}{2} i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), k \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof. The proof is decomposed in three steps:
Step 1.

$$\left(\frac{1-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) + i\lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right) = 0. \tag{39}$$

We known that

$$xJ'_\nu(x) = \nu J(x) - xJ_{\nu+1}(x). \tag{40}$$

Then (39) is equivalent to

$$\begin{aligned} f(\lambda) &= \left(\tfrac{1-\gamma}{2} + \nu_\gamma \tfrac{2-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma} i\lambda\right) - i\lambda J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma} i\lambda\right) \\ &= -i\lambda\left(J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma} i\lambda\right) - \frac{1}{i\lambda}\left(\frac{1-\gamma}{2} + \nu_\gamma\frac{2-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right)\right) = 0. \end{aligned} \tag{41}$$

We set

$$\tilde{f}(\lambda) = J_{\nu_\gamma+1}\left(\frac{2}{2-\gamma} i\lambda\right) - \frac{1}{i\lambda}\left(\frac{1-\gamma}{2} + \nu_\gamma\frac{2-\gamma}{2} + \beta + \varrho\lambda(\lambda+\eta)^{\alpha-1}\right) J_{\nu_\gamma}\left(\frac{2}{2-\gamma} i\lambda\right). \tag{42}$$

We will use the following classical asymptotic development (see **[25]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|argz| \leq \pi - \delta$:

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{43}$$

Then

$$\tilde{f}(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \frac{e^{-iz}}{2i}\tilde{\tilde{f}}(\lambda), \tag{44}$$

where

$$\tilde{z} = \frac{2}{2-\gamma} i\lambda, \quad z = \frac{2}{2-\gamma} i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}$$

and

$$\begin{aligned} \tilde{\tilde{f}}(\lambda) &= (e^{2iz} - 1) - \frac{\varrho}{\lambda^{1-\alpha}}(e^{2iz} + 1) + o\left(\frac{1}{\lambda^{1-\alpha}}\right) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\tfrac{1}{\lambda^{1-\alpha}}\right), \end{aligned} \tag{45}$$

where

$$f_0(\lambda) = e^{2iz} - 1. \tag{46}$$

$$f_1(\lambda) = -\frac{\varrho}{\lambda^{1-\alpha}}(e^{2iz} + 1). \tag{47}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (46), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2iz} - 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\gamma}i\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) = 2ik\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (41) the unknown λ by $u = 2iz$ then (41) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)$$

The roots of f_0 are $u_k = -\dfrac{2-\gamma}{2}i\left(k + \dfrac{\nu_\gamma}{2} + \dfrac{1}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\dfrac{2-\gamma}{2}i\left(k + \dfrac{\nu_\gamma}{2} + \dfrac{1}{4}\right)\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\gamma}{2}i\left(k + \frac{\nu_\gamma}{2} + \frac{1}{4}\right)\pi + \varepsilon_k. \tag{48}$$

Using (48), we get

$$\begin{aligned} e^{2i\lambda_k} &= e^{-\frac{4}{2-\gamma}\varepsilon_k} \\ &= 1 - \tfrac{4}{2-\gamma}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{49}$$

Substituting (49) into (45), using that $\widetilde{\tilde{f}}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -\frac{4}{2-\gamma}\varepsilon_k - \frac{2\varrho}{(-\frac{2-\gamma}{2}ik\pi)^{1-\alpha}} + o(\varepsilon_k) + o\left(\frac{1}{k^{1-\alpha}}\right) = 0 \tag{50}$$

and hence

$$\begin{aligned} \varepsilon_k &= -\frac{(2-\gamma)\varrho}{2(-\frac{2-\gamma}{2}ik\pi)^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right) \\ &= \begin{cases} -\left(\dfrac{2-\gamma}{2}\right)^\alpha \dfrac{\varrho}{(k\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} - i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{1-\alpha}}\right) \text{ for } k \geq 0, \\ -\left(\dfrac{2-\gamma}{2}\right)^\alpha \dfrac{\varrho}{(-k\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} + i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{1-\alpha}}\right) \text{ for } k \leq 0. \end{cases} \end{aligned} \tag{51}$$

From (51) we have in that case $|k|^{1-\alpha}\Re\lambda_k \sim \beta$, with

$$\beta = -\left(\frac{2-\gamma}{2}\right)^\alpha \frac{\varrho}{\pi^{1-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

Remark 4.6. *1) Similarly, we can prove Lack of exponential stability when $\nu_\gamma \in \mathbb{N}$. In this case we define Bessel's functions of order ν_γ of the second kind as following*

$$Y_{\nu_\gamma}(y) = \lim_{\nu \to \nu_\gamma} \frac{J_\nu(y)\cos\nu\pi - J_{-\nu}(y)}{\sin\nu\pi}.$$

Then, J_{ν_γ} and Y_{ν_γ} forms a fundamental system of solutions $(36)_1$.
2) Similarly, we can prove Lack of exponential stability when $\gamma \in [0,1[$.

B) General form of $a(x)$: There exists only few works concerning explicit representation for solutions of Sturm-Liouville equations (see **[23]** and **[24]**).

We consider the following eigenvalues problem

$$\begin{cases} \lambda^2 u - (a(x)u_x)_x = 0 \\ (a(x)u_x)(0) = 0 \\ u_x(1) + (\varrho\lambda(\lambda+\eta)^{\alpha-1} + \beta)u(1) = 0. \end{cases} \tag{52}$$

Define the functions

$$l(x) = \int_0^x \frac{1}{\sqrt{a(s)}}\,ds, \quad \rho(x) = a(x)^{1/4}$$

and

$$Q(x) = \frac{a(x)}{4}\left[\left(\frac{a'(x)}{a(x)}\right)' + \frac{3}{4}\left(\frac{a'(x)}{a(x)}\right)^2\right].$$

In **[23]**, the authors have derive a Neumann series of Bessel functions (NSBF) representation for solutions of Sturm-Liouville equation with variable coefficients as the following:

Let g be a solution of the equation

$$(a(x)g')' = 0, \quad x \in [0,1].$$

Then the following two families of auxiliary functions are well defined

$$\begin{aligned} \tilde{Y}^{(0)}(x) &\equiv Y^{(0)}(x) \equiv 1, \\ Y^{(n)}(x) &= \begin{cases} n\int_0^x Y^{(n-1)}(x)\frac{1}{g^2(s)a(s)}\,ds, & n \text{ odd}, \\ n\int_0^x Y^{(n-1)}(x)g^2(s)\,ds, & n \text{ even} \end{cases} \\ \tilde{Y}^{(n)}(x) &= \begin{cases} n\int_0^x \tilde{Y}^{(n-1)}(x)g^2(s)\,ds, & n \text{ odd}, \\ n\int_0^x \tilde{Y}^{(n-1)}(x)\frac{1}{g^2(s)a(s)}\,ds, & n \text{ even}. \end{cases} \end{aligned}$$

We define the formal powers associated to equation (52)

$$\Phi_k(x) = \begin{cases} g(x)Y^{(k)}(x), & k \text{ odd}, \\ g(x)\tilde{Y}^{(k)}(x), & k \text{ even}, \end{cases} \quad \Psi_k(x) = \begin{cases} \frac{1}{g(x)}Y^{(k)}(x), & k \text{ even}, \\ \frac{1}{g(x)}\tilde{Y}^{(k)}(x), & k \text{ odd}. \end{cases}$$

Then two linearly independent solutions v_1 and v_2 of equation (52) for $\lambda \neq 0$ can be written in the form

$$v_1(x) = \frac{\cos(i\lambda l(x))}{a(x)^{1/4}} + 2\sum_{n=0}^{\infty}(-1)^n \sigma_{2n}(x) j_{2n}(i\lambda l(x)),$$

$$v_2(x) = \frac{\sin(i\lambda l(x))}{a(x)^{1/4}} + 2\sum_{n=0}^{\infty}(-1)^n \sigma_{2n+1}(x) j_{2n+1}(i\lambda l(x)),$$

the coefficients σ_n being defined by the equalities

$$\sigma_n(x) = \frac{2n+1}{2}\left(\sum_{k=0}^{n} \frac{l_{k,n}\Phi_k(x)}{l^k(x)} - \frac{1}{a(x)^{1/4}}\right),$$

where $l_{k,n}$ is the corresponding coefficient of x^k in the Legendre polynomial of order n. Moreover, we obtain

$$\begin{aligned} v_1'(x) &= \frac{1}{\sqrt{a(x)}}\left(\frac{1}{a(x)^{1/4}}(G_1(x)\cos(i\lambda l(x)) - i\lambda\sin(i\lambda l(x))) + 2\sum_{n=0}^{\infty}(-1)^n\mu_{2n}(x)j_{2n}(i\lambda l(x))\right) \\ &\quad -\frac{(a(x)^{1/4})'}{a(x)^{1/4}}v_1(x), \\ v_2'(x) &= \frac{1}{\sqrt{a(x)}}\left(\frac{1}{a(x)^{1/4}}(G_2(x)\sin(i\lambda l(x)) + i\lambda\cos(i\lambda l(x))) + 2\sum_{n=0}^{\infty}(-1)^n\mu_{2n+1}(x)j_{2n+1}(i\lambda l(x))\right) \\ &\quad -\frac{(a(x)^{1/4})'}{a(x)^{1/4}}v_2(x), \end{aligned}$$

where

$$G_1(x) = h + \frac{1}{2}\int_0^{l(x)} Q(s)\,ds, \quad G_2(x) = \frac{1}{2}\int_0^{l(x)} Q(s)\,ds,$$

where

$$h = \lim_{x\to 0}\sqrt{a(x)}\left(\frac{g'(x)}{g(x)} + \frac{\rho'(x)}{\rho(x)}\right),$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x)$$

and

$$\begin{aligned} \mu_n(x) = \frac{2n+1}{2\rho(x)}&\left(\sum_{k=0} n\frac{l_{k,n}}{l^k(x)}\left(k\frac{\Psi_{k-1}(x)}{\rho(x)} + \rho(x)\sqrt{a(x)}\left(\frac{g'(x)}{g(x)} + \frac{(\rho'(x))}{\rho(x)}\right)\Phi_k(x)\right)\right. \\ &\left. -\frac{n(n+1)}{2l(x)} - G_2(x) - \frac{h}{2}(1+(-1)^n)\right) \end{aligned}$$

Now using this explicit representation together with asymptotic behavior of the spherical Bessel function j_n, we can deduce lack of exponential stability of solutions.

Remark 4.7. *We mention here the work of Baouendi and Goulaouic* [7]. *They studied a degeneate elliptic problem in an open domain of* $\mathbb{R}^n$ *and they gave an estimate of the spectral behavior.*

□

5. Asymptotic Stability

5.1. Strong stability of the system

In this part, we use a general criteria of Theorem 4.3 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the wave system (P) in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 5.1. *The* C_0*-semigroup* $e^{t\mathcal{A}}$ *is strongly stable in* $\mathcal{H}$*; i.e, for all* $U_0 \in \mathcal{H}$*, the solution of (18) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 5.1, we need the following two lemmas.

Lemma 5.2. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof
We will argue by contraction. Let us suppose that there $\lambda \in \mathbb{R}$.
•**Case 1**: $\lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (a(x)u_x)_x = 0, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0, \end{cases} \tag{53}$$

Then, from (22) we have

$$\phi \equiv 0. \tag{54}$$

From $(53)_3$, we have

$$v(1) = 0. \tag{55}$$

Hence, from $(53)_1$ we obtain

$$u(1) = 0 \text{ and } u_x(1) = 0. \tag{56}$$

From $(53)_1$ and $(53)_2$, we have

$$-\lambda^2 u - (a(x)u_x)_x = 0. \tag{57}$$

Hence

$$\begin{cases} \lambda^2 u + (a(x)u_x)_x = 0, \\ u(1) = u_x(1) = 0, \\ \begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1), \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2). \end{cases} \end{cases} \tag{58}$$

Multiplying equation $(58)_1$ by $\overline{u}$, using Green formula, Proposition 2.2-(iii) and the boundary conditions, we get

$$\lambda^2 \int_0^1 |u|^2 \, dx - \int_0^1 a(x)|u_x|^2 \, dx = 0. \tag{59}$$

Multiplying equation $(58)_1$ by $x\overline{u}_x$, we get

$$\lambda^2 \int_0^1 xu\overline{u}_x \, dx + \int_0^1 x\overline{u}_x(a(x)u_x)_x \, dx = 0. \tag{60}$$

$U \in D(\mathcal{A})$, then the regularity is sufficiently for applying an integration on the second integral in the left hand side in equation (60). Then we obtain

$$\frac{\lambda^2}{2} \int_0^1 x\frac{d}{dx}|u|^2 \, dx - \int_0^1 a(x)|u_x|^2 \, dx - \frac{1}{2}\int_0^1 xa(x)\frac{d}{dx}|u_x|^2 \, dx = 0. \tag{61}$$

Using Green formula and the boundary conditions, we get

$$\lambda^2 \int_0^1 |u|^2 \, dx + \int_0^1 (a(x) - xa'(x))|u_x|^2 \, dx = 0 \tag{62}$$

Multiplying equations (59) by $-\mu_a/2$, and tacking the sum of this equation and (62), we get

$$\frac{2-\mu_a}{2}\lambda^2\int_0^1 |u|^2\,dx + \int_0^1 \left(a(x) - xa'(x) + \frac{\mu_a}{2}a(x)\right)|u_x|^2\,dx = 0. \tag{63}$$

By definition of μ_a, we have

$$(2-\mu_a)a(x) \le 2(a(x) - xa'(x)) + \mu_a a(x).$$

This, togheter with (63), gives

$$\frac{2-\mu_a}{2}\lambda^2\int_0^1 |u|^2\,dx + \frac{2-\mu_a}{2}\int_0^1 a(x)|u_x|^2\,dx \le 0. \tag{64}$$

we deduce that

$$u = 0. \tag{65}$$

Using equation $(53)_1$, we obtain

$$v = 0. \tag{66}$$

Consequently, using equations (65), (66) and (54), we obtain $U = 0$, which contradict the hypothesis $U \neq 0$. The proof has been completed.

•**Case 2**: $\lambda = 0$. The system (53) becomes

$$\begin{cases} v = 0, \\ (a(x)u_x)_x = 0, \\ (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0. \end{cases} \tag{67}$$

From $(67)_1$ and $(67)_3$, we have

$$v \equiv 0, \quad \phi \equiv 0. \tag{68}$$

Multiplying equation $(67)_2$ by $\overline{u}$, using Green formula, Proposition 2.2-(iii) and the boundary conditions, we get

$$\int_0^1 a(x)|u_x|^2\,dx + \beta|u(1)|^2 = 0. \tag{69}$$

Then

$$(a|u_x|^2)(x) = 0 \quad \forall x \in (0,1). \tag{70}$$

Moreover, if $\mu_a \in [1,2)$, then $u(1) = 0$. Hence $(au_x)(1) = 0$ and consequently

$$u_x(1) = 0. \tag{71}$$

Moreover, from (70), we have

$$u_x(x) = 0 \ \text{ on } (0,1).$$

Hence u is constant in $(0,1)$. As $u(1) = 0$, then

$$u \equiv 0$$

Now, if $\mu_a \in [0,1)$, we have $u(0) = 0$. Hence $u \equiv 0$. and consequently, we obtain $U = 0$, which contradict the hypothesis $U \neq 0$. The proof has been completed.

We have Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues. So the condition (*i*) of Theorem 4.3 holds. The condition (*ii*) of Theorem 4.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We have the following lemma.

Lemma 5.3. *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Proof

•**Case 1**: $\lambda \neq 0$.

We will prove that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we seek $X = (u, v, \phi)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda I - \mathcal{A})X = F. \tag{72}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (a(x)u_x)_x = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{73}$$

From $(73)_1$ and $(73)_2$, we have

$$-\lambda^2 u - (a(x)u_x)_x = (f_2 + i\lambda f_1). \tag{74}$$

Solving system (74) is equivalent to finding $u \in H_a^2 \cap H_*^1(0,1)$ such that

$$\int_0^1 (-\lambda^2 u\overline{w} - (a(x)u_x)_x\overline{w})\, dx = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx \tag{75}$$

for all $w \in H_*^1(0,1)$. Then, we get

$$\begin{cases} \displaystyle\int_0^1 (-\lambda^2 u\overline{w} + (a(x)u_x)\overline{w}_x)\, dx + (i\lambda\tilde{\zeta} + \beta)u(1)\,\overline{w}(1) \\ \displaystyle = \int_\Omega (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\overline{w}\, d\xi + \tilde{\zeta} f_1(1)\overline{w}(1)\,. \end{cases} \tag{76}$$

We can rewrite (76) as

$$-(L_\lambda u, w)_{H_*^1} + (u, w)_{H_*^1} = l(w), \tag{77}$$

with the inner product defined by

$$(u, w)_{H_*^1} = \int_0^1 a(x)u_x\overline{w}_x\, dx + \beta u(1)\,\overline{w}(1)$$

and

$$(L_\lambda u, w)_{H_*^1} = \int_\Omega \lambda^2 u\overline{w}\, dx - i\lambda\tilde{\zeta}u(1)\,\overline{w}(1).$$

Using the compactness embedding from $L^2(0,1)$ into $H_*^{-1}(0,1)$ and from $H_*^1(0,1)$ into $L^2(0,1)$ we deduce that the operator L_λ is compact from $L^2(0,1)$ into $L^2(0,1)$. Consequently, by Fredholm alternative, proving the existence of u solution of (77) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $u \neq 0$, such that

$$(L_\lambda u, w)_{H_*^1} = (u, w)_{H_*^1} \quad \forall w \in H_*^1. \tag{78}$$

In particular for $w = u$, it follows that

$$\lambda^2 \|u(x)\|^2_{L^2(0,1)} - i\lambda\tilde{\zeta}|u(1)|^2 = \| \sqrt{a(x)}u_x(x)\|^2_{L^2(0,1)} + \beta|u(1)|^2.$$

Hence, we have

$$u(1) = 0. \tag{79}$$

From (78), we obtain

$$(au_x)(1) = 0 \tag{80}$$

Then

$$\begin{cases} \lambda^2 u + (a(x)u_x)_x = 0 \text{ on } (0,1), \\ \begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1) \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2) \end{cases} \\ u(1) = 0 \ u_x(1) = 0. \end{cases} \tag{81}$$

We deduce that $U = 0$.
•**Case 2**: $\lambda = 0$ and $\eta \neq 0$.
The system (73) is reduced to the following system

$$\begin{cases} v = -f_1, \\ -(a(x)u_x)_x = f_2, \\ (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3. \end{cases} \tag{82}$$

Solving system (82) is equivalent to finding $u \in H^1_*(0,1)$ such that

$$-\int_0^1 (a(x)u_x)_x \overline{w}\, dx = \int_0^1 f_2 \overline{w}\, dx \tag{83}$$

for all $w \in H^1_*(0,1)$. Then, we get

$$\begin{aligned} \int_0^1 a(x)u_x\overline{w}_x\, dx + \beta u(1)\overline{w}(1) \quad &= \quad \int_0^1 f_2\overline{w}\, dx + \varrho\eta^{\alpha-1} f_1(1)\overline{w}(1) \\ &\quad -\zeta \int_{-\infty}^{\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta}\, d\xi \overline{w}(1). \end{aligned} \tag{84}$$

Consequently, problem (84) is equivalent to the problem

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{85}$$

where the bilinear form $\mathcal{B} : H^1_*(0,1) \times H^1_*(0,1) \to \mathbb{C}$ and the linear form $\mathcal{L} : H^1_*(0,1) \to \mathbb{C}$ are defined by

$$\mathcal{B}(u,w) = \int_0^1 (a(x)u_x\overline{w}_x)\, dx + \beta u(1)\overline{w}(1) \tag{86}$$

and

$$\mathcal{L}(w) = \int_0^1 f_2\overline{w}\, dx + \varrho\eta^{\alpha-1} f_1(1)\overline{w}(1) - \zeta \int_{-\infty}^{\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta}\, d\xi \overline{w}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So by applying the Lax-Milgram theorem, we deduce that for all $w \in H^1_*(0,1)$ problem (85) admits a unique solution $u \in H^1_*(\Omega)$. Applying the classical elliptic regularity, it follows from (84) that $u \in H^2_a(0,1)$. Therefore, the operator $\mathcal{A}$ is surjective. □

5.2. Residual spectrum of $\mathcal{A}$

Lemma 5.4. *Let $\mathcal{A}$ be defined by (19). Then*

$$\mathcal{A}^*\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} -v \\ -(a(x)u_x)_x \\ -(\xi^2+\eta)\phi - v(1)\mu(\xi) \end{pmatrix} \tag{87}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (u,v,\phi)^T \text{ in } \mathcal{H} : u \in H_a^2 \cap H_*^1(0,1), v \in H_*^1(0,1), \\ -(\xi^2+\eta)\phi - v(1)\mu(\xi) \in L^2(-\infty,+\infty), \\ (au_x)(1) + \beta u(1) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \tag{88}$$

Proof
Let $U = (u,v,\phi)^T$ and $V = (\tilde{u},\tilde{v},\tilde{\phi})^T$. We have
$< \mathcal{A}U, V >_{\mathcal{H}} = < U, \mathcal{A}^*V >_{\mathcal{H}}$.

$$\begin{aligned}
&< \mathcal{A}U, V >_{\mathcal{H}} = \int_0^1 a(x)v_x\overline{\tilde{u}_x}\, dx + \int_0^1 (a(x)u_x)_x\overline{\tilde{v}}\, dx + \zeta\int_{-\infty}^{+\infty} [-(\xi^2+\eta)\phi + v(1)\mu(\xi)]\overline{\tilde{\phi}}\, d\xi + \beta v(1)\overline{\tilde{u}}(1) \\
&= -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\, dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\, dx + [(a(x)\overline{\tilde{u}_x})v]_0^1 + [(a(x)u_x)\overline{\tilde{v}}]_0^1 - \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)\phi\overline{\tilde{\phi}}\, d\xi \\
&+\zeta v(1)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\tilde{\phi}}\, d\xi + \beta v(1)\overline{\tilde{u}}(1) \\
&= -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\, dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\, dx + (a(x)\overline{\tilde{u}_x})(1)v(1) - \beta u(1)\overline{\tilde{v}(1)} - \zeta\overline{\tilde{v}(1)}\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi \\
&-\zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)\phi\overline{\tilde{\phi}}\, d\xi + \zeta v(1)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\tilde{\phi}}\, d\xi + \beta v(1)\overline{\tilde{u}}(1).
\end{aligned}$$

If we set

$$(a(x)\tilde{u}_x)(1) + \beta\tilde{u}_x(1) + \zeta\int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}\, d\xi = 0,$$

we get

$$< \mathcal{A}U, V >_{\mathcal{H}} = -\int_0^1 (a(x)\overline{\tilde{u}_x})_x v\, dx - \int_0^1 (a(x)u_x)\overline{\tilde{v}_x}\, dx - \zeta\int_{-\infty}^{+\infty} \overline{[(\xi^2+\eta)\tilde{\phi} + \tilde{v}(1)\mu(\xi)]}\phi\, d\xi - \beta u(1)\overline{\tilde{v}(1)}.$$

Theorem 5.5. *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$. It is defined as*

$$\sigma_r(\mathcal{A}) = \{\lambda \in \mathbb{C} : ker(\lambda I - \mathcal{A}) = 0 \text{ and } Im(\lambda I - \mathcal{A}) \text{ is not dense in } \mathcal{H}\}.$$

Proof. Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. obviously this is because the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (87), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (u,v,\phi) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda u + v = 0, \\ \lambda v + (a(x)u_x)_x = 0, \\ \lambda\phi + (\xi^2+\eta)\phi + v(1)\mu(\xi) = 0. \end{cases} \tag{89}$$

From $(89)_1$ and $(89)_2$, we find

$$\lambda^2 u - (a(x)u_x)_x = 0. \tag{90}$$

As $(au_x)(1) = -\zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi - \beta u(1)$, we deduce from $(89)_3$ and $(88)_3$ that

$$(au_x)(1) = \zeta v(1) \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi - \beta u(1) = -\varrho\lambda(\lambda+\eta)^{\alpha-1} u(1) - \beta u(1) \tag{91}$$

with the following conditions

$$\begin{cases} u(0) = 0 & \text{if } \mu_a \in [0,1) \\ (a(x)u_x)(0) = 0 & \text{if } \mu_a \in [1,2) \end{cases} \tag{92}$$

Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete.

Remark 5.6. *When $\eta = 0$, then $\lambda = 0$ is in the continuous spectrum.*

□

5.3. Polynomial Stability (for $\eta \neq 0$)

Theorem 5.7. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\alpha)+\frac{\xi}{2}}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (a(x)u_x)_x = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = f_3, \end{cases} \tag{93}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$.
•**Step 1** Taking inner product in $\mathcal{H}$ with U and using (22) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{94}$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{95}$$

and, applying $(93)_1$, we obtain

$$\Big| |\lambda||u(1)| - |f_1(1)| \Big|^2 \leq |v(1)|^2.$$

We deduce that

$$|\lambda|^2|u(1)|^2 \leq c|f_1(1)|^2 + c|v(1)|^2. \tag{96}$$

Moreover, from $(93)_4$, we have

$$(au_x)(1) = -\beta u(1) - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi.$$

Then

$$\begin{aligned} |(au_x)(1)|^2 &\leq 2\beta^2|u(1)|^2 + 2\zeta^2 \left| \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi \right|^2 \\ &\leq 2\beta^2|u(1)|^2 + 2\zeta^2 \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi \right)\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \right) \\ &\leq 2\beta^2|u(1)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned} \tag{97}$$

From $(93)_3$, we obtain

$$v(1)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{98}$$

By multiplying (98) by $(i\lambda + \xi^2 + \eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}$ (for $\varepsilon > 0$), we get

$$(i\lambda + \xi^2 + \eta)^{-1}v(1)\mu(\xi)|\xi|^{\frac{1-\varepsilon}{2}} = |\xi|^{\frac{1-\varepsilon}{2}}\phi - (i\lambda + \xi^2 + \eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}} f_3(\xi). \tag{99}$$

Hence, by taking absolute values of both sides of (99), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$\mathcal{S}|v(1)| \leq \mathcal{U}\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi|^2\, d\xi \right)^{\frac{1}{2}} + \sqrt{2}\mathcal{V}\left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2\, d\xi \right)^{\frac{1}{2}}, \tag{100}$$

where

$$\mathcal{S} = \left| \int_{-\infty}^{+\infty} (i\lambda + \xi^2 + \eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}\mu(\xi)\, d\xi \right| = \frac{\pi}{\sin(\frac{2(\alpha+1)-\varepsilon}{4})\pi}|i\lambda + \eta|^{\frac{(\alpha-1)}{2}-\frac{\varepsilon}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\xi|^{1-\varepsilon}\, d\xi \right)^{\frac{1}{2}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2}|\xi|^{1-\varepsilon}\, d\xi \right)^{\frac{1}{2}} = \left(\frac{\varepsilon}{2}\frac{\pi}{\sin(\frac{2-\varepsilon}{2})\pi}(|\lambda|+\eta)^{-(1+\frac{\varepsilon}{2})} \right)^{1/2}.$$

Thus, by using the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, again, we get

$$\mathcal{S}^2|v(1)|^2 \leq 2\mathcal{U}^2\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi|^2\, d\xi \right) + 4\mathcal{V}^2\left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2\, d\xi \right). \tag{101}$$

We deduce that

$$|v(1)|^2 \leq c|\lambda|^{1-\alpha+\frac{\varepsilon}{2}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{102}$$

•**Step 2** Now we use the classical multiplier method.
Let us introduce the following notation

$$\mathcal{I}_u(\alpha) = |\sqrt{a(x)}u_x(\alpha)|^2 + |v(\alpha)|^2,$$

$$\mathcal{E}_u = \int_0^1 \mathcal{I}_u(s)\, ds.$$

Lemma 5.8. *We have that*

$$\begin{aligned}&\int_0^1 \left[\left((a(x)-xa'(x))+\frac{\mu_a}{2}a(x)\right)|u_x|^2+\left(1-\frac{\mu_a}{2}\right)|v(x)|^2\right]dx\\&=[x\mathcal{I}_u]_0^1+\tfrac{\mu_a}{2}[a(x)u_x\overline{u}]_0^1+R,\end{aligned}\tag{103}$$

where R satisfies

$$|R|\le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

for a positive constant C.

Proof

To get (103), let us multiply the equation $(93)_2$ by $x\overline{u}_x$ Integrating on $(0,1)$ we obtain

$$i\lambda\int_0^1 vx\overline{u}_x\,dx-\int_0^1(a(x)u_x)_x x\overline{u}_x\,dx=\int_0^L f_2x\overline{u}_x\,dx$$

or

$$-\int_0^1 vx(\overline{i\lambda u_x})\,dx-\int_0^1 x(a(x)u_x)_x\overline{u}_x\,dx=\int_0^1 f_2x\overline{u}_x\,dx.$$

Since $i\lambda u_x=v_x+f_{1x}$ taking the real part in the above equality results in

$$\begin{aligned}&-\frac{1}{2}\int_0^1 x\frac{d}{dx}|v|^2\,dx+\frac{1}{2}\int_0^1 xa(x)\frac{d}{dx}|u_x|^2\,dx-[xa(x)|u_x|^2]_0^1+\int_0^1 a(x)|u_x|^2\,dx\\&=Re\int_0^1 vx\overline{f}_{1x}\,dx+Re\int_0^1 f_2x\overline{u}_x\,dx.\end{aligned}$$

Performing an integration by parts we get

$$\int_0^1[|\sqrt{a(x)}u_x|^2+|v(x)|^2]\,dx-\int_0^1 xa'(x)|u_x(x)|^2\,dx=[x(|\sqrt{a(x)}u_x|^2+|v(x)|^2)]_0^1+R_1,\tag{104}$$

where

$$R_1=2Re\int_0^1 xf_2\overline{u}_x\,dx+2Re\int_0^1 xv\overline{f}_{1x}\,dx.$$

Multiplying $(93)_2$ by $\overline{u}$ and integrating over $(0,1)$ and using integration by parts we get

$$\int_0^1 a(x)|u_x|^2dx-\int_0^1|v|^2dx-[a(x)u_x\overline{u}]_0^1=\int_0^1 v\overline{f_1}\,dx+\int_0^1 f_2\overline{u}dx.\tag{105}$$

Multiplying (105) by $\mu_a/2$ and summing with (104) we get

$$\begin{aligned}&\int_0^1((a(x)-xa'(x))+\frac{\mu_a}{2}a(x))|u_x|^2+(1-\frac{\mu_a}{2})|v(x)|^2]\,dx\\&=[x\mathcal{I}_u]_0^1+\tfrac{\mu_a}{2}[a(x)u_x\overline{u}]_0^1+R\end{aligned}\tag{106}$$

with:

$$R=R_1+R_2$$

and

$$R_2=\frac{\mu_a}{2}\int_0^1 v\overline{f_1}\,dx+\frac{\mu_a}{2}\int_0^1 f_2\overline{u}dx.$$

Moreover

$$\int_0^1 x^2|u_x|^2\,dx \le \int_0^1 x^{\mu_a}|u_x|^2\,dx \le \frac{1}{a(1)}\int_0^1 a(x)|u_x|^2\,dx.$$

Then

$$\left|\int_0^1 x f_2\overline{u}_x\,dx\right| \le C\|f_2\|_{L^2(0,1)}\|xu_x\|_{L^2(0,1)} \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 xv\overline{f}_{1x}\,dx\right| \le C\|v\|_{L^2(0,1)}\|xf_{1x}\|_{L^2(0,1)} \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

$$\left|\int_0^1 v\overline{f_1}\,dx\right| \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$$

and

$$\left|\int_0^1 f_2\overline{u}dx\right| \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Hence, we deduce that

$$|R| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{107}$$

•**Step 3** We have

$$(a(x)u_x\overline{u})_{x=0} = 0,\ (x|v(x)|^2)_{x=0} = 0,\ (xa(x)|u_x|^2)_{x=0} = 0.$$

By definition of μ_a, we have

$$(2-\mu_a)a \le 2(a - xa') + \mu_a a.$$

This, together with (106), gives

$$\frac{2-\mu_a}{2}\int_0^1 (a(x)|u_x|^2 + |v|^2)\,dx \le \mathcal{I}_u(1) + \frac{\mu_a}{2}a(1)|u_x(1)\overline{u}(1)| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{108}$$

Then

$$\mathcal{E}_u \le c|u(1)|^2 + c'|u_x(1)|^2 + c''|v(1)|^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{109}$$

As

$$\begin{aligned}
|f_1(1)|^2 &\le 2\int_0^1 |f_1|^2\,dx + 2\int_0^1 |f_1(x) - f_1(1)|^2\,dx\\
&\le 2\int_0^1 |f_1|^2\,dx + \frac{2}{a(1)(2-\mu_a)}\int_0^1 a(x)|f_{1x}|^2\,dx\\
&\le C\|F\|^2_{\mathcal{H}}
\end{aligned}$$

Moreover, from (96) we deduce

$$|u(1)|^2 \le c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2.$$

From (97) we deduce

$$|u_x(1)|^2 \le c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2 + c'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Since that

$$\int_{-\infty}^{+\infty} |\phi(\xi)|^2\,d\xi \le C\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\,d\xi \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Hence

$$\|U\|^2_{\mathcal{H}} \le c\frac{1}{\lambda^2}\|F\|^2 + c\frac{1}{\lambda^2}|v(1)|^2 + c|v(1)|^2 + c'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{110}$$

Substitution of inequalities (102) into (110), we obtain that

$$\|U\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha+\frac{\varepsilon}{2}}\|F\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 4.2.

□

6. Optimality of Energy Decay when $a(x) = x^{\gamma}$ and $\eta \neq 0$

By Lemma 4.5, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues, since Lemma 4.5 shows an expected optimal behavior of resolvent like

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \equiv |\lambda|^{1-\alpha}.$$

We can expect a decay rate (optimal) of the energy at $t^{-2/(1-\alpha)}$. Unfortunately we were not able to prove this optimal decay rate by frequency domain method based on multiplier method for general function a. In Theorem 5.7, we obtain an upper estimate of resolvent like

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq |\lambda|^{2(1-\alpha)} \text{ as } |\lambda| \to \infty$$

which is less better. In this section, for $a(x) = x^{\gamma}, 0 \leq \gamma < 2$, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of the theorem by Borichev and Tomilov, we prove an optimal decay rate. We treat only the case $\gamma \in [1, 2)$ and $\nu_{\gamma} \notin \mathbb{N}$. The cases $\gamma \in [1, 2)$ and $\nu_{\gamma} \in \mathbb{N}$ and $\gamma \in [0, 1)$ are similar with some modifications.

Let us consider the resolvant equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^{\gamma} u_x)_x = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = f_3, \end{cases} \tag{111}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$. From $(111)_1$ and $(111)_2$, we have

$$\lambda^2 u + (x^{\gamma} u_x)_x = -(f_2 + i\lambda f_1) \tag{112}$$

with

$$\begin{cases} (x^{\gamma} u_x)_{x=0} = 0 \\ u_x(1) + \beta u(1) + \zeta \displaystyle\int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \end{cases} \tag{113}$$

The substitution of ϕ given by $(111)_3$ into $(113)_2$ give us

$$u_x(1) + \beta u(1) + \varrho(i\lambda + \eta)^{\alpha-1} v(1) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi = 0. \tag{114}$$

Moreover, from $(111)_1$, we have

$$v(1) = i\lambda u(1) - f_1(1).$$

Then, the condition (114) become

$$u_x(1) + (\beta + \varrho i\lambda(i\lambda + \eta)^{\alpha-1}) u(1) = \varrho(i\lambda + \eta)^{\alpha-1} f_1(1) - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{115}$$

Assume that Φ is a solution of (112), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\gamma}{2}}\Psi\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \tag{116}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\gamma-1}{2-\gamma}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\gamma})^2(\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{3-\gamma}{2-\gamma}}\left(f_2\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right) + i\lambda f_1\left((\tfrac{2-\gamma}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\gamma}}\right)\right). \end{aligned} \tag{117}$$

The solution can be written as

$$\Psi(y) = AJ_{\nu_\gamma}(y) + BJ_{-\nu_\gamma}(y) + \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\int_0^y \frac{f(s)}{s}\left(J_{\nu_\gamma}(s)J_{-\nu_\gamma}(y) - J_{\nu_\gamma}(y)J_{-\nu_\gamma}(s)\right)ds.$$

Thus,

$$\begin{aligned} &u(x) = Ax^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) + Bx^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right) \\ &-\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)x^{\frac{1-\gamma}{2}}\int_0^x s^{\frac{1-\gamma}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)\right. \\ &\left.-J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right)J_{-\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda s^{\frac{2-\gamma}{2}}\right)\right)ds. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) \;=\; & A\Phi_+(x) + B\Phi_-(x) \\ & -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x) - \Phi_+(x)\Phi_-(s))\,ds, \end{aligned} \tag{118}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\gamma}{2}}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\gamma}{2}}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda x^{\frac{2-\gamma}{2}}\right).$$

From where it follows

$$\begin{aligned} u_x(x) \;=\; & A\Phi'_+(x) + B\Phi'_-(x) \\ & -\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^x (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(x) - \Phi'_+(x)\Phi_-(s))\,ds \end{aligned} \tag{119}$$

From (115), (119) and (118), we conclude that

$$\begin{aligned} &A(\Phi'_+(1) + (\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\Phi_+(1)) = \varrho(i\lambda+\eta)^{\alpha-1}f_1(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \\ &+\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}\left(\frac{2}{2-\gamma}\right)\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds \\ &+(\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds, \end{aligned} \tag{120}$$

where

$$\begin{aligned} \Phi'_+(1) &= \frac{1-\gamma}{2}J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right), \\ \Phi'_-(1) &= \frac{1-\gamma}{2}J_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{-\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right). \end{aligned}$$

Let us set

$$\begin{aligned} D &= \Phi'_+(1) + (\beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})\Phi_+(1) \\ &= (\frac{1-\gamma}{2} + \beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1})J_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) + \lambda J'_{\nu_\gamma}\left(\frac{2}{2-\gamma}\lambda\right) \\ &= \left(\frac{1-\gamma}{2} + \frac{2-\gamma}{2}\nu_\gamma + \beta + \varrho i\lambda(i\lambda+\eta)^{\alpha-1}\right)J_{\nu_\gamma}\left(\tfrac{2}{2-\gamma}\lambda\right) - \lambda J_{\nu_\gamma+1}\left(\tfrac{2}{2-\gamma}\lambda\right) \\ &= \frac{i^\alpha\sqrt{2-\gamma}}{\sqrt{\pi}}\lambda^{\alpha-1/2}\cos\left(\frac{2}{2-\gamma}\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{i^\gamma\sqrt{2-\gamma}}{\sqrt{\pi}}\lambda^{1/2}\sin\left(\frac{2}{2-\gamma}\lambda - \nu_\gamma\frac{\pi}{2} - \frac{\pi}{4}\right) \\ &\quad + O\left(\frac{1}{\lambda^{1/2}}\right). \end{aligned}$$

It is clear that

$$|D| \geq c|\lambda|^{\alpha-1/2} \text{ for large } \lambda.$$

The constant A in (120) satisfies

$$\begin{aligned} |A||D(\lambda)| &\leq o(1) + \beta\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \\ &\quad +\varrho|\lambda|^\alpha\frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \\ &\quad \frac{2\nu_\gamma}{\sin\nu_\gamma\pi}(\frac{2}{2-\gamma})\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds\right| \\ &\leq o(1) + c\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right), \end{aligned} \tag{121}$$

where we have used the fact that $f_1 \in H^1_*(0,1)$ and

$$\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right),$$

$$\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi'_-(1) - \Phi'_+(1)\Phi_-(s))\,ds\right| \leq \left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

Then, we conclude that

$$|A| \leq c|\lambda|^{1/2-\alpha}.$$

Then

$$\|u\|_{L^2(0,1)} \leq c|\lambda|^{-\alpha}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From (119), we deduce that

$$\|x^{\gamma/2}u_x\|_{L^2(0,1)} \leq c|\lambda|^{1-\alpha}\left(\|f_1\|_{H^1_*(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From $(111)_1$ and (118), we get

$$\|v\|_{L^2(0,1)} \leq c|\lambda|^{1-\alpha}\left(\|f_1\|_{H^1(0,1)} + \|f_2\|_{L^2(0,1)}\right).$$

From $(111)_3$, we get

$$\begin{aligned} \|\phi\|_{L^2(-\infty,\infty)} &\leq |v(1)|\left\|\frac{\mu(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)} + \left\|\frac{f_3(\xi)}{i\lambda+\xi^2+\eta}\right\|_{L^2(-\infty,\infty)} \\ &\leq c|\lambda|^{-1/2}\left(\|f_1\|_{H^1(0,1)} + \|f_2\|_{L^2(0,1)}\right) + c\frac{1}{|\lambda|}\|f_3\|_{L^2(-\infty,\infty)}. \end{aligned}$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha} \text{ as } |\lambda| \to \infty.$$

The conclusion then follows by applying Theorem 4.2.

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