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# On Some Conditions for $p$-Valency

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**Abstract.** In this paper we consider analytic functions in the unit disc $\mathbb{D}$ satisfying the Ozaki's condition that

$\Re\mathfrak{e}\left\{f^{(p)}(z)\right\} > 0, \quad |z| < 1.$

We prove some implications of this condition and we estimate the order of strongly starlikeness of $f^{(p-3)}(z)$.

## 1. Introduction

A function $f$ analytic in a domain $D \in \mathbb{C}$ is called $p$-valent in $D$, if for every complex number $w$, the equation $f(z) = w$ has at most $p$ roots in $D$, so that there exists a complex number $w_0$ such that the equation $f(z) = w_0$ has exactly $p$ roots in $D$. We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit unit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}_p$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}).$$

Let $\mathcal{A} = \mathcal{A}(1)$. Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_p^*(\alpha)$ and $C_p(\alpha)$ be the subclasses of $\mathcal{A}(p)$ consisting of all $p$-valent functions which are strongly starlike and strongly convex of order $\alpha$, $0 \le \alpha < 1$, defined as

$$\mathcal{S}_p^*(\alpha) = \left\{f(z) \in \mathcal{A}(p) : \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D}\right\},$$

$$C_p(\alpha) = \left\{f(z) \in \mathcal{A}(p) : zf'(z)/p \in \mathcal{S}_p^*(\alpha)\right\}.$$

Note that $\mathcal{S}_1^*(1) = \mathcal{S}^*$ and $C_1(1) = C$, where $\mathcal{S}^*$ and $C$ are usual classes of starlike and convex functions respectively.

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The known Ozaki's condition says that

$$\Re e\left\{f^{(p)}(z)\right\} > 0, \quad (z \in \mathbb{D})$$

follows that $f(z)$ is at most $p$-valent in $\mathbb{D}$. We prove that under additional assumption $p \geq 3$ the above condition follows that $f(z)$ is at most $p$-valent convex in $\mathbb{D}$.

## 2. Preliminaries

In this paper we need the following lemmas.

**Lemma 2.1.** *[2, Th.5] If $f(z) \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have*

$$\Re e\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \dots, p\}: \quad \Re e\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0. \tag{2.1}$$

**Lemma 2.2.** *[2, Th.1] If $f(z) \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have*

$$\Re e\left\{p + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0 \quad (z \in \mathbb{D}), \tag{2.2}$$

*then f(z) is p-valent in $\mathbb{D}$ and*

$$\forall k \in \{1, \dots, p-1\}: \quad \Re e\left\{k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

**Lemma 2.3.** *[4] Let $p(z) = 1 + \sum_{n\geq m}^{\infty} c_n z^n$, $c_m \neq 0$ be analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point $z_0$, $|z_0| < 1$, such that*

$$|\arg\{p(z)\}| < \frac{\pi\beta}{2} \;\; for \;\; |z| < |z_0|$$

*and*

$$|\arg\{p(z_0)\}| = \frac{\pi\beta}{2}$$

*for some $\beta > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

*for some $k \geq m(a + a^{-1})/2 > m$, where*

$$\{p(z_0)\}^{1/\beta} = \pm ia, \;\; and \;\; a > 0.$$

## 3. Main Results

**Theorem 3.1.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$ and*

$$\Re e\{f^{(p)}(z)\} > 0, \quad (z \in \mathbb{D}), \tag{3.1}$$

*then*

$$\left|\arg\left\{\frac{f^{(p-1)}(z)}{z}\right\}\right| < \frac{\alpha_1\pi}{2}, \quad (z \in \mathbb{D}),$$

*where $\alpha_1 = 0.638322\ldots$ is the unique root of of the equation*

$$\frac{\alpha\pi}{2} + \tan^{-1}\frac{\alpha}{2} = \frac{\pi}{2}. \tag{3.2}$$

*Proof.* If we put

$$g_1(z) = \frac{1}{p!z} f^{(p-1)}(z), \quad g_1(0) = 1, \quad (z \in \mathbb{D}),$$

then it follows that

$$\begin{aligned} f^{(p)}(z) &= p!(g_1(z) + z g_1'(z)) \\ &= p! g_1(z) \left( 1 + \frac{z g_1'(z)}{g_1(z)} \right). \end{aligned}$$

If there exists a point $z_1 \in \mathbb{D}$, such that

$$|\arg\{g_1(z)\}| < \frac{\alpha_1 \pi}{2}, \quad (|z| < |z_1|)$$

and

$$|\arg\{g_1(z_1)\}| = \frac{\alpha_1 \pi}{2},$$

then from Lemma 2.3, we have

$$\frac{z_1 g_1'(z_1)}{g_1(z_1)} = \frac{2ik \arg\{g_1(z_1)\}}{\pi}$$

for some $k \geq m(a + a^{-1})/2 > 1$, where

$$\{g_1(z_1)\}^{1/\alpha_1} = \pm ia, \text{ and } a > 0.$$

For the case $\arg\{g_1(z_1)\} = \alpha_1 \pi/2$, we have

$$\begin{aligned} \arg\{f^{(p)}(z_1)\} &= \arg\left\{ p! g_1(z_1) \left( 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right) \right\} \\ &= \arg\{g_1(z_1)\} + \arg\left\{ 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right\} \\ &= \frac{\alpha_1 \pi}{2} + \arg\left\{ 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right\} \\ &= \frac{\alpha_1 \pi}{2} + \arg\{1 + i\alpha_1 k\} \\ &\geq \frac{\alpha_1 \pi}{2} + \tan^{-1} \alpha_1 \\ &= \frac{\pi}{2} \end{aligned}$$

because of (3.2), but this contradicts hypothesis (3.1). For the case $\arg\{g_1(z_1)\} = -\alpha_1 \pi/2$, we have

$$\begin{aligned} \arg\{f^{(p)}(z_1)\} &= \arg\left\{ p! g_1(z_1) \left( 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right) \right\} \\ &= \arg\{g_1(z_1)\} + \arg\left\{ 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right\} \\ &= -\frac{\alpha_1 \pi}{2} + \arg\left\{ 1 + \frac{z_1 g_1'(z_1)}{g_1(z_1)} \right\} \\ &= \frac{\alpha_1 \pi}{2} + \arg\{1 - i\alpha_1 k\} \\ &\leq \frac{-\alpha_1 \pi}{2} - \tan^{-1} \alpha_1 \\ &= -\frac{\pi}{2} \end{aligned}$$

because of (3.2) and this contradicts hypothesis (3.1) too. This shows that

$$\left|\arg\left\{\frac{f^{(p-1)}(z)}{z}\right\}\right| < \frac{\alpha_1\pi}{2}, \quad (z \in \mathbb{D}). \tag{3.3}$$

□

**Theorem 3.2.** *If* $f(z) \in \mathcal{A}_p$, $p \geq 2$ *and*

$$\Re\mathfrak{e}\{f^{(p)}(z)\} > 0, \quad (z \in \mathbb{D}), \tag{3.4}$$

*then*

$$\left|\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| < \frac{\alpha_2\pi}{2}, \quad (z \in \mathbb{D}),$$

*where* $\alpha_2 = 0.486434\ldots$ *is the unique root of of the equation*

$$\frac{\alpha\pi}{2} + \tan^{-1}\frac{\alpha}{2} = \frac{\alpha_1\pi}{2} \tag{3.5}$$

*and where* $\alpha_1 = 0.638322\ldots$ *is the unique solution of the equation* (3.2).

*Proof.* Let us put

$$g_2(z) = \frac{2!}{p!z^2}f^{(p-2)}(z), \quad g_2(0) = 1, \quad (z \in \mathbb{D}),$$

then it follows that $f^{(p-2)}(z) = p!z^2g_2(z)/2!$ and

$$f^{(p-1)}(z) = \frac{p!}{2!}(2zg_2(z) + z^2g_2'(z))$$

and so

$$\frac{f^{(p-1)}(z)}{p!z} = g_2(z) + \frac{1}{2}zg_2'(z).$$

If there exists a point $z_2 \in \mathbb{D}$, such that

$$|\arg\{g_2(z)\}| < \frac{\alpha_2\pi}{2}, \quad (|z| < |z_2|)$$

and

$$|\arg\{g_2(z_2)\}| = \frac{\alpha_2\pi}{2},$$

then from Lemma 2.3, we have

$$\frac{z_2g_2'(z_2)}{g_2(z_2)} = \frac{2ik\arg\{g_2(z_2)\}}{\pi}$$

for some $k \geq m(a + a^{-1})/2 > 1$, where

$$\{g_2(z_2)\}^{1/\alpha_2} = \pm ia, \text{ and } a > 0.$$

Therefore, applying the same method as in the proof of Theorem 3.1, we can get

$$\begin{aligned}
\left|\arg\left\{\frac{f^{(p-1)}(z_2)}{p!z_2}\right\}\right| &= \left|\arg\left\{\frac{f^{(p-1)}(z_2)}{z_2}\right\}\right| \\
&= \left|\arg\left\{p!g_2(z_2)\left(1+\frac{z_2g_2'(z_2)}{2g_2(z_2)}\right)\right\}\right| \\
&= \left|\arg\{g_2(z_2)\}+\arg\left\{1+\frac{z_2g_2'(z_2)}{2g_2(z_2)}\right\}\right| \\
&= \left|\frac{\alpha_2\pi}{2}+\arg\left\{1+\frac{z_2g_2'(z_2)}{2g_2(z_2)}\right\}\right| \\
&= \left|\frac{\alpha_2\pi}{2}+\arg\left\{1+i\frac{\alpha_2 k}{2}\right\}\right| \\
&\geq \frac{\alpha_2\pi}{2}+\tan^{-1}\frac{\alpha_2}{2} \\
&= \frac{\alpha_1\pi}{2}
\end{aligned}$$

because of (3.5). On the other hand this contradicts Theorem 3.1. This shows that

$$\left|\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| < \frac{\alpha_2\pi}{2}, \quad (z\in\mathbb{D}). \tag{3.6}$$

□

**Theorem 3.3.** *If* $f(z)\in\mathcal{A}_p$, $p\geq 3$ *and*

$$\mathfrak{Re}\{f^{(p)}(z)\} > 0, \quad (z\in\mathbb{D}), \tag{3.7}$$

*then*

$$\left|\arg\left\{\frac{f^{(p-3)}(z)}{z^3}\right\}\right| < \frac{\alpha_3\pi}{2}, \quad (z\in\mathbb{D}),$$

*where* $\alpha_3 = 0.401696\ldots$ *is the unique root of of the equation*

$$\frac{\alpha\pi}{2}+\tan^{-1}\frac{\alpha}{3} = \frac{\alpha_2\pi}{2} \tag{3.8}$$

*and where* $\alpha_2 = 0.486434\ldots$ *is is described in Theorem 3.5.*

*Proof.* Applying the same method as in the above proofs and putting

$$g_3(z) = \frac{3!}{p!z^3}f^{(p-3)}(z), \quad g_3(0)=1, \quad (z\in\mathbb{D}),$$

follows that

$$\begin{aligned}
\frac{f^{(p-2)}(z)}{p!z^2} &= g_3(z)+\frac{1}{3}zg_3'(z) \\
&= g_3(z)\left(1+\frac{1}{3}\frac{zg_3'(z)}{g_3(z)}\right).
\end{aligned}$$

If there exists a point $z_3\in\mathbb{D}$, such that

$$|\arg\{g_3(z)\}| < \frac{\alpha_3\pi}{2}, \quad (|z|<|z_3|)$$

and

$$|\arg\{g_3(z_3)\}| = \frac{\alpha_3\pi}{2},$$

then from Lemma 2.3, we have

$$\frac{z_3 g_3'(z_3)}{g_3(z_3)} = \frac{2ik\arg\{g_3(z_3)\}}{\pi}$$

for some $k \geq m(a+a^{-1})/2 > 1$, where

$$\{g_3(z_3)\}^{1/\alpha_2} = \pm ia, \text{ and } a > 0.$$

Therefore, we have

$$\begin{aligned}
\left|\arg\left\{\frac{f^{(p-2)}(z_3)}{p!z_3^2}\right\}\right| &= \left|\arg\left\{\frac{f^{(p-2)}(z_3)}{z_3^2}\right\}\right| \\
&= \left|\arg\left\{p!g_3(z_3)\left(1+\frac{z_3 g_3'(z_3)}{3g_3(z_3)}\right)\right\}\right| \\
&= \left|\arg\{g_3(z_3)\} + \arg\left\{1+\frac{z_3 g_3'(z_3)}{3g_3(z_3)}\right\}\right| \\
&= \left|\frac{\alpha_3\pi}{2} + \arg\left\{1+\frac{z_3 g_3'(z_3)}{3g_3(z_3)}\right\}\right| \\
&= \left|\frac{\alpha_3\pi}{2} + \arg\left\{1+i\frac{\alpha_3 k}{3}\right\}\right| \\
&\geq \frac{\alpha_3\pi}{2} + \tan^{-1}\frac{\alpha_3}{3} \\
&= \frac{\alpha_2\pi}{2}
\end{aligned}$$

because of (3.8), but this contradicts Theorem 3.2. This shows that

$$\left|\arg\left\{\frac{f^{(p-3)}(z)}{z^3}\right\}\right| < \frac{\alpha_3\pi}{2}, \quad (z \in \mathbb{D}).$$

□

**Corollary 3.4.** *If* $f(z) \in \mathcal{A}_p$, $p \geq 3$ *and*

$$\Re\mathfrak{e}\{f^{(p)}(z)\} > 0, \quad (z \in \mathbb{D}),$$

*then we have*

$$\left|\arg\left\{\frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)}\right\}\right| < \frac{\pi}{2}(\alpha_2+\alpha_3), \quad (z \in \mathbb{D}).$$

*This means that* $3!f^{(p-3)}(z)/p! = z^3 + \ldots$ *is in the class* $\mathcal{S}_3^*(\alpha_2+\alpha_3)$ *of* 3*-valent strongly starlike functions of order* $\alpha_2+\alpha_3 = 0.488\ldots$.

*Proof.* Applying the above results, we have

$$\begin{aligned}
\left|\arg\left\{\frac{\frac{f^{(p-2)}(z)}{z^2}}{\frac{f^{(p-3)}(z)}{z^3}}\right\}\right| &= \left|\arg\left\{\frac{zf^{(p-2)}}{f^{(p-3)}}\right\}\right| \\
&\leq \left|\arg\left\{\frac{zf^{(p-2)}(z)}{z^2}\right\}\right| + \left|\arg\left\{\frac{zf^{(p-3)}(z)}{z^3}\right\}\right| \\
&< \frac{\pi}{2}(\alpha_2+\alpha_3).
\end{aligned}$$

This completes the proof of Corollary 3.4. □

**Theorem 3.5.** *If* $f(z) \in \mathcal{A}_p$, $p \geq 3$ *and*

$$\Re e\{f^{(p)}(z)\} > 0, \quad (z \in \mathbb{D}). \tag{3.9}$$

*Then* $f(z)$ *is p-valently starlike in* $\mathbb{D}$ *and also,* $f(z)$ *is p-valently convex in* $\mathbb{D}$.

*Proof.* From Theorem 3.1, we have

$$\begin{aligned}\left|\arg \frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)}\right| &= \left|\arg \frac{z(f^{(p-3)}(z))'}{f^{(p-3)}(z)}\right| \\ &\leq \frac{\pi}{2}(\alpha_2 + \alpha_3) \\ &< \frac{\pi}{2}0.89 < \frac{\pi}{2}, \quad (z \in \mathbb{D}).\end{aligned}$$

This shows that

$$\frac{3!f^{(p-3)}(z)}{p!} = z^3 + \cdots$$

is 3-valently starlike in $\mathbb{D}$. Applying Lemma 2.1 to the function $f^{(p-3)}(z)$ gives

$$\Re e\frac{zf'(z)}{f(z)} > 0, \quad (z \in \mathbb{D}),$$

therefore, $f(z)$ is $p$-valently starlike in $\mathbb{D}$. On the other hand, it is trivial that

$$\begin{aligned}\left|\arg\left\{1 + \frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)}\right\}\right| &< \left|\arg \frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)}\right| \\ &\leq \frac{\pi}{2}(\alpha_2 + \alpha_3) \\ &\leq \frac{\pi}{2}, \quad (z \in \mathbb{D}).\end{aligned}$$

This shows that $3!f^{(p-3)}(z)/p!$ is 3-valently convex in $\mathbb{D}$. Then it is trivial that

$$3 + \Re e\frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)} > 1 + \Re e\frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)} > 0, \quad (z \in \mathbb{D}).$$

Therefore, applying Lemma 2.2 to the function $f^{(p-3)}(z)$ gives

$$\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

It completes the proof of Theorem 3.5. □

Notice here the well-known Noshiro-Warschawski theorem and some related results. The Noshiro-Warschawski theorem [1, 10], says that if $f \in \mathcal{H}$ satisfies

$$\Re e\left\{e^{i\alpha}f'(z)\right\} > 0, \quad (z \in \mathbb{D}) \tag{3.10}$$

for some real $\alpha$, then $f(z)$ is univalent in $\mathbb{D}$. Ozaki [5], generalized the above theorem for $f \in \mathcal{A}_p$: if

$$\Re e\left\{e^{i\alpha}f^{(p)}(z)\right\} > 0, \quad (z \in \mathbb{D}) \tag{3.11}$$

for some real $\alpha$, then $f(z)$ is at most $p$-valent in $\mathbb{D}$. Also in [3, 454] it was shown that if $f \in \mathcal{A}_p$, $p \geq 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}), \tag{3.12}$$

then $f$ is at most $p$-valent in $\mathbb{D}$.

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