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Optimal Solutions and Applications to Nonlinear Matrix and Integral Equations via Simulation Function

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Abstract. Based on the concepts of α-proximal admissible mappings and simulation function, we establish some best proximity point and coupled best proximity point results in the context of b-complete b-metric spaces. We also provide some concrete examples to illustrate the obtained results. Moreover, we prove the existence of the solution of nonlinear integral equation and positive definite solution of nonlinear matrix equation $X = Q + \sum_{i=1}^{m} A_i^*\gamma(X)A_i - \sum_{i=1}^{m} B_i^*\gamma(X)B_i$. The given results not only unify but also generalize a number of existing results on the topic in the corresponding literature.

1. Introduction and Preliminaries

In 1989, Bakhtin [6] introduced the concept of b-metric space and presented the generalization of Banach contraction principle (see Banach [7]) (see also Czerwik [10]). Subsequently, several researchers studied fixed point theory for single-valued and set-valued mappings in b-metric spaces (see [1, 8, 15] and references therein).

Definition 1.1. [6, 10] Let X be a nonempty set, and let $k \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq k(d(x, z) + d(z, y))$.

In this case pair (X, d) is called b-metric space with coefficient k.

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There exist many examples in the literature (see [6, 10]) showing that the class of b-metrics is effectively larger than that of metric spaces. The notions of convergence, compactness, closedness and completeness in b-metric spaces are given in the same way as in metric spaces. For more work on fixed point theory in b-metric spaces, we refer to [1, 8, 11] and the references therein.

Example 1.2. [6] The space $l_p, (0 < p < 1), l_p = \left\{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$ together with the function $d : l_p \times l_p \to \mathbb{R}$, given by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$, where $x = (x_n), y = (y_n) \in l_p$, is a b-metric space. Indeed, by an elementary calculation we obtain

$$d(x, z) \leq 2^{\frac{1}{p}} [d(x, y) + d(y, z)],$$

hence $k = 2^{\frac{1}{p}}$ in this case.

Example 1.3. [6] The space $L_p, (0 < p < 1)$ for all real function $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}.$$

Khojasteh *et al.* [18] introduced the notion of simulation function:

Definition 1.4. [18] A simulation function is a mapping $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ which satisfies the following conditions:

($\xi_1{}^*$) $\xi(0, 0) = 0$;

($\xi_2{}^*$) $\xi(s, t) < t - s$ for all $s, t > 0$;

($\xi_3{}^*$) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} \sup \xi(s_n, t_n) < 0.$$

Later, Argoubi *et al.* [5] slightly modified the definition of simulation function by withdrawing a condition $\left(\xi_1^*\right)$.

Definition 1.5. [5] A simulation function is a mapping $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ which satisfies the following conditions:

(ξ_1) $\xi(s, t) < t - s$ for all $s, t > 0$;

(ξ_2) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} \sup \xi(s_n, t_n) < 0.$$

Let $\mathcal{Z}^*$ denote the class of simulation functions in the sense of Argoubi *et al.* [5].

Example 1.6. [5] Let $\xi_\lambda : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function defined by

$$\xi_\lambda(t, s) = \begin{cases} 1 & if \ (t, s) = (0, 0), \\ \lambda s - t & \text{otherwise}, \end{cases}$$

where $\lambda \in (0, 1)$. Then, $\xi_\lambda \in \mathcal{Z}^*$.

Example 1.7. [13] Let $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function defined by $\xi(t,s) = \psi(s) - \phi(t)$ for all $s,t \geq 0$, where $\psi : [0,\infty) \to \mathbb{R}$ is an upper semi-continuous function and $\varphi : [0,\infty) \to \mathbb{R}$ is a lower semi-continuous function such that $\psi(t) < t \leq \varphi(t)$, for all $t > 0$. Then $\xi \in \mathcal{Z}^*$.

Khojasteh *et al.* [18] presented a new contractive condition generalizing the contraction principal via simulation function. Later on, many authors presented the contractive conditions involving simulation functions (see, e.g. [3, 13, 20, 33] and references therein).

Definition 1.8. [18] Assume that (X,d) is a metric space and $\xi \in \mathcal{Z}^*$. A self mapping T on X is called $\mathcal{Z}$-contraction with respect to ξ, whenever the inequality

$$\xi(d(Tx,Ty),d(x,y)) \geq 0 \quad \text{for all } x,y \in X$$

is satisfied.

Theorem 1.9. *[18] Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ξ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence x_n, where $x_n = Tx_{n+1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

On the other hand, best proximity point theory analyze the existence of an approximate solution that is optimal. Let A be a non-empty subset of a metric space (X,d) and $f : A \to X$ has a fixed point in A if the fixed point equation $fx = x$ has at least one solution. If the fixed point equation $fx = x$ does not possess a solution, then $d(x,fx) > 0$ for all $x \in A$. In that case, we aim to find an element $x \in A$ such that $d(x,fx)$ is minimum as much as possible.

Let A and B be two non-empty subsets of a metric space (X,d) and $T : A \to B$ is a non-self mapping, then $d(x,Tx) \geq d(A,B)$ for all $x \in A$. In general for a non- self mapping $T : A \to B$, the fixed point equation $Tx = x$ may not have a solution. In this case, it is focused on the possibility of finding an element x that is in closed proximity to Tx in some sense, i.e., to find an approximate solution $x \in A$ such that the error $d(x,Tx)$ is minimum, possibly $d(x,Tx) = d(A,B)$. A point $x \in A$ is called best proximity point of $T : A \to B$ if $d(x,Tx) = d(A,B)$, where $d(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}$.

The purpose of this paper is to define the notion of modified α-type $\mathcal{Z}$-contraction and to prove the existence of best proximity point results in the frame work of complete b-metric spaces. Moreover we obtain best proximity point results in b-metric spaces endowed with binary relation through our main results. As an application we obtain some fixed point and coupled fixed point results for such contractions in b-complete b-metric and b-metric spaces endowed with binary relation. Examples are given to prove the validity of our results. Moreover, we show the existence of solution of nonlinear integral and matrix equations.

In the sequel, (X,d) a b-metric space and $x \in X$, define

$$\begin{aligned} A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a,b) = d(A,B)\} \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d(a,b) = d(A,B)\} \\ d(x,A) &= \inf\{d(x,a) : a \in A\}. \end{aligned}$$

Definition 1.10. [28] Let (A,B) be a pair of nonempty subsets of a b-metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1,x_2 \in A$ and $y_1,y_2 \in B$,

$$\left.\begin{aligned} d(x_1,y_1) &= d(A,B) \\ d(x_2,y_2) &= d(A,B) \end{aligned}\right\} \text{ implies } d(x_1,x_2) = d(y_1,y_2).$$

Definition 1.11. [17] Let $T : A \to B$ and $\alpha : A \times A \to [0,\infty)$. We say that T is α-proximal admissible if

$$\left.\begin{aligned} \alpha(x_1,x_2) &\geq 1 \\ d(u_1,Tx_1) &= \mathrm{d}(A,B) \\ d(u_2,Tx_2) &= \mathrm{d}(A,B) \end{aligned}\right\} \text{ implies } \alpha(u_1,u_2) \geq 1,$$

for all $x_1,x_2,u_1,u_2 \in A$.

Definition 1.12. [31] We denote by Ψ the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

1. ψ is continuous and
2. $\psi(t) = 0$ if and only if $t = 0$.

2. Best Proximity Point Results

We begin this section with the following definition:

Definition 2.1. Let (X, d) be a b-metric space and A, B be two nonempty subsets of X. Let $\alpha : A \times A \to \mathbb{R}^+$ be a function. Suppose that $T : A \to B$ be a mapping. Then T is called modified α-type $\mathcal{Z}$-contraction with respect to ξ if there are $\xi \in \mathcal{Z}^*$, such that

$$\xi(\alpha(x,y)d(Tx,Ty), \lambda M(x,y)) \geq 0 \tag{2.1}$$

for each $x, y \in A$, $\lambda \in (0,1)$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Example 2.2. Let $X = \mathbb{R}$ with b-metric $d(x,y) = |x-y|^2$ for all $x, y \in X$ with $k = 2$. Let $A = [0,1]$ and $B = [2,3]$, then $d(A,B) = 1$. Define $T : A \to B$ by

$$Tx = 2 + \frac{x^2}{2} \text{ for all } x \in A,$$

$\alpha : A \times A \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

and $\xi(x,y) = y - \frac{2+x}{1+x}x$ for all $x, y \geq 0$. Now for $x, y \in A$, let $x = \frac{1}{2}$ and $y = \frac{1}{6}$ then $Tx = 2 + \frac{1}{8}$, $Ty = 2 + \frac{1}{72}$ and $d(Tx,Ty) = |\frac{1}{8} - \frac{1}{72}|^2 = \frac{1}{81}$.

$$\begin{aligned} M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\} \\ &= \max\{|\frac{1}{2} - \frac{1}{6}|^2, |\frac{1}{2} - 2 - \frac{1}{8}|^2, |\frac{1}{6} - 2 - \frac{1}{72}|^2\} \\ &= \max\{0.027777, 2.640625, 3.412229\} \\ &= 3.412229. \end{aligned}$$

Then for $\lambda = \frac{1}{2}$

$$\begin{aligned} \xi(\alpha(x,y)d(Tx,Ty), \lambda M(x,y)) &= \frac{1}{2}M(x,y) - \frac{2 + \alpha(x,y)d(Tx,Ty)}{1 + \alpha(x,y)d(Tx,Ty)}\alpha(x,y)d(Tx,Ty) \\ &= \frac{1}{2}3.412229 - \frac{2 + \frac{1}{81}}{1 + \frac{1}{81}} \times \frac{1}{81} \\ &= 1.7061145 - \frac{163}{82} \times \frac{1}{81} \\ &= 1.7061145 - 0.0245408 > 0. \end{aligned}$$

This implies that T is modified α-type $\mathcal{Z}$-contraction.

We now present our first main result:

Theorem 2.3. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$. Suppose that $T : A \to B$ a non-self mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 1;$$

(iv) *T is continuous modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B).$$

Proof. From assumption (iii), there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Since T is α-proximal admissible, we have $\alpha(x_1, x_2) \geq 1$. Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B)$$

and since T is α-proximal admissible, we get $\alpha(x_2, x_3) \geq 1$. Continuing this process, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \alpha(x_n, x_{n+1}) \geq 1, \tag{2.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since T is modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$, for all $n \in \mathbb{N}$, we have

$$\xi(\alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), \lambda M(x_{n-1}, x_n)) \geq 0, \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

So (2.3) gives

$$\xi(\alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \geq 0. \tag{2.4}$$

By P-property, (2.4) become

$$\xi(\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \geq 0. \tag{2.5}$$

If there is some $n_0 \in \mathbb{N} \cup \{0\}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then from (2.2), we get that $d(x_{n_0}, Tx_{n_0}) = d(A, B)$, that is, x_{n_0} is a best proximity point and so the proof is complete. Suppose now that $d(x_n, x_{n+1}) > 0$, for all $n = 0, 1, 2, \cdots$. Therefore, from (2.5) and (ξ_1), we have

$$0 < \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} - \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}).$$

Consequently, we derive that

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(x_n, x_{n+1}) < \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

for all $n \geq 1$. *Necessarily, we have*

$$\lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = \lambda d(x_{n-1}, x_n), \tag{2.6}$$

for all $n \geq 1$. *Consequently, we obtain*

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n), \quad \textit{for all } n \geq 1. \tag{2.7}$$

Hence according to Lemma 2.2 of [23], we get that $\{x_n\}$ *is a Cauchy sequence in* (X, d). *Since* (X, d) *is b-complete and* A *is closed, there exists* $x^* \in A$ *such that* $x_n \to x^*$ *as* $n \to \infty$. *On the other hand continuity of* T *implies* $Tx_n \to Tx^*$ *as* $n \to \infty$ *and from (2.2), we have*

$$d(A, B) = d(x^*, Tx^*)$$

as $n \to \infty$. *This complete the proof.* □

Example 2.4. Let $X = [0, \infty) \times [0, \infty)$ with b-metric $d(x, y) = |x_1 - x_2|^2 + |y_1 - y_2|^2$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $k = 2$. Suppose $A = \{(\frac{1}{2}, a) : 0 \leq a < \infty\}$ and $B = \{(0, a) : 0 \leq a < \infty\}$, then $d(A, B) = \frac{1}{2}$. Define $T : A \to B$ by

$$T\left(\frac{1}{2}, a\right) = \begin{cases} \left(0, \frac{a}{2}\right) & if \;\; a \leq 1 \\ \left(0, a^2\right) & if \;\; a > 1, \end{cases}$$

$\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if \;\; x, y \in \left\{(\frac{1}{2}, a) : 0 \leq a \leq 1\right\} \\ 0 & \text{otherwise,} \end{cases}$$

and $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\xi(t, s) = \frac{1}{2}s - t$. Notice that $A_0 = A$, $B_0 = B$ and $T(A_0) \subseteq B_0$. Also the pair (A, B) satisfies P−property. Let $u_1, u_2 \in \{(\frac{1}{2}, a) : 0 \leq a \leq 1\}$, then $Tu_1, Tu_1 \in \{(0, \frac{a}{2}) : 0 \leq a \leq 1\}$. Consider $v_1, v_2 \in A$ such that $d(v_1, Tu_1) = d(A, B)$ and $d(v_2, Tu_2) = d(A, B)$. Then we have $v_1, v_2 \in \{(\frac{1}{2}, a) : 0 \leq a \leq 1\}$. Hence T is α-proximal admissible mapping. For $x_0 = (\frac{1}{2}, 1) \in A_0$ and $(0, \frac{1}{2}) = Tx_0$ in B_0, we have $x_1 = (\frac{1}{2}, \frac{1}{2}) \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) = 1$. If $x = (\frac{1}{2}, \frac{1}{2}), y = (\frac{1}{2}, \frac{1}{3}) \in A_0$, then we have $Tx = (0, \frac{1}{4})$ and $Ty = (0, \frac{1}{6})$. Now for $\lambda = \frac{1}{2}$ we have

$$0 \leq \xi(\alpha(x, y) d(Tx, Ty), \frac{1}{2}M(x, y)) = \frac{1}{2}M(x, y) - \alpha(x, y) d(Tx, Ty) \tag{2.8}$$

where

$$\begin{aligned} M(x, y) &= \max\{|\frac{1}{2} - \frac{1}{2}|^2 + |\frac{1}{2} - \frac{1}{3}|^2, (|\frac{1}{2} - 0|^2 + |\frac{1}{2} - \frac{1}{4}|^2), (|\frac{1}{2} - 0|^2 + |\frac{1}{3} - \frac{1}{6}|^2)\} \\ &= \frac{5}{16}. \end{aligned}$$

So (2.8) become

$$\begin{aligned} \xi(\alpha(x, y) d(Tx, Ty), \frac{1}{2}M(x, y)) &= \frac{1}{2} \times \frac{5}{16} - (1)|\frac{1}{4} - \frac{1}{6}|^2 \\ &= \frac{43}{288} > 0. \end{aligned}$$

Hence T is modified α-type $\mathcal{Z}$-contraction. Therefore, all the conditions of Theorem 2.3 holds and T has a best proximity point $(\frac{1}{2}, 1)$.

If we remove the condition of continuity on T in Theorem 2.3 and replace it with condition $\mathcal{H}$, then we have the following best proximity point result:

$(\mathcal{H})$: If $\{x_n\}$ is a sequence in A converges to $x \in A$ such that $\alpha(x_n, x_{n+1}) \geq 1$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\alpha(x_{n_k}, x) \geq 1$ for all k.

Theorem 2.5. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and $T : A \to B$ a non-self mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ with $d(x_1, Tx_0) = d(A, B)$, $\alpha(x_0, x_1) \geq 1$;*

(iv) *T is modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$;*

(v) *$(\mathcal{H})$ holds.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. *Following the proof of Theorem 2.3, we have a Cauchy sequence $\{x_n\}$ in A such that $x_n \to x^* \in A$ as $n \to \infty$ and $\alpha(x_n, x_n + 1) \geq 1$ for all $n \in \mathbb{N}$. Since $(\mathcal{H})$ holds, we have that*

$$\alpha(x_{n_k}, x^*) \geq 1 \quad \text{for all } \ k \in \mathbb{N}. \tag{2.9}$$

Regarding (i), we note that $Tx^* \in B_0$ and hence

$$d(y_1, Tx^*) = d(A, B), \text{for some } y_1 \in A_0.$$

Since (A, B) satisfies P-property, and

$$d(y_1, Tx^*) = d(x_{n_{k+1}}, Tx_{n_k}) = d(A, B) \tag{2.10}$$

we get that

$$d(x_{n_{k+1}}, y_1) = d(Tx_{n_k}, Tx^*). \tag{2.11}$$

Therefore, by condition (iv), we get

$$\begin{aligned} 0 &\leq \xi(\alpha(x_{n_k}, x^*)d(Tx_{n_k}, Tx^*), \lambda M(x_{n_k}, x^*) \\ &< \lambda M(x_{n_k}, x^*) - \alpha(x_{n_k}, x^*)d(Tx_{n_k}, Tx^*) \\ &= \lambda M(x_{n_k}, x^*) - \alpha(x_{n_k}, x^*)d(x_{n_{k+1}}, y_1), \end{aligned} \tag{2.12}$$

where

$$M(x_{n_k}, x^*) = \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*)\}.$$

This implies

$$d(x_{n_{k+1}}, y_1) \leq \alpha(x_{n_k}, x^*)d(x_{n_{k+1}}, y_1) < \lambda M(x_{n_k}, x^*) = d(x_{n_k}, x^*)$$

and so

$$\lim_{k\to\infty} d(x_{n_{k+1}}, y_1) \to 0.$$

Thus $y_1 = x^*$ and (2.10) gives

$$d(x^*, Tx^*) = d(A, B).$$

This complete the proof. □

Now, we prove the uniqueness of best proximity point. For this, we need the following additional condition:

$$(\mathcal{U}) : \alpha(x, y) \geq 1 \text{ for all best proximity points } x, y \text{ of } T.$$

Theorem 2.6. *Adding condition ($\mathcal{U}$) to the hypothesis of Theorem 2.3 (2.5), we obtain that x^* is the unique best proximity point of T.*

Proof. *We argue with contradiction, that there exist $x^*, y^* \in A_0$ such that*

$$\begin{aligned} d(x^*, Tx^*) &= d(A, B) \\ d(y^*, Ty^*) &= d(A, B) \end{aligned}$$

with $x^* \neq y^*$. Since pair (A, B) satisfies P-property, then

$$d(x^*, y^*) = d(Tx^*, Ty^*).$$

By assumption ($\mathcal{U}$), we have $\alpha(x^*, y^*) \geq 1$. So, by (ξ_1), we get

$$\begin{aligned} 0 &\leq \xi(\alpha(x^*, y^*)d(Tx^*, Ty^*), \lambda M(x^*, y^*)) \\ &= \xi(\alpha(x^*, y^*)d(x^*, y^*), \lambda M(x^*, y^*)) \\ &< \lambda M(x^*, y^*) - \alpha(x^*, y^*)d(x^*, y^*). \end{aligned}$$

Consequently, we derive that

$$\begin{aligned} d(x^*, y^*) &\leq \alpha(x^*, y^*)d(x^*, y^*) \\ &< \lambda M(x^*, y^*) \\ &= \lambda \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*)\} \\ &= \lambda d(x^*, y^*), \end{aligned}$$

a contradiction. Hence, $x^* = y^*$. □

Corollary 2.7. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$. Suppose that $T : A \to B$ is s non-self continuous mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 1;$$

(iv) *for all $x, y \in A$ and $\psi \in \Psi$, $\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$, where*

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B).$$

Proof. *Define $\xi_A(s, t) = \psi(t) - s$ for each $s, t \in [0, \infty)$. It is clear that the mapping T is modified α-type $\mathcal{Z}$-contraction with respect to $\xi_A \in \mathcal{Z}^*$. Therefore by taking $\xi_A = \xi$ in Theorem 2.3 (respectively in, 2.5, 2.6), we have the required result.* □

3. Best Proximity Point Results in *b*-Metric Space Endowed with an Arbitrary Binary Relation

Let (X, d) be a b-metric space and $\mathcal{R}$ be a binary relation on X. Denote

$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1};$$

this is a symmetric relation attached to $\mathcal{R}$. Clearly,

$$x, y \in X, \quad x\mathcal{S}y \text{ if and only if } x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 3.1. [17] We say that $T : A \to B$ is proximal comparative mapping if

$$\left.\begin{array}{rcl} & x_1\mathcal{S}x_2 & \\ d(u_1, Tx_1) & = & \mathrm{d}(A, B) \\ d(u_2, Tx_2) & = & \mathrm{d}(A, B) \end{array}\right\} \text{ implies } \ u_1\mathcal{S}u_2,$$

for all $x_1, x_2, u_1, u_2 \in A$.

We now prove our second new result.

Theorem 3.2. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) with A_0 is nonempty. Let $\mathcal{R}$ be a binary relation on X. Suppose that $T : A \to B$ is a non-self continuous mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0\mathcal{S}x_1;$$

(iv) *for some $x, y \in A : x\mathcal{S}y$ implies $\xi(d(Tx, Ty), \lambda M(x, y)) \geq 0$.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. *Define a mapping $\alpha : A \times A \to [0, \infty)$ by*

$$\alpha(x, y) = \begin{cases} 1 & x\mathcal{S}y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Suppose that

$$\begin{cases} \alpha(x_1, x_2) \geq 1, & \\ d(u_1, Tx_1) = d(A, B), & \\ d(u_2, Tx_2) = d(A, B). & \end{cases}$$

For some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{cases} x_1\mathcal{S}x_2, & \\ d(u_1, Tx_1) = d(A, B), & \\ d(u_2, Tx_2) = d(A, B). & \end{cases}$$

By (ii) we have $u_1\mathcal{S}u_2$, and by the definition of α, we get $\alpha(u_1, u_2) \geq 1$ and so T is α-proximal admissible. Condition (iii) implies that

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1.$$

From (iv), we have

$$\begin{aligned} 0 &\leq \xi(d(Tx,Ty),\lambda M(x,y)) \\ &< \lambda M(x,y) - d(Tx,Ty) \end{aligned}$$

this implies

$$\begin{aligned} 0 \leq d(Tx,Ty) &\leq \alpha(x,y)d(Tx,Ty) \leq \lambda M(x,y) \\ &\leq \lambda M(x,y) - \alpha(x,y)d(Tx,Ty) \\ &\leq \xi(\alpha(x,y)d(Tx,Ty),\lambda M(x,y)) \end{aligned}$$

for all $x, y \in A$, that is, T is modified α-type $\mathcal{Z}$-contraction. All the conditions of Theorem 2.3 are satisfied, hence the result follows. □

Condition of continuity can be replaced with the one:

($\mathcal{P}$): If the sequence $\{x_n\}$ in A and the point $x \in A$ are such that $x_n \mathcal{S} x_{n+1}$ for all n and $\lim_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \mathcal{S} x$ for all k.

Theorem 3.3. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) with A_0 is nonempty. Let $\mathcal{R}$ be a binary relation on X. Suppose that $T : A \to B$ is a non-self mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \;\; x_0 \mathcal{S} x_1;$$

(iv) *for some $x, y \in A$, $x\mathcal{S}y$ implies $\xi(d(Tx,Ty),\lambda M(x,y)) \geq 0$;*

(v) *($\mathcal{P}$) holds.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. *The result follows from Theorem 2.5 by considering the mapping α given by* (3.1) *and by observing that condition ($\mathcal{P}$) implies condition ($\mathcal{R}$).* □

3.1. Coupled best proximity point results in b-metric space endowed with an arbitrary binary relation

We now apply the results obtained in previous section to prove the existence of coupled best proximity points.

Definition 3.4. [34] Let A and B be two subsets of a b-metric space (X, d). An element $(x^*, y^*) \in A \times A$ is called coupled best proximity point of the mapping $F : A \times A \to B$ if $d(x^*, F(x^*, y^*)) = d(A, B)$ and $d(y^*, F(y^*, x^*)) = d(A, B)$.

We need the following notations:

$$\mathcal{X} := X \times X, \quad \mathcal{A} := A \times A, \quad \mathcal{B} := B \times B.$$

Define the non-self mapping $T : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ by

$$T(x, y) = (F(x, y), F(y, x)), \quad \text{for all } (x, y) \in \mathcal{A}.$$

We endow the product set $\mathcal{X}$ with the b-metric d_1 given by

$$d_1((x, y), (u, v)) = d(x, u) + d(y, v), \quad \text{for all } (x, y), (u, v) \in \mathcal{X}.$$

Clearly, if (X, d) is complete, then $(\mathcal{X}, d_1)$ is complete.

Definition 3.5. [17] We say that $F : A \times A \to B$ is bi-proximal comparative mapping if

$$\left.\begin{array}{lcl} x_1\mathcal{S}x_2, y_1\mathcal{S}y_2 & & \\ d(u_1, F(x_1, y_1)) & = & \mathrm{d}(A, B) \\ d(u_2, F(x_2, y_2)) & = & \mathrm{d}(A, B) \end{array}\right\} \quad \text{implies} \quad u_1\mathcal{S}u_2,$$

for all $x_1, x_2, y_1, y_2, u_1, u_2 \in A$.

We now prove the following coupled best proximity point result:

Theorem 3.6. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) with A_0 is nonempty. Let $\mathcal{R}$ be a binary relation on X. Suppose that $F : A \times A \to B$ a continuous mapping satisfying the following conditions:*

(i) *$F(A_0 \times A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *F is bi-proximal comparative mappings;*

(iii) *there exist elements x_0, x_1, y_0, y_1 in A_0 such that*

$$d(x_1, F(x_0, y_0)) = d(y_1, F(y_0, x_0)) = d(A, B) \text{ and } x_0\mathcal{S}x_1, \ \ y_0\mathcal{S}y_1;$$

(iv) *there exist $\xi \in \mathcal{Z}^*$ such that $x, y, u, v \in A, x\mathcal{S}u, \ y\mathcal{S}v$ implies*

$$\xi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)), \lambda M(x, y, u, v)) \geq 0,$$

where

$$M(x, y, u, v) = \max\{d(x, u) + d(y, v), d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\}.$$

Then there exist $x^*, y^* \in A_0$ such that

$$d(x^*, F(x^*, y^*)) = d(y^*, F(y^*, x^*)) = d(A, B).$$

Proof. *Define the binary relation $\mathcal{R}_1$ over $\mathcal{X}$ by*

$$(x, y), (u, v) \in \mathcal{X}, \ \ (x, y)\mathcal{R}_1(u, v) \Leftrightarrow x\mathcal{S}u, y\mathcal{S}v.$$

If we denote by $\mathcal{S}_1$ the symmetric relation attached to $\mathcal{R}_1$, clearly, we have $\mathcal{S}_1 = \mathcal{R}_1$.
To complete the proof we have to show that $T : \mathcal{A} \to \mathcal{B}$ has a best proximity point $z^* = (x^*, y^*) \in A_0 \times A_0$, that is ,

$$d_1(z^*, Tz^*) = d_1(\mathcal{A}, \mathcal{B}).$$

Since F is continuous, it follows that T is continuous.
Now define $\mathcal{A}_0$ and $\mathcal{B}_0$ by

$$\begin{array}{lcl} \mathcal{A}_0 & = & \{(a_1, a_2) \in \mathcal{A} : d_1((a_1, a_2), (b_1, b_2)) = d_1(\mathcal{A}, \mathcal{B}) \ \ for\ some\ (b_1, b_2) \in \mathcal{B}\} \\ \mathcal{B}_0 & = & \{(b_1, b_2) \in \mathcal{B} : d_1((a_1, a_2), (b_1, b_2)) = d_1(\mathcal{A}, \mathcal{B}) \ \ for\ some\ (a_1, a_2) \in \mathcal{A}\}. \end{array}$$

We can observe that

$$d_1(\mathcal{A}, \mathcal{B}) = 2d(A, B).$$

In fact, we have

$$\begin{array}{lcl} d_1(\mathcal{A}, \mathcal{B}) & = & \inf\{d_1((a_1, a_2), (b_1, b_2)) : (a_1, a_2) \in \mathcal{A}, (b_1, b_2) \in \mathcal{B}\} \\ & = & \inf\{d(a_1, b_1) + d(a_2, b_2) : (a_1, b_1) \in A \times B, (a_2, b_2) \in A \times B\} \\ & = & \inf\{d(a_1, b_1) : (a_1, b_1) \in A \times B\} + \inf\{d(a_2, b_2) : (a_2, b_2) \in A \times B\} \\ & = & d(A, B) + d(A, B) \\ & = & 2d(A, B). \end{array}$$

Now, let $(a_1, a_2) \in \mathcal{A}_0$. Then, there exists $(b_1, b_2) \in \mathcal{B}$ such that

$$d_1((a_1, a_2), (b_1, b_2)) = d_1(\mathcal{A}, \mathcal{B}),$$

that is,

$$d(a_1, b_1) + d(a_2, b_2) = 2d(A, B).$$

Thus we have

$$\begin{cases} d(a_1, b_1) + d(a_2, b_2) = 2d(A, B), \\ \qquad d(a_1, b_1) \geq d(A, B), \\ \qquad d(a_2, b_2) \geq d(A, B), \end{cases}$$

which implies that

$$d(a_1, b_1) = d(a_2, b_2) = d(A, B).$$

Similarly, if $(a_1, a_2) \in A_0 \times A_0$, we have $(a_1, a_2) \in \mathcal{A}_0$. Thus we proved that

$$\mathcal{A}_0 = A_0 \times A_0.$$

Similarly we can show that

$$\mathcal{B}_0 = B_0 \times B_0.$$

Since A_0 is nonempty, then $\mathcal{A}_0$ is nonempty. On the other hand, from condition (i), we have

$$T(\mathcal{A}_0) = \{(F(x, y), F(y, x)) : (x, y) \in A_0 \times A_0\} \subset F(A_0 \times A_0) \subseteq \mathcal{B}_0.$$

Suppose now that for some $(a_1, a_2), (x_1, x_2) \in \mathcal{A}$ and $(b_1, b_2), (y_1, y_2) \in \mathcal{B}$, we have

$$\begin{aligned} d_1((a_1, a_2), (b_1, b_2)) &= d_1(\mathcal{A}, \mathcal{B}), \\ d_1((x_1, x_2), (y_1, y_2)) &= d_1(\mathcal{A}, \mathcal{B}). \end{aligned}$$

Since (A, B) satisfies the P-property, we get that

$$d(a_1, x_1) = d(b_1, y_1) \text{ and } d(a_2, x_2) = d(b_2, y_2),$$

which implies that

$$d_1((a_1, a_2), (x_1, x_2)) = d_1((b_1, b_2), (y_1, y_2)).$$

Thus we have proved that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the P-property.

Suppose that for some $(a_1, a_2), (x_1, x_2), (u_1, u_2), (v_1, v_2) \in \mathcal{A}$, we have

$$\begin{aligned} &(a_1, a_2)\mathcal{S}_1(x_1, x_2), \\ d_1((u_1, u_2), T(a_1, a_2)) &= d(\mathcal{A}, \mathcal{B}), \\ d_1((v_1, v_2), T(x_1, x_2)) &= d(\mathcal{A}, \mathcal{B}). \end{aligned}$$

This implies that

$$\begin{aligned} &a_1\mathcal{S}x_1, a_2\mathcal{S}x_2 \\ d(u_1, F(a_1, a_2)) &= d(A, B), \\ d(v_1, F(x_1, x_2)) &= d(A, B), \end{aligned}$$

and

$$\begin{aligned} &a_2\mathcal{S}x_2, a_1\mathcal{S}x_1 \\ d(u_2, F(a_2, a_1)) &= d(A, B), \\ d(v_2, F(x_2, x_1)) &= d(A, B). \end{aligned}$$

Since F is bi-proximal comparative mapping, we get that
$u_1\mathcal{S}v_1$ *and* $u_2\mathcal{S}v_2$,
that is, $(u_1, u_2)\mathcal{S}_1(v_1, v_2)$. Thus we proved that T is proximal comparative mapping. Now, from condition (iii), we have

$$d(x_1, F(x_0, y_0)) + d(y_1, F(y_0, x_0)) = 2d(A, B) \text{ and } (x_0, y_0)\mathcal{S}_1(x_1, y_1),$$

which implies that

$$d_1((x_1, y_1), T(x_0, y_0)) = d_1(\mathcal{A}, \mathcal{B}) \text{ and } (x_0, y_0)\mathcal{S}_1(x_1, y_1).$$

Now let $p = (x, y), q = (u, v) \in \mathcal{A}_0$ such that $p\mathcal{S}_1 q$, that is $x\mathcal{S}u$ and $y\mathcal{S}v$, then

$$\begin{aligned} d_1(p, q) &= d(x, u) + d(y, v), \\ d_1(Tp, Tq) &= d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)), \\ d_1(p, Tp) &= d(x, F(x, y)) + d(y, F(y, x)), \\ d_1(q, Tq) &= d(u, F(u, v)) + d(v, F(v, u)). \end{aligned}$$

So, condition (iv) is reduced to

$$\xi(d_1(Tp, Tq), \lambda M(p, q)) \geq 0, \text{ where}$$

$$M(p, q) = \max\{d_1(p, q), d_1(p, Tp), d_1(q, Tq)\}.$$

Hence T satisfies all the hypothesis of Theorem 3.2 and thus T has a best proximity point in $\mathcal{A}_0$, that is, there exist an element $z^* = (x^*, y^*) \in \mathcal{A}_0$, such that

$$d_1(z^*, Tz^*) = d_1(\mathcal{A}, \mathcal{B});$$

which implies that,

$$d_1((x^*, y^*), T(x^*, y^*)) = d_1(A \times A, B \times B).$$

That is

$$d(x^*, F(x^*, y^*) + d(y^*, F(y^*, x^*) = d(A, B) + d(A, B)$$

from which we have

$$d(x^*, F(x^*, y^*) = d(A, B) \text{ and } d(y^*, F(y^*, x^*) = d(A, B).$$

Therefore $(x^*, y^*) \in A_0 \times A_0$ is coupled best proximity point of F. □

Similarly, from Theorem 3.3, we get the following result:

Theorem 3.7. *Let A and B be nonempty closed subsets of a b-complete b-metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation on X. Suppose that $F : A \times A \to B$ is a mapping satisfying the following conditions:*

(i) *$F(A_0 \times A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *F is a bi-proximal comparative mappings;*

(iii) *there exist elements x_0, x_1, y_0, y_1 in A_0 such that*

$$d(x_1, F(x_0, y_0)) = d(y_1, F(y_0, x_0)) = d(A, B) \text{ and } x_0\mathcal{S}x_1, \quad y_0\mathcal{S}y_1;$$

there exist $\xi \in \mathcal{Z}^*$ such that $x, y, u, v \in A, x\mathcal{S}u, y\mathcal{S}v$ implies

$$\xi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)), \lambda M(x, y, u, v)) \geq 0,$$

where

$$M(x, y, u, v) = \max\{d(x, u) + d(y, v), d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\}.$$

(iv) *$(\mathcal{P})$ holds.*

Then there exist $x^*, y^* \in A_0$ such that

$$d(x^*, F(x^*, y^*)) = d(y^*, F(y^*, x^*)) = d(A, B).$$

4. Some Fixed Point Results

Taking $A = B = X$ in Theorem 2.3 (respectively in Theorem 2.5, 2.6), we obtain the following fixed point results:

Theorem 4.1. *Let (X, d) be a b-complete b-metric space and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ a self mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *T is continuous modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$.*

Then T has a fixed point.

Theorem 4.2. *Let (X, d) be a b-complete b-metric space and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ is a self mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *T is continuous modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$;*

(iii) *$(\mathcal{H})$ holds.*

Then T has a fixed point.

Theorem 4.3. *Let (X, d) be a b-complete b-metric space and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ is a self mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *T is continuous modified α-type $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}^*$;*

(iii) *$(\mathcal{U})$ holds.*

Then T has a unique fixed point.

4.1. Fixed point results in b-metric space endowed with an arbitrary binary relation

Taking $A = B = X$ in Theorem 3.2 (respectively in Theorem 3.3, 3.6), we obtain the following fixed point results:

Definition 4.4. [30] A mapping $T : X \to X$ is called comparative mapping if it maps comparable elements into comparable elements, that is,

$$x, y \in X, \quad x\mathcal{S}y \text{ implies } Tx\mathcal{S}Ty.$$

Theorem 4.5. *Assume that $T : X \to X$ is a continuous comparative map, and*

$$x, y \in X, \quad x\mathcal{S}y \text{ implies } \xi(d(Tx, Ty), \lambda M(x, y)) \geq 0,$$

where $\xi \in \mathcal{Z}^*$. Suppose that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. Then T has a fixed point.

Theorem 4.6. *Assume that $T : X \to X$ is a comparative map satisfying*

$$x, y \in X, \quad x\mathcal{S}y \text{ implies } \xi(d(Tx, Ty), \lambda M(x, y)) \geq 0,$$

for some $\xi \in \mathcal{Z}^*$. Suppose that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$ and $(\mathcal{P})$ holds. Then T has a fixed point.

Theorem 4.7. *In addition to the hypothesis of Theorem 4.5 (resp. Theorem 4.6), suppose that for all $(x, y) \in X \times X$ with $(x, y) \notin \mathcal{S}$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then T has a unique fixed point.*

4.2. Coupled fixed point results in b-metric space endowed with an arbitrary binary relation

We continue to use same notation as in Section 4.1. We need the following definitions:

Definition 4.8. [17] Let $F : X \times X \to X$ be a given mapping. We say that F is a bi-comparative mapping if

$(x_1, y_1), (x_2, y_2) \in X \times X, x_1 \mathcal{S} x_2, y_1 \mathcal{S} y_2$ implies $F(x_1, y_1) \mathcal{S} F(x_2, y_2)$.

Definition 4.9. [34] Let $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$x = F(x, y) \text{ and } y = F(y, x).$$

We now have the following coupled fixed point results:

Theorem 4.10. *Suppose that $F : X \times X \to X$ is a continuous mapping satisfying the following conditions:*

(i) *F is a bi-comparative mapping;*

(ii) *there exist elements x_0, y_0 in X such that*

$$x_0 \mathcal{S} F(x_0, y_0), \quad y_0 \mathcal{S} F(y_0, x_0);$$

(iii) *there exist $\xi \in \mathcal{Z}^*$ and $\lambda \in (0, 1)$ such that $x, y, u, v \in X, x\mathcal{S}u, y\mathcal{S}v$ implies*

$$\xi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)), \lambda M(x, y, u, v)) \geq 0,$$

where

$$M(x, y, u, v) = \max\{d(x, u) + d(y, v), d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\}.$$

Then F has a coupled fixed point.

Proof. *It follows immediately from Theorem 3.6 by taking $A = B = X$ and that a bi-proximal comparative mapping is a bi-comparative mapping.* □

Theorem 4.11. *Suppose that $F : X \times X \to X$ is a mapping satisfying the following conditions:*

(i) *F is a bi-comparative mapping;*

(ii) *there exist elements x_0, y_0 in X such that*

$$x_0 \mathcal{S} F(x_0, y_0), \quad y_0 \mathcal{S} F(y_0, x_0);$$

(iii) *there exist $\xi \in \mathcal{Z}^*$ and $\lambda \in (0, 1)$ such that $x, y, u, v \in X, x\mathcal{S}u, y\mathcal{S}v$ implies*

$$\xi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)), \lambda M(x, y, u, v)) \quad \geq \quad 0,$$

where

$$M(x, y, u, v) = \max\{d(x, u) + d(y, v), d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\};$$

(iv) *$(\mathcal{P})$ holds.*

Then F has a coupled fixed point.

Theorem 4.12. *In addition to the hypothesis of Theorem 4.10 (resp. Theorem 4.11), suppose that for all $(x, y) \in X \times X$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then T has a unique coupled fixed point $(x^*, y^*) \in X \times X$. Moreover, we have $x^* = y^*$.*

5. Application to Integral Equations

We now apply Theorem 4.1 to prove the existence of solution to the nonlinear integral equations.

Theorem 5.1. *Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, the b-metric d with $k = 2^{p-1}$ defined by $d(u,v) = \sup_{t\in[a,b]} |u(t) - v(t)|^p$ for all $u, v \in C[a,b]$ and some $p > 1$. Consider the nonlinear integral equation*

$$u(t) = g(t) + \int_a^b K(t,x,u(x))dx,$$

where $t \in [a,b]$, $g : [a,b] \to \mathbb{R}$, $K : [a,b] \times [a,b] \times u[a,b] \to \mathbb{R}$ for each $u \in C[a,b]$.
Suppose that the following hold:

(i) *g is continuous on $[a,b]$ and $K(t,x,u(x))$ is integrable with respect to x on $[a,b]$;*

(ii) *$Tu \in C[a,b]$ for all $u \in [a,b]$, where $Tu(t) = g(t) + \int_a^b K(t,x,u(x))dx$ for all $t \in [a,b]$;*

(iii) *for all $u \in C[a,b]$ and $u(x) \geq 0$ for all $x \in [a,b]$, we have $Tu(x) \geq 0$ for all $x \in [a,b]$;*

(iv) *For all $x, t \in [a,b]$ and $u, v \in C[a,b]$ such that $u(x), v(x) \in [0,\infty)$ for all $x \in [a,b]$, we have*

$$|K(t,x,u(x)) - K(t,x,v(x))| \leq \mu(t,x) \max\left\{|u(x) - v(x)|, |u(x) - Tu(x)|, |v(x) - Tv(x)|\right\},$$

where $\mu : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function satisfying

$$\sup_{t\in[a,b]} \left(\int_a^b \mu^p(t,x)dx \right) < \frac{\lambda}{2^p(b-a)^{p-1}}, \text{ where } \lambda \in (0,1);$$

(v) *there exist $u_1 \in C[a,b]$ such that $u_1(t) \geq 0$ and $Tu_1(t) \geq 0$ for all $t \in [a,b]$.*

Then the given integral equation has a unique solution in $C[a,b]$.

Proof. *Define a mapping $T : C[a,b] \to C[a,b]$ by*

$$Tu(t) = g(t) + \int_a^b K(t,x,u(x))dx$$

for all $u \in C[a,b]$ and for all $t \in [a,b]$. It follows from hypothesis (i) and (ii) that T is well-defined. Notice that the existence of solution of given integral equation is equivalent to the existence of fixed point of T. Now, we will show that all hypothesis of Theorem 4.1 are satisfied.

Define a mapping $\alpha : C[a,b] \times C[a,b] \to \mathbb{R}$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } u(x), v(x) \in [0,\infty) \text{ for all } x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that T is α-proximal admissible mapping. Indeed, for $u, v \in C[a,b]$ such that $\alpha(u,v) \geq 1$, we have $u(x), v(x) \geq 0$ for all $x \in [a,b]$. It follows from condition (iii) that $Tu(x), Tv(x) \geq 0$. Therefore $\alpha(Tu(x), Tv(x)) \geq 1$ and hence T is α-proximal admissible mapping.

We claim that T is α-type modified $\mathcal{Z}$-contraction mapping. That is, there exist $\xi \in \mathcal{Z}^*$ such that

$$\xi(\alpha(x,y)d(Tx,Ty), \lambda M(x,y)) \geq 0,$$

for each $x, y \in C[a,b]$, where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Let $\xi(s,t) = t - 2s$. Indeed, let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. From condition (4), for all $u, v \in C[a,b]$ such that $u(x), v(x) \in [0,\infty)$ for all $x \in [a,b]$, we have

$$\begin{aligned}
& 2^p \alpha(u,v)|Tu(x) - Tv(x)|^p \\
= \;& 2^p |Tu(x) - Tv(x)|^p \\
\leq \;& 2^p \left| \int_a^b K(t,x,u(x))dx - \int_a^b K(t,x,v(x))dx \right|^p \\
\leq \;& 2^p \left| \int_a^b (K(t,x,u(x)) - K(t,x,v(x)))dx \right|^p \\
\leq \;& 2 \times 2^{p-1} \left(\int_a^b |K(t,x,u(x)) - K(t,x,v(x))| dx \right)^p \\
\leq \;& 2 \times \left[2^{p-1} \left(\int_a^b dx \right)^{\frac{1}{q}} \left(\int_a^b |K(t,x,u(x)) - K(t,x,v(x))|^p dx \right)^{\frac{1}{p}} \right]^p \\
\leq \;& 2^{p-1}(b-a)^{p-1} \left(\int_a^b \mu^p(t,x)dx \right) \left(\max\left\{ |u(x) - v(x)|^p, |u(x) - Tu(x)|^p, |v(x) - Tv(x)|^p \right\} \right) \\
\leq \;& 2^{p-1}(b-a)^{p-1} \left(\int_a^b \mu^p(t,x)dx \right) \left(\max\left\{ \sup_{x\in[a,b]} |u(x) - v(x)|^p, \sup_{x\in[a,b]} |u(x) - Tu(x)|^p, \sup_{x\in[a,b]} |v(x) - Tv(x)|^p \right\} \right) \\
\leq \;& 2^{p-1}(b-a)^{p-1} \left(\int_a^b \mu^p(t,x)dx \right) (\max\{d(x,y), d(x,Tx), d(y,Ty)\}) \\
= \;& 2^p (b-a)^{p-1} \left(\int_a^b \mu^p(t,x)dx \right) M(u,v) \\
\leq \;& 2^p (b-a)^{p-1} \sup_{t\in[a,b]} \left(\int_a^b \mu^p(t,x)dx \right) M(u,v) \\
< \;& \lambda M(u,v),
\end{aligned}$$

where $2^p(b-a)^{p-1} \sup_{t\in[a,b]}(\int_a^b \mu^p(t,x)dx) < \lambda$. This implies that

$$2^p \alpha(u,v)|Tu(x) - Tv(x)|^p \leq \lambda M(u,v).$$

Now

$$\begin{aligned}
\xi(\alpha(u,v)d(Tu(x),Tv(x)), \lambda M(x,y)) &= \lambda M(u,v) - 2\alpha(u,v)d(Tu(x),Tv(x)) \\
&= \lambda M(u,v) - 2^p \alpha(u,v)|Tu(x) - Tv(x)|^p > 0.
\end{aligned}$$

Therefore, T is α-type modified $\mathcal{Z}$-contraction mapping.

Let $\{u_n\} \subset C[a,b]$ such that $\alpha(u_n, u_{n+1}) \geq 1$ and $\lim_{n\to\infty} u_n = u \in C[a,b]$. Then $u(x), u_n(x) \in [0,\infty)$ for all $x \in [a,b]$ and $n \geq 0$. Therefore, $\alpha(u_n, u) \geq 1$ for all $n \geq 1$.

Therefore, we conclude that all the hypothesis of Theorem 4.1 are satisfied. Thus, T has a fixed point $u \in C[a,b]$ and hence given integral equation has a solution $u \in C[a,b]$. □

6. Application to Matrix Equations

In this section, an illustration of Theorem 4.10 to guarantee the existence of positive definite solution of nonlinear matrix equations is given. We shall use the following notations: Let $\mathcal{M}(n)$ be the set of all $n \times n$ complex matrices, $H(n) \subseteq \mathcal{M}(n)$ be the class of all $n \times n$ Hermitian matrices, $P(n) \subseteq H(n)$ be the set of all $n \times n$ Hermitian positive definite matrices, $H^+(n) \subseteq H(n)$ be the set of all $n \times n$ positive semidefinite matrices. Instead of $X \in P(n)$ we will write $X \succ 0$.

Furthermore, $X \succeq 0$ means $X \in H^+(n)$. Also we will use $X \succeq Y(X \preceq Y)$ instead of $X - Y \succeq 0(Y - X \succeq 0)$. Also, for every $X, Y \in H(n)$ there is a greatest lower bound and a least upper bound. The symbol $\|.\|$ denotes the spectral norm of the matrix A, that is, $\|A\| = \sqrt{\lambda^+(A^*A)}$ such that $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A where A^* is the conjugate transpose of A. We denote by $\|.\|_\tau$ the Ky Fan norm defined by $\|A\|_\tau = \sum_{i=1}^{n} s_i(A) = tr((A^*A)^{\frac{1}{2}})$, where $s_i(A)$, $i = 1, ..., n$, are the singular values of $A \in \mathcal{M}(n)$ and $tr(A)$ for (Hermitian) nonnegative matrices. For a given $Q \in P(n)$ we denote the modified norm $\|.\|_{\tau,Q}$ by $\|A\|_{\tau,Q} = \|Q^{\frac{1}{2}} A Q^{\frac{1}{2}}\|_\tau$. The set $H(n)$ equipped with the metric induced by $\|.\|_{1,Q}$ is a complete metric space for any positive definite matrix Q. Moreover, $H(n)$ is a partially ordered set with partial order $\preceq$ where $X \preceq Y \Leftrightarrow Y \preceq X$.
Denote $d(X, Y) = \|Y - X\|_{\tau,Q} = tr(Q^{\frac{1}{2}}(\gamma(Y) - \gamma(X))Q^{\frac{1}{2}})$. We consider the following class of non-linear matrix equation:

$$X = Q + \sum_{i=1}^{m} A_i^* \gamma(X) A_i - \sum_{i=1}^{m} B_i^* \gamma(X) B_i, \tag{6.1}$$

where $Q \in P(n)$, $A_i, B_i, i = 1, 2, ...m$, are arbitrary $n \times n$ matrices and a continuous mapping $\gamma : H(n) \to H(n)$ which maps $P(n)$ into $P(n)$. Matrix equations of this type often arise from many areas, such as ladder networks [2, 4], dynamic programming [24], control theory [14], etc. Assume that γ is an order-preserving (γ is order preserving if $A, B \in H(n)$ with $A \preceq B$ implies that $\gamma(A) \preceq \gamma(B)$.

Lemma 6.1. *[29] Let $A \succeq 0$ and $B \succeq 0$ be $n \times n$ matrices. Then $0 \leq tr(AB) \leq \|A\|.tr(B)$.*

Lemma 6.2. *[22] Let $A \in H(n)$ satisfy $A \prec I$; then $\|A\| < 1$.*

Theorem 6.3. *Let $\gamma : H(n) \to H(n)$ be an order-preserving mapping which maps $P(n)$ into $P(n)$ and $Q \in P(n)$. Assume that*

(i) *there exist a positive number R for which*
$\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i Q A_i^* Q^{-\frac{1}{2}} \preceq RI_n$, $\quad \sum_{i=1}^{m} Q^{-\frac{1}{2}} B_i Q B_i^* Q^{-\frac{1}{2}} \preceq RI_n$,

(ii) $\sum_{i=1}^{m} A_i^* \gamma(Q) A_i \succ 0$, $\quad \sum_{i=1}^{m} B_i^* \gamma(Q) B_i \succ 0$,

(iii) *for all $X \preceq U$, $Y \preceq V$ and $\lambda \in (0, 1)$, we have*

$$d(\gamma(X), \gamma(U)) \leq \frac{\lambda}{2R} M(X, Y, U, V) \text{ and } d(\gamma(Y), \gamma(V)) \leq \frac{\lambda}{2R} M(X, Y, U, V),$$

where

$$M(X, Y, U, V) = \max\{d(X, U) + d(Y, V), [d(X, F(X, Y)) + d(Y, F(Y, X))], [d(U, F(U, V)) + d(V, F(V, U))]\}.$$

Then (6.1) *has a solution in $P(n)$.*

Proof. *We claim that there exist $(X, Y) \in H(n) \times H(n)$, a solution to the system*

$$\begin{cases} X = Q + \sum_{i=1}^{m} A_i^* \gamma(X) A_i - \sum_{i=1}^{m} B_i^* \gamma(Y) B_i, \\ Y = Q + \sum_{i=1}^{m} A_i^* \gamma(Y) A_i - \sum_{i=1}^{m} B_i^* \gamma(X) B_i. \end{cases} \tag{6.2}$$

Define $F : H(n) \times H(n) \to H(n)$ by

$$F(X, Y) = Q + \sum_{i=1}^{m} A_i^* \gamma(X) A_i - \sum_{i=1}^{m} B_i^* \gamma(Y) B_i, \tag{6.3}$$

for all $X, Y \in H(n)$. It is clear that F is a mapping having the bi-comparative property with respect to symmetric relation $\mathcal{S}$. Then a coupled fixed point of F is a solution of (6.2). *Let $X, Y, U, V \in H(n)$ with $X \preceq U, Y \preceq V$, then $\gamma(X) \preceq \gamma(Y)$, $\gamma(Y) \preceq \gamma(V)$. So, we have*

$$
\begin{aligned}
d(F(X,Y)-F(U,V)) &= \|F(X,Y)-F(U,V)\|_{\tau,Q}\\
&= tr(Q^{\frac{1}{2}}(F(X,Y)-F(U,V))Q^{\frac{1}{2}})\\
&= tr\left(Q^{\frac{1}{2}}\left(\sum_{i=1}^{m} A_i^*\gamma(X)A_i - \sum_{i=1}^{m} B_i^*\gamma(Y)B_i - \sum_{i=1}^{m} A_i^*\gamma(U)A_i + \sum_{i=1}^{m} B_i^*\gamma(V)B_i\right)Q^{\frac{1}{2}}\right)\\
&= \sum_{i=1}^{m} tr(Q^{\frac{1}{2}}(A_i^*(\gamma(X)-\gamma(U))A_i) + (B_i^*(\gamma(V)-\gamma(Y))B_i)Q^{\frac{1}{2}})\\
&= \sum_{i=1}^{m} tr(Q^{\frac{1}{2}}(A_i^*(\gamma(X)-\gamma(U))A_i)Q^{\frac{1}{2}}) + tr(Q^{\frac{1}{2}}(B_i^*(\gamma(V)-\gamma(Y))B_i)Q^{\frac{1}{2}})\\
&= \sum_{i=1}^{m} tr(A_iQA_i^*(\gamma(X)-\gamma(U))) + tr(B_iQB_i^*(\gamma(V)-\gamma(Y)))\\
&= \sum_{i=1}^{m} tr(A_iQA_i^*Q^{-\frac{1}{2}}Q^{\frac{1}{2}}(\gamma(X)-\gamma(U))Q^{\frac{1}{2}}Q^{-\frac{1}{2}}) + tr(B_iQB_i^*Q^{-\frac{1}{2}}Q^{\frac{1}{2}}(\gamma(V)-\gamma(Y))Q^{\frac{1}{2}}Q^{-\frac{1}{2}})\\
&= \sum_{i=1}^{m} tr(Q^{-\frac{1}{2}}A_iQA_i^*Q^{-\frac{1}{2}}Q^{\frac{1}{2}}(\gamma(X)-\gamma(U))Q^{\frac{1}{2}}) + tr(Q^{-\frac{1}{2}}B_iQB_i^*Q^{-\frac{1}{2}}Q^{\frac{1}{2}}(\gamma(V)-\gamma(Y))Q^{\frac{1}{2}})\\
&= tr\left(\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_iQA_i^*Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}(\gamma(X)-\gamma(U))Q^{\frac{1}{2}})\right) + tr\left(\sum_{i=1}^{m} Q^{-\frac{1}{2}}B_iQB_i^*Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}(\gamma(V)-\gamma(Y))Q^{\frac{1}{2}})\right)\\
&\leq \left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_iQA_i^*Q^{-\frac{1}{2}}\right\| \|\gamma(X)-\gamma(U)\|_{\tau,Q} + \left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}}B_iQB_i^*Q^{-\frac{1}{2}}\right\| \|\gamma(V)-\gamma(Y)\|_{\tau,Q}\\
&\leq \frac{\lambda\left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_iQA_i^*Q^{-\frac{1}{2}}\right\|}{R} M(Y,X,U,V) + \frac{\lambda\left\|\sum_{i=1}^{m} Q^{-\frac{1}{2}}B_iQB_i^*Q^{-\frac{1}{2}}\right\|}{R} M(Y,X,U,V)\\
&\leq \frac{\lambda M(X,Y,U,V)}{2} + \frac{\lambda M(X,Y,U,V)}{2} = \lambda M(X,Y,U,V).
\end{aligned}
$$

Thus, the contractive condition of Theorem (4.10) *is satisfied for all $X, Y, U, V \in H(n)$ with $X \preceq U$ and $Y \preceq V$. Now from Theorem* (4.10), *we conclude that F has a coupled fixed point and hence matrix equation* (6.2) *has a solution in $H(n) \times H(n)$.*

Since, (X, Y) is coupled fixed point of F and let $X = Y = \hat{X}$. Thus $\hat{X} \in H(n)$ is a solution of (6.1). □

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