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Additive Perturbations and Multiplicative Perturbations for the Core Inverse of Bounded Linear Operator in Hilbert Space

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Abstract. In this paper, we present some characteristics and expressions of the core inverse $A^{\#\!\!\!\bigcirc}$ of bounded linear operator A in Hilbert spaces. Additive perturbations of core inverse are investigated under the condition $R(\bar{A})\cap N(A^{\#}) = \{0\}$ and an upper bound of $\|\bar{A}^{\#\!\!\!\bigcirc} - A^{\#\!\!\!\bigcirc}\|$ is obtained. We also discuss the multiplicative perturbations. The expressions of core inverse of perturbed operator $T = EAF$ and the upper bounds of $\|T^{\#\!\!\!\bigcirc} - A^{\#\!\!\!\bigcirc}\|$ are obtained too.

1. Introduction

Let H, K be Hilbert spaces throughout this paper. Let $B(H, K)$ denote the set of all bounded linear operators from H to K, $B(H)$ denote $B(H, H)$. For an operator $A \in B(H, K)$, $R(A)$, $N(A)$ denote the range space and null space of A, respectively. Let $P_{M,L}$ be a projector onto M along a complementary subspace L. If $L = M^{\perp}$, then $P_{M,M^{\perp}}$ is an orthogonal projector.

An operator $X \in B(K, H)$ which satisfies $AXA = A$ is called the inner inverse of A, denoted by A^{-}. If $XAX = X$ holds too, then X is called a reflexive generalized inverse, denoted by A^{r}. Assume that $R(A)$ is closed and $P_{N(A),\mathcal{S}}$, $Q_{R(A),\mathcal{M}}$ are continuous projectors. If X satisfies

$$AXA = A, \quad XAX = X, \quad AX = Q_{R(A),\mathcal{M}}, \quad XA = I - P_{N(A),\mathcal{S}},$$

then X is called the oblique projector generalized inverse, denoted by $A^{r}_{P_{N(A),\mathcal{S}},Q_{R(A),\mathcal{M}}}$. In this case,

$$H = N(A) \oplus \mathcal{S}, \quad K = R(A) \oplus \mathcal{M}.$$

Specially, if $\mathcal{S} = N(A)^{\perp}$, $\mathcal{M} = R(A)^{\perp}$, i.e. $P_{N(A),N(A)^{\perp}}$, $Q_{R(A),R(A)^{\perp}}$ are orthogonal projectors, then X is Moore-Penrose inverse of A, denoted by $A^{\dagger}$. In this case, the equalities $AX = Q_{R(A),R(A)^{\perp}}$ and $XA = I - P_{N(A),N(A)^{\perp}}$ are equivalent to $AX = (AX)^{*}$ and $XA = (XA)^{*}$ if $R(A)$ is closed, respectively. We know from [8, 10] that if $R(A)$ closed, then $A^{\dagger}$ exists and

$$A^{\dagger} = (I - A^{r}A - (A^{r}A)^{*})^{-1}A^{r}(I - AA^{r} - (AA^{r})^{*})^{-1}.$$

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The group inverse of $A \in B(H)$ is an operator $X \in B(H)$ such that

$$AXA = A, \quad XAX = X, \quad AX = XA.$$

The group inverse of A is unique if it exits and denoted by $A^{\#}$. The idempotent operator $I - AA^{\#}$ is denoted by A^{π}. The group inverse $A^{\#}$ exists if and only if $ind(A) \leq 1$ if and only if $R(A) \oplus N(A) = H$. We have from [11, 25] that if $A^{\#}$ exists, then

$$A^{\#} = (I + A - AA^{r})^{-2}A = A(I + A - A^{r}A)^{-2}.$$

The core inverse of matrices was introduced in [2] by Baksalary and Trenkler. A matrix $A^{\circledast} \in \mathbf{C}^{n\times n}$ is called core inverse of A if $AA^{\circledast} = P_{R(A),R(A)^{\perp}}$ and $R(A^{\circledast}) \subseteq R(A)$. In [12], D.S. Rakić, N.Č. Dinčić and D.S. Djordjević extended the notion of core inverse to operator on a Hilbert space as follows:

$$AA^{\circledast}A = A, \quad R(A^{\circledast}) = R(A), \quad N(A^{\circledast}) = N(A^{*}).$$

They show their definition is equivalent to the definition by Baksalary and Trenkler in a finite dimensional Hilbert space, but in arbitrary Hilbert space the first definition does not imply the second definition. The core inverse was been extended to rings with involution in [13]. Let $\mathcal{R}$ be a ring with involution, $a \in \mathcal{R}$. An element $a^{\circledast} \in \mathcal{R}$ satisfying

$$aa^{\circledast}a = a, \quad a^{\circledast}\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a^{\circledast} = \mathcal{R}a^{*}$$

is called core inverse of a. Moreover, the authors characterized the core invertibility by the following equations:

$$axa = a, \quad xax = x, \quad (ax)^{*} = ax, \quad xa^{2} = a, \quad ax^{2} = x.$$

This characterization was simplified in [24]. Specifically, $a \in \mathcal{R}$ is core invertible if and only if there exists $x \in \mathcal{R}$ such that

$$ax^{2} = x, \quad xa^{2} = a, \quad (ax)^{*} = ax.$$

Clearly, if the core inverse exists then it is unique.

The core inverse combines some of the properties of the group inverse and the Moore-Pennrose inverse. Various properties of core inverse have been investigated in [1, 12, 13, 24].

The perturbation analyses of generalized inverses have been studied from various perspectives in last decades. In general, there are two kinds of perturbation analysis of generalized inverse, one is additive perturbation analysis, the other is multiplicative perturbation analysis. Most classic reference consider additive perturbation analysis and many results have been obtained. The paper [19] and [21] discuss various results on perturbations till 1976. Using the perturbation bounds for outer inverse, M. Nashed and X. Chen present a Kantorovich-type analysis for Newton-like methods for singular operator equations in [22]. Subsequent research has been the subject of many papers. For example, in [8, 26], Xue and Chen have investigated the stable perturbation and got the error upper bounds of perturbation of Moore-Penrose inverse. Xue has systematically studied additive perturbation of generalized inverse of linear operators in his book [25]. For more details of additive perturbation analysis, the reader can refer to more reference.

Lately, there has been an increase of interest in multiplicative perturbation analysis. For example, Cai et.al, in [7], have studied the Moore-Penrose inverse of $\tilde{A} = D_1^{*}AD_2$, where $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$ are nonsingular matrices and obtained the upper bound of $\|\tilde{A}^{\dagger} - A^{\dagger}\|$; N.Castro-González, et.al, in [5, 6], have discussed the Moore-Penrose inverse of a matrix of the form $\tilde{A} = (I + E)A(I + F) \in \mathbb{C}^{m\times n}$, where $(I + E) \in \mathbb{C}^{m\times m}$ and $(I + F) \in \mathbb{C}^{n\times n}$ are nonsingular matrices and improved the related results in [7]; L.Meng and B.Zheng have studied multiplicative perturbation of group inverse in [16], ect. Please see [4, 23, 28] for more details of multiplicative perturbation theory.

Our goal of this paper is to consider the expressions, additive perturbations and multiplicative perturbations of core inverse of bounded linear operators in Hilbert space. The paper is organized as follows. In Section 2, we recall some lemmas and basic results which will be used in the context. In Section 3, we present explicit expressions of the core inverse and consider additive perturbations of the core inverse. In section 4, we investigate expressions and multiplicative perturbations of the core inverse.

2. Preliminaries

In this section , we recall some lemmas and basic results which will be used in this paper. we start with the definition of the core inverse of a linear operator. D.S. Rakić, et.al, introduced the core inverse of bounded linear operator in [12] as follows:

Definition 2.1. [12] *Let $A \in B(H)$. An operator $A^{\circledast} \in B(H)$ is called a core inverse if*

$$AA^{\circledast}A = A, \quad R(A^{\circledast}) = R(A) \quad \textit{and} \quad N(A^{\circledast}) = N(A^*).$$

In [12], the authors showed that this definition is an extension of the definition of the core inverse of matrix and characterized the core inverse using the following five equations(see [12, Theorem 3.5] for details):

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad XA^2 = A \quad \text{and} \quad AX^2 = X.$$

Motivated by [24, Theorem 3.1], we also characterize the core inverse of linear operator as follows:

Proposition 2.2. *Let $A \in B(H)$. Then $X \in B(H)$ is a core inverse of A equivalent to X is a solution of the following equations*

$$AX^2 = X, \quad XA^2 = A, \quad (AX)^* = AX.$$

Proof. The result follows from the proof of [24, Theorem 3.1] and [12, Theorem 3.5]. □

Many properties of core inverse have been investigated. We summarize some of these properties in the next lemma.

Lemma 2.3. [2, 12, 13] *Suppose the core inverse $A^{\circledast}$ of $A \in B(H)$ exists. Then*

(1) $A^{\circledast} \in A\{1, 2\}$.

(2) $A^{\circledast} = A^{\#}P_{R(A),R(A)^{\perp}}$.

(3) $A^{\circledast} = A^{\#}AA^{\dagger}$.

(4) $(A^{\circledast})^2A = A^{\#}$.

(5) $A^{\circledast}A = AA^{\#}$.

(6) $AA^{\circledast} = AA^{\dagger}$.

(7) $A^{\circledast} = 0 \Leftrightarrow A = 0$.

(8) $A^{\circledast} = P_{R(A),R(A)^{\perp}} \Leftrightarrow A^2 = A$.

From property (2) and property (4) in Lemma 2.3, it follows that $A^{\circledast}$ exists if and only if $A^{\#}$ exists.

3. Additive Perturbation Analysis

In this section, we consider expressions and additive perturbations of the core inverse of a bounded linear operator. First of all, we present some characteristics of the core inverse.

Lemma 3.1. *Suppose the core inverse $A^{\circledast}$ of $A \in B(H)$ exists. Then*

$$A^{\circledast} = P_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}}$$

is independent of the choice of A^{-}.

Proof. Set $X = P_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}}$. Then

$$\begin{aligned} AXA &= AP_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}}A \\ &= AA^{-}A \\ &= A, \\ XAX &= P_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}}AP_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}} \\ &= P_{R(A),N(A)}A^{-}Q_{R(A),R(A)^{\perp}} \\ &= X, \\ R(X) &= R(XA) = R(P_{R(A),N(A)}A^{-}A) = R(P_{R(A),N(A)}) = R(A), \\ N(X) &= N(AX) = N(AA^{-}Q_{R(A),R(A)^{\perp}}) = R(A)^{\perp} = N(A^{*}). \end{aligned}$$

□

Lemma 3.2. *Let $A \in B(H)$ and suppose $A^{\circledast}$ exists. Then $A^{\circledast}$ is a oblique projector generalized inverse with respect to $P_{N(A),R(A)}$ and $Q_{R(A),R(A)^{\perp}}$, i.e,*

$$A^{\circledast} = A^{r}_{P_{N(A),R(A)},Q_{R(A),R(A)^{\perp}}}.$$

Proof. The result follows from Lemma 3.1. □

In order to get a representation of core inverse, we need the following expressions for $A^{\#}$.

Lemma 3.3. [11] *Let $A \in B(H)$ and suppose $A^{\#}$ exists. Then*

$$\begin{aligned} A^{\#} &= A(I + A - A^{r}A)^{-2} \\ &= (I + A - AA^{r})^{-2}A \\ &= (I + A - AA^{r})^{-1}A(I + A - A^{r}A)^{-1}. \end{aligned}$$

Using expressions of the group inverse of A, we give some explicit expressions of the core inverse.

Lemma 3.4. *Let $A \in B(H)$ and suppose $A^{\circledast}$ exists. Then*

$$A^{\circledast} + A^{\pi} = (I + A - AA^{\dagger})^{-1}.$$

Proof. The result follows from simple calculations. □

Remark 3.5. *Using Lemma 2.3 and Lemma 3.4, it is easy to verify the following equations.*

$$\begin{aligned} AA^{\dagger} &= AA^{\circledast} = A(I + A - AA^{\dagger})^{-1}, \\ AA^{\#} &= A^{\circledast}A = (I + A - AA^{\dagger})^{-1}A. \end{aligned}$$

Lemma 3.6. *Let $A \in B(H)$ and suppose $A^{\circledast}$ exists. Then*

$$A^{\circledast} = (I + A - AA^{\dagger})^{-1}AA^{\dagger} = A(I + A - A^{\dagger}A)^{-1}A^{\dagger}.$$

Proof. Noting that $(I + A - AA^{\dagger})A = A^{2}$, by Lemma 3.3 and Lemma 2.3, we have

$$\begin{aligned} A^{\circledast} &= A^{\#}AA^{\dagger} \\ &= (I + A - AA^{\dagger})^{-2}A^{2}A^{\dagger} \\ &= (I + A - AA^{\dagger})^{-1}AA^{\dagger}. \end{aligned}$$

Using the equality $(I + A - A^{\dagger}A)A^{\dagger} = AA^{\dagger}$, we get

$$A^{\circledast} = A(I + A - A^{\dagger}A)^{-2}AA^{\dagger} = A(I + A - A^{\dagger}A)^{-1}A^{\dagger}.$$

□

Lemma 3.7. *Let $A \in B(H)$ and suppose $A^{\circledast}$ exists. Then*

$$\begin{aligned} A^{\circledast} &= (I + A - AA^r)^{-1}AA^r(AA^r + (AA^r)^* - I)^{-1} \\ &= A(I + A - A^rA)^{-1}A^r(AA^r + (AA^r)^* - I)^{-1}. \end{aligned}$$

Proof. Using [10, Lemma 2.2], we have

$$AA^{\dagger} = AA^r(AA^r + (AA^r)^* - I)^{-1}.$$

Noting that

$$(I + A - AA^r)A = A^2, \quad (I + A - A^rA)A^r = AA^r,$$

we have, by Lemma 2.3 and Lemma 3.3,

$$\begin{aligned} A^{\circledast} &= A^{\#}AA^{\dagger} \\ &= (I + A - AA^r)^{-2}A^2A^r(AA^r + (AA^r)^* - I)^{-1} \\ &= (I + A - AA^r)^{-1}AA^r(AA^r + (AA^r)^* - I)^{-1} \end{aligned}$$

and

$$\begin{aligned} A^{\circledast} &= A^{\#}AA^{\dagger} \\ &= A(I + A - A^rA)^{-2}AA^{\dagger} \\ &= A(I + A - A^rA)^{-2}AA^r(AA^r + (AA^r)^* - I)^{-1} \\ &= A(I + A - A^rA)^{-1}A^r(AA^r + (AA^r)^* - I)^{-1}. \end{aligned}$$

□

Next, we consider additive perturbations of the core inverse. The following proposition gives an expression of the core inverse of the perturbed operator.

Proposition 3.8. *Let $A \in B(H)$ and $\bar{A} = A + \delta A \in B(H)$. Let $Z = I + A^r\delta A, U = (I - AA^r)\delta A, F_U = I - U^rU, S = A^rAZF_U$. Then $\bar{A}^{\circledast}$ exists if and only if $\bar{A} + F_UF_S$ is invertible. In this case*

$$\bar{A}^{\circledast} = \bar{A}(\bar{A} + F_UF_S)^{-2}\bar{A}\bar{A}^{\dagger}.$$

Proof. We have from [11, Corollary 3.5],

$$\bar{A}^{\#} = \bar{A}(\bar{A} + F_UF_S)^{-2}.$$

The result then follows from $\bar{A} = \bar{A}^{\#}\bar{A}\bar{A}^{\dagger}$. □

Xue and Chen have investigated stable perturbation of group inverse in [27]. In virtue of those results, we consider the error estimation of additive perturbations of the core inverse of a bounded linear operator in Hilbert space.

Theorem 3.9. *Suppose that the core inverse $A^{\circledast}$ of $A \in B(H)$ exists and $\bar{A} = A+\delta A \in B(H)$. Assume $R(\bar{A})\cap N(A^{\#}) = \{0\}$ and $\|A^{\#}\|\|\delta A\| < \dfrac{1}{1 + \|A^{\pi}\|}$. Then $\bar{A}^{\circledast}$ exists and*

$$\bar{A}^{\circledast} = (I + C(A))D(A)AA^{\circledast}\{\bar{A}(I + A^{\#}\delta A)^{-1}A^{\circledast} + ((I + \delta AA^{\#})^*)^{-1}(I - AA^{\circledast})\}^{-1}.$$

Moreover,

$$\|\bar{A}^{\circledast}\| \le \frac{\|A^{\#}\|}{[1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|]^2},$$

$$\|\bar{A}^{\circledast} - A^{\circledast}\| \le \frac{\|A^{\#}\|^2\|\delta A\|}{1 - \|A^{\#}\|\|\delta A\|}\{\frac{2\|A^{\pi}\|}{1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|} + \|A^{\#}A\|\frac{1 + \sqrt{5}}{2}\}.$$

Here,

$$C(A) = A^{\pi}\delta A(I + A^{\#}\delta A)^{-1}A^{\#},$$
$$D(A) = (I + A^{\#}\delta A)^{-1}A^{\#}\Phi^{-1}(A),$$
$$\Phi(A) = I + \delta A(I - AA^{\#})\delta A[(I + A^{\#}\delta A)^{-1}A^{\#}]^{2}.$$

Proof. Since $R(\bar{A}) \cap N(A^{\#}) = \{0\}$ and $\|A^{\#}\|\|\delta A\| < \dfrac{1}{1 + \|A^{\pi}\|}$, from [27], we have

$$\bar{A}^{\#} = (I + C(A))[D(A) + D(A)^{2}\delta AA^{\pi}](I - C(A)).$$

It follows from [8] that $\bar{A}^{r} = A^{\#}(I + \delta AA^{\#})^{-1} = (I + A^{\#}\delta A)^{-1}A^{\#}$ and

$$\bar{A}^{\dagger} = (I - \bar{A}^{r}\bar{A} - (\bar{A}^{r}\bar{A})^{*})^{-1}\bar{A}^{r}(I - \bar{A}\bar{A}^{r} - (\bar{A}\bar{A}^{r})^{*})^{-1}$$

and

$$\bar{A}\bar{A}^{\dagger} = -\bar{A}\bar{A}^{r}(I - \bar{A}\bar{A}^{r} - (\bar{A}\bar{A}^{r})^{*})^{-1}.$$

Let $\bar{Q} = \bar{A}\bar{A}^{r} = (I + \delta AA^{\#})AA^{\#}(I + \delta AA^{\#})^{-1}$, then, by simple calculations, we have

$$\bar{Q} = C(A) + AA^{\#} = (I + C(A))AA^{\#}.$$

Note that $AA^{\#} + (AA^{\#})^{*} - I = A^{\oplus}A + (A^{\oplus}A)^{*} - I$ is invertible and

$$(A^{\oplus}A + (A^{\oplus}A)^{*} - I)^{-1} = AA^{\dagger} + A^{\dagger}A - I.$$

Using these results and the following computations

$$\begin{aligned}
C(A)(AA^{\dagger} + A^{\dagger}A - I) &= A^{\pi}\delta A(I + A^{\#}\delta A)^{-1}A^{\#}(AA^{\dagger} + A^{\dagger}A - I)\\
&= A^{\pi}\delta A(I + A^{\#}\delta A)^{-1}A^{\#}AA^{\dagger}\\
&= A^{\pi}(I + \delta AA^{\#})^{-1}\delta AA^{\#}AA^{\dagger}\\
&= A^{\pi}(I + \delta AA^{\#})^{-1}(I + \delta AA^{\#} - I)AA^{\dagger}\\
&= A^{\pi}AA^{\dagger} - A^{\pi}(I + \delta AA^{\#})^{-1}AA^{\dagger}\\
&= -A^{\pi}(I + \delta AA^{\#})^{-1}AA^{\dagger},\\
(C(A))^{*}(AA^{\dagger} + A^{\dagger}A - I) &= [(AA^{\dagger} + A^{\dagger}A - I)C(A)]^{*}\\
&= [(AA^{\dagger} - I)\delta A(I + A^{\#}\delta A)^{-1}A^{\#}]^{*}\\
&= -[(I - AA^{\dagger})\delta AA^{\#}(I + \delta AA^{\#})^{-1}]^{*},
\end{aligned}$$

we have

$$\begin{aligned}
\bar{A}\bar{A}^{\dagger} &= (I + C(A))AA^{\#}[(C(A) + AA^{\#}) + (C(A) + AA^{\#})^{*} - I]^{-1}\\
&= (I + C(A))AA^{\#}[C(A) + (C(A))^{*} + AA^{\#} + (AA^{\#})^{*} - I]^{-1}\\
&= (I + C(A))AA^{\#}(AA^{\dagger} + A^{\dagger}A - I)[(C(A) + (C(A))^{*})(AA^{\dagger} + A^{\dagger}A - I) + I]^{-1}\\
&= (I + C(A))AA^{\dagger}\{I - A^{\pi}(I + \delta AA^{\#})^{-1}AA^{\dagger} - [(I - AA^{\dagger})\delta AA^{\#}(I + \delta AA^{\#})^{-1}]^{*}\}^{-1}\\
&= (I + C(A))AA^{\dagger}\{I - A^{\pi}(I + \delta AA^{\#})^{-1}AA^{\dagger} - [(I - AA^{\dagger})(I - (I + \delta AA^{\#})^{-1})]^{*}\}^{-1}\\
&= (I + C(A))AA^{\dagger}\{AA^{\dagger} - A^{\pi}(I + \delta AA^{\#})^{-1}AA^{\dagger} + [(I + \delta AA^{\#})^{-1}]^{*}(I - AA^{\dagger})\}^{-1}\\
&= (I + C(A))AA^{\dagger}\{(\delta AA^{\#} + AA^{\#})(I + \delta AA^{\#})^{-1}AA^{\dagger} + [(I + \delta AA^{\#})^{-1}]^{*}(I - AA^{\dagger})\}^{-1}\\
&= (I + C(A))AA^{\dagger}\{\bar{A}A^{\#}(I + \delta AA^{\#})^{-1}AA^{\dagger} + [(I + \delta AA^{\#})^{-1}]^{*}(I - AA^{\dagger})\}^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{A}^{\#} &= \bar{A}^{\#}\bar{A}\bar{A}^{\dagger} \\
&= (I + C(A))[D(A) + D(A)^2\delta A A^{\pi}](I - C(A)) \\
&\times (I + C(A))AA^{\dagger}\{\bar{A}A^{\#}(I + \delta A A^{\#})^{-1}AA^{\dagger} + [(I + \delta A A^{\#})^{-1}]^*(I - AA^{\dagger})\}^{-1} \\
&= (I + C(A))D(A)AA^{\dagger}\{\bar{A}(I + A^{\#}\delta A)^{-1}A^{\#}AA^{\dagger} + [(I + \delta A A^{\#})^{-1}]^*(I - AA^{\dagger})\}^{-1} \\
&= (I + C(A))D(A)AA^{\circledast}\{\bar{A}(I + A^{\#}\delta A)^{-1}A^{\circledast} + ((I + \delta A A^{\#})^*)^{-1}(I - AA^{\circledast})\}^{-1}.
\end{aligned}$$

It follows from [27], [8, Theorem 1] and [9, Corollary 3.1], respectively that

$$\|\bar{A}^{\circledast}\| = \|\bar{A}^{\#}\bar{A}\bar{A}^{\dagger}\| \le \|\bar{A}^{\#}\| \le \frac{\|A^{\#}\|}{[1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|]^2}.$$

and

$$\|\bar{A}^{\dagger}\| \le \frac{\|A^{\#}\|}{1 - \|A^{\#}\|\|\delta A\|}, \quad \|\bar{A}^{\dagger} - A^{\dagger}\| \le \frac{1 + \sqrt{5}}{2}\frac{\|A^{\#}\|^2\|\delta A\|}{1 - \|A^{\#}\|\|\delta A\|}.$$

It also follows from the proof of [27, Theorem 3.1], that

$$\|\bar{A}^{\#}\bar{A} - A^{\#}A\| \le \frac{2\|A^{\pi}\|\|A^{\#}\|\|\delta A\|}{1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|}.$$

Using the following equality

$$\bar{A}^{\circledast} - A^{\circledast} = (\bar{A}^{\#}\bar{A} - A^{\#}A)\bar{A}^{\dagger} + A^{\#}A(\bar{A}^{\dagger} - A^{\dagger}),$$

we obtain

$$\begin{aligned}
\|\bar{A}^{\circledast} - A^{\circledast}\| &\le \|\bar{A}^{\#}\bar{A} - A^{\#}A\|\|\bar{A}^{\dagger}\| + \|A^{\#}A\|\|\bar{A}^{\dagger} - A^{\dagger}\| \\
&\le \frac{2\|A^{\pi}\|\|A^{\#}\|\|\delta A\|}{1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|}\frac{\|A^{\#}\|}{1 - \|A^{\#}\|\|\delta A\|} + \|A^{\#}A\|\frac{1 + \sqrt{5}}{2}\frac{\|A^{\#}\|^2\|\delta A\|}{1 - \|A^{\#}\|\|\delta A\|} \\
&= \frac{\|A^{\#}\|^2\|\delta A\|}{1 - \|A^{\#}\|\|\delta A\|}\{\frac{2\|A^{\pi}\|}{1 - (1 + \|A^{\pi}\|)\|A^{\#}\|\|\delta A\|} + \|A^{\#}A\|\frac{1 + \sqrt{5}}{2}\}.
\end{aligned}$$

□

4. Multiplicative Perturbation Analysis

Multiplicative perturbation theory play an important role in analysis of High Relative Accuracy algorithms for a wide range of Numerical Linear Algebra problems. Recently, Multiplicative perturbation analysis of Moore-Penrose inverse has received an increase interest, in part, due to its application to the error estimate of least squares problems.

In this section, we consider multiplicative perturbation analysis of the core inverse of a bounded linear operator. First, we consider expressions of the core inverse of the product of operators.

We start with the notion of the gap between subspaces which plays an important role in perturbation analysis of generalized inverse.

Definition 4.1. [14] *Let M, N be the subspaces of Banach space X. Let*

$$\delta(M, N) = \begin{cases} \sup\{dist(x, N) \,|\, x \in M, \ \|x\| = 1\}, & M \neq \{0\} \\ 0 & M = \{0\} \end{cases},$$

here, $dist(x, N) = \inf\{\|x - y\|, \forall y \in N\}$. We call $\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$ the gap between M and N.

It is easy to verify $dist(x,N) \leq \|x\|\delta(M,N)(\forall x \in M)$ from the definition of the gap between subspaces.

The upper bound of difference of two oblique projectors was presented in the following lemma in term of the gap between subspaces.

Lemma 4.2. [25, Lemma 4.4.1] *Let $P, Q \in B(H)$ be idempotents. Then*

$$\|P - Q\| \leq \|I - P\|\|Q\|\delta(R(Q), R(P)) + \|P\|\|I - Q\|\delta(N(Q), N(P)).$$

Z. Boulmaarouf, et.al., have studied the projections in [3], and presented the following useful lemma:

Lemma 4.3. [3, Proposition 1.7] *Let $P \in B(H)$ be projection. $P \neq 0, P \neq 1$. Then*

$$\|P\| = \|I - P\| = \|I - P - P^*\| = \|I + P - P^*\| = \|I - P + P^*\|.$$

Lemma 4.4. *Let $A, E, F \in B(H)$ and suppose that $R(A), R(E), R(F)$ are closed. Let $T = EAF$. Assume that $E^\dagger EA = A$ and $AFF^\dagger = A$. Then $T^+ = F^\dagger A^\dagger E^\dagger$.*

Proof. The assertion is easy to verify. □

Proposition 4.5. *Let $A, E, F \in B(H)$ and suppose that $R(A), R(E), R(F)$ are closed. Let $T = EAF$. Assume that $E^\dagger EA = A$ and $AFF^\dagger = A$ and $T^{\#}$ exists. Then*

$$T^{\#} = (I + EAF - EAA^\dagger E^\dagger)^{-1} EAA^\dagger E^\dagger (EAA^\dagger E^\dagger + (EAA^\dagger E^\dagger)^* - I)^{-1}.$$

Proof. Noting that $TT^r = EAA^\dagger E^\dagger$, by Lemma 3.7, we have

$$T^{\#} = (I + EAF - EAA^\dagger E^\dagger)^{-1} EAA^\dagger E^\dagger (EAA^\dagger E^\dagger + (EAA^\dagger E^\dagger)^* - I)^{-1}.$$

□

Corollary 4.6. *Let $A, E, F \in B(H)$ with $R(A)$ closed, and let $T = EAF$. Assume E, F are invertible in $B(H)$ and $T^{\#}$ exists. Then*

$$T^{\#} = \{AF + (I - AA^\dagger)E^{-1}\}^{-1} AA^\dagger \{E(AA^\dagger - I) + (E^* EAA^\dagger E^{-1})^*\}^{-1}.$$

Proof. Since E, F are invertible in $B(H)$, by Proposition 4.5, we get

$$\begin{aligned} T^{\#} &= \{I + EAF - EAA^\dagger E^{-1}\}^{-1} EAA^\dagger E^{-1} \{EAA^\dagger E^{-1} + (EAA^\dagger E^{-1})^* - I\}^{-1} \\ &= \{E^{-1} + AF - AA^\dagger E^{-1}\}^{-1} AA^\dagger \{EAA^\dagger + (EAA^\dagger E^{-1})^* E - E\}^{-1} \\ &= \{AF + (I - AA^\dagger)E^{-1}\}^{-1} AA^\dagger \{E(AA^\dagger - I) + (E^* EAA^\dagger E^{-1})^*\}^{-1}. \end{aligned}$$

□

Remark 4.7. *It is known that $T^{\#}$ exists if and only if $T^\#$ exists if and only if $R(T) \oplus N(T) = H$. So, if E, F are invertible, then*

$$R(T) = R(EAF) = R(EA), N(T) = N(EAF) = N(AF).$$

Consequently, $T^{\#}$ exists if and only if $R(EA) \oplus N(AF) = H$.

Next, we consider multiplication perturbation of the core inverse of a bounded linear operator.

Proposition 4.8. *Let $A, E, F \in B(H)$ and $R(A), R(E), R(F)$ be closed subspaces, and let $T = EAF$. Assume that $E^\dagger EA = A$ and $AFF^\dagger = A$ and $A^\#, T^\#$ exist. Then $E^\dagger P_{R(T),N(T)} E$ and $I - FP_{R(T),N(T)} F^\dagger$ are projectors onto $R(A)$ and $N(A)$, respectively. Moreover,*

$$A^\pi E^\dagger (I - T^\pi) E = 0, \qquad (I - E^\dagger (I - T^\pi) E)(I - A^\pi) = 0,$$

$$F(I - T^\pi) F^\dagger A^\pi = 0, \qquad (I - A^\pi)(I - F(I - T^\pi) F^\dagger) = 0.$$

Proof. It is easy to verify using the equalities $E^\dagger EA = A$ and $AFF^\dagger = A$ that $E^\dagger P_{R(T),N(T)}E$ and $FP_{R(T),N(T)}F^\dagger$ are idempotents and $R(A) = R(AF)$ and $N(EA) = N(A)$.

Noting that $R(E^\dagger P_{R(T),N(T)}E) \subseteq R(AF)$ and $E^\dagger P_{R(T),N(T)}EAF = AF$, we have

$$R(E^\dagger P_{R(T),N(T)}E) = R(AF) = R(A).$$

That is, $E^\dagger P_{R(T),N(T)}E$ is a projector onto $R(A)$.

Since $N(EA) \subseteq N(FP_{R(T),N(T)}F^\dagger)$ and $EAFP_{R(T),N(T)}F^\dagger = EA$, we have

$$N(FP_{R(T),N(T)}F^\dagger) = N(EA) = N(A).$$

So, $I - FP_{R(T),N(T)}F^\dagger$ is projector onto $N(A)$.

Since $P_{R(A),N(A)} = I - A^\pi, P_{R(T),N(T)} = I - T^\pi$, the following equations hold.

$$A^\pi E^\dagger(I - T^\pi)E = 0, \qquad (I - E^\dagger(I - T^\pi)E)(I - A^\pi) = 0,$$
$$F(I - T^\pi)F^\dagger A^\pi = 0, \qquad (I - A^\pi)(I - F(I - T^\pi)F^\dagger) = 0.$$

□

Corollary 4.9. *Let $A, E, F \in B(H)$ with $R(A)$ closed and let $T = EAF$. Assume E, F are invertible in $B(H)$ and $A^\#, T^\#$ exist. Then $E^{-1}P_{R(T),N(T)}E$ and $I - FP_{R(T),N(T)})F^{-1}$ are projectors onto $R(A)$ and $N(A)$, respectively. Moreover,*

$$A^\pi E^{-1}(I - T^\pi) = 0, \qquad T^\pi E(I - A^\pi) = 0,$$
$$(I - T^\pi)F^{-1}A^\pi = 0, \qquad (I - A^\pi)FT^\pi = 0.$$

Proof. The results follow from Proposition 4.8. □

Lemma 4.10. *Let $A, E, F \in B(H)$ and suppose that $R(A), R(E), R(F)$ are closed. Let $T = EAF$. Assume that $E^\dagger EA = A$ and $AFF^\dagger = A$. Then*

(1) $\delta(R(A), R(T)) \le \|I - E\|$,
(2) $\delta(N(T), N(A)) \le \|I - F\|$.

Proof. (1) For any $u \in R(A)$ with $\|u\| = 1$, there exists a $y \in Y$ such that $AA^\dagger y = u$. Thus, by $AFF^\dagger = A$, we have

$$\begin{aligned} dist(u, R(T)) &\le \|u - EAF(F^\dagger)A^\dagger y\| \\ &= \|(I - E)AA^\dagger y\| \\ &\le \|I - E\|\|AA^\dagger y\| \\ &\le \|I - E\|\|u\|. \end{aligned}$$

Consequently, $\delta(R(A), R(T)) \le \|I - E\|$.

(2) Noting that $E^\dagger EA = A$, we have $N(EA) = N(A)$. For any $u \in N(T)$ with $\|u\| = 1$, we have $Tu = EAFu = 0$. So, $Fu \in N(EA) = N(A)$. Thus,

$$dist(u, N(A)) \le \|u - Fu\| \le \|I - F\|.$$

Consequently, $\delta(N(T), N(A)) \le \|I - F\|$. □

Corollary 4.11. *Let $A, E, F \in B(H)$ with $R(A)$ closed and let $T = EAF$. Assume E, F are invertible in $B(H)$. Then*

(1) $\delta(R(A), R(T)) \le \|I - E\|$,
(2) $\delta(N(T), N(A)) \le \|I - F\|$.

Utilize the results of Lemma 4.10, we give an estimation of the norm of the oblique projector $P_{R(T),N(T)}$.

Lemma 4.12. *Let $A, E, F \in B(H)$ and suppose that $R(A), R(E), R(F)$ are closed. Let $T = EAF$. Assume that $E^{\dagger}EA = A$ and $AFF^{\dagger} = A$ and $A^{\#}$, $T^{\#}$ exist. Set*

$$\tau = 1 - \|P_{R(A),N(A)}\|(\|I - E\| + \|I - F^{\dagger}\|).$$

If $\tau > 0$, then

$$\|P_{R(T),N(T)}\| \leq \frac{\|P_{R(A),N(A)}\|}{\tau}.$$

Proof. Since $E^{\dagger}EA = A$ and $AFF^{\dagger} = A$, we have $A = E^{\dagger}TF^{\dagger}$. It is easy to check that $EE^{\dagger}T = T, TF^{\dagger}F = T$. Thus, by Lemma 4.10 (2), we have

$$\delta(N(A), N(T)) \leq \|I - F^{\dagger}\|.$$

Therefore, from Lemma 4.2, Lemma 4.3 and Lemma 4.10, we have

$$\begin{aligned}
\|P_{R(T),N(T)} - P_{R(A),N(A)}\| &\leq \|I - P_{R(T),N(T)}\|\|P_{R(A),N(A)}\|\delta(R(A), R(T)) \\
&\quad + \|P_{R(T),N(T)}\|\|I - P_{R(A),N(A)}\|\delta(N(A), N(T)) \\
&\leq \|P_{R(T),N(T)}\|\|P_{R(A),N(A)}\|[\delta(R(A), R(T)) + \delta(N(A), N(T))] \\
&\leq \|P_{R(T),N(T)}\|\|P_{R(A),N(A)}\|(\|I - E\| + \|I - F^{\dagger}\|).
\end{aligned}$$

So,

$$\|P_{R(T),N(T)}\| \leq \|P_{R(A),N(A)}\| + \|P_{R(T),N(T)}\|\|P_{R(A),N(A)}\|(\|I - E\| + \|I - F^{\dagger}\|).$$

Since

$$\tau = 1 - \|P_{R(A),N(A)}\|(\|I - E\| + \|I - F^{\dagger}\|) > 0,$$

we obtain

$$\|P_{R(T),N(T)}\| \leq \frac{\|P_{R(A),N(A)}\|}{\tau}.$$

□

Remark 4.13. *Replacing the assumption of existence of $A^{\#}$ by the assumption that $R(A)$ is closed and replacing $P_{R(A),N(A)}$ by the orthogonal projection $P_{R(A),R(A)^{\perp}}$ in Lemma 4.12, we have*

$$\|P_{R(T),N(T)}\| \leq \frac{1}{\tau},$$

where $\tau = 1 - (\|I - E\| + \|I - F^{\dagger}\|) > 0$.

Corollary 4.14. *Let $A, E, F \in B(H)$ and let $T = EAF$. Assume E, F are invertible in $B(H)$ and $A^{\#}$, $T^{\#}$ exist. Set*

$$\tau = 1 - \|P_{R(A),N(A)}\|(\|I - E\| + \|I - F^{-1}\|).$$

If $\tau > 0$, then

$$\|P_{R(T),N(T)}\| \leq \frac{\|P_{R(A),N(A)}\|}{\tau}.$$

Theorem 4.15. *Let $A, E, F \in B(H)$ with $R(A), R(E), R(F)$ closed and let $T = EAF$ and $A^{\circledast}$, $T^{\circledast}$ exist. Assume that $E^{\dagger}EA = A$ and $AFF^{\dagger} = A$. Set*

$$\tau = 1 - \|A^{\circledast}A\|(\|I - E\| + \|I - F^{\dagger}\|).$$

If $\tau > 0$, then

(1) $T^{\circledast} = P_{R(T),N(T)}F^{\dagger}A^{\circledast}E^{\dagger}Q_{R(T),R(T)^{\perp}} = (I + \theta_F)A^{\circledast}(I + \theta_E)$,

(2) $\|T^{\#}\| \le \dfrac{\|A^{\#}A\|\|F^{\dagger}\|\|A^{\#}\|\|E^{\dagger}\|}{\tau}$

(3) $\dfrac{\|T^{\#} - A^{\#}\|}{\|A^{\#}\|} \le \|E - I\| + \|I - E^{\dagger}\| + \dfrac{\|A^{\#}A\|}{\tau}(\|E - I\| + \|I - F^{\dagger}\|)(\|E\| + \|E^{\dagger}\|)$.

Here,

$$\theta_F = (I - P_{R(T),N(T)})(E - I) - P_{R(T),N(T)}(I - F^{\dagger}),$$
$$\theta_E = (E^* - I)(I - Q_{R(T),R(T)^{\perp}}) - (I - E^{\dagger})Q_{R(T),R(T)^{\perp}}.$$

Proof. Put

$$\theta_F = (I - P_{R(T),N(T)})(E - I) - P_{R(T),N(T)}(I - F^{\dagger}),$$
$$\theta_E = (E^* - I)(I - Q_{R(T),R(T)^{\perp}}) - (I - E^{\dagger})Q_{R(T),R(T)^{\perp}}.$$

Since $R(T) = R(EAF) \subseteq R(EA) = R(EAFF^{\dagger}) \subseteq R(T)$, then $R(T) = R(EA)$. Thus, $(I - P_{R(T),N(T)})EA = 0$ and $(I - Q_{R(T),R(T)^{\perp}})EA = 0$. So, we have $(I - P_{R(T),N(T)})EA^{\#} = 0$ and $(I - Q_{R(T),R(T)^{\perp}})EQ_{R(A),R(A)^{\perp}} = 0$ which is equivalent to $Q_{R(A),R(A)^{\perp}}[I + (E - I)^*](I - Q_{R(T),R(T)^{\perp}}) = 0$. Therefore,

$$Q_{R(A),R(A)^{\perp}}[I + (E - I)^*(I - Q_{R(T),R(T)^{\perp}})] = Q_{R(A),R(A)^{\perp}}Q_{R(T),R(T)^{\perp}}.$$

Noting that

$$(I + \theta_F)A^{\#} = P_{R(T),N(T)}F^{\dagger}A^{\#}$$

and

$$\begin{aligned} A^{\dagger}E^{\dagger}Q_{R(T),R(T)^{\perp}} &= A^{\dagger}[I + E^{\dagger} - I]Q_{R(T),R(T)^{\perp}} \\ &= A^{\dagger}[Q_{R(A),R(A)^{\perp}} + E^{\dagger} - I]Q_{R(T),R(T)^{\perp}} \\ &= A^{\dagger}[Q_{R(A),R(A)^{\perp}}(I + (E - I)^*(I - Q_{R(T),R(T)^{\perp}})) + (E^{\dagger} - I)Q_{R(T),R(T)^{\perp}}] \\ &= A^{\dagger}[I + (E - I)^*(I - Q_{R(T),R(T)^{\perp}}) + (E^{\dagger} - I)Q_{R(T),R(T)^{\perp}}] \\ &= A^{\dagger}(I + \theta_E), \end{aligned}$$

we have

$$T^{\#} = P_{R(T),N(T)}F^{\dagger}A^{\#}E^{\dagger}Q_{R(T),R(T)^{\perp}} = (I + \theta_F)A^{\#}(I + \theta_E).$$

Then, by Lemma 4.12,

$$\|T^{\#}\| \le \|P_{R(T),N(T)}\|\|F^{\dagger}\|\|A^{\#}\|\|E^{\dagger}\| = \frac{\|A^{\#}A\|\|F^{\dagger}\|\|A^{\#}\|\|E^{\dagger}\|}{\tau}.$$

Using Lemma 4.3, we have

$$\begin{aligned} \|\theta_F\| &\le \|P_{R(T),N(T)}\|(\|E - I\| + \|I - F^{\dagger}\|) \\ \|\theta_E\| &\le \|E - I\| + \|I - E^{\dagger}\| \\ \|I + \theta_E\| &= \|E^*(I - Q_{R(T),R(T)^{\perp}}) + E^{\dagger}Q_{R(T),R(T)^{\perp}}\| \\ &\le \|E\| + \|E^{\dagger}\|. \end{aligned}$$

It is easy to verify

$$T^{\#} - A^{\#} = A^{\#}\theta_E + \theta_F A^{\#} + \theta_F A^{\#}\theta_E.$$

Thus,

$$\begin{aligned} \|T^{\#} - A^{\#}\| &= \|A^{\#}\theta_E + \theta_F A^{\#}(I + \theta_E)\| \\ &\le \|A^{\#}\|(\|\theta_E\| + \|\theta_F\|\|I + \theta_E\|) \\ &\le \|A^{\#}\|\{\|E - I\| + \|I - E^{\dagger}\| + \|P_{R(T),N(T)}\|(\|E - I\| + \|I - F^{\dagger}\|)(\|E\| + \|E^{\dagger}\|)\} \\ &\le \|A^{\#}\|\{\|E - I\| + \|I - E^{\dagger}\| + \frac{\|A^{\#}A\|}{\tau}(\|E - I\| + \|I - F^{\dagger}\|)(\|E\| + \|E^{\dagger}\|)\}. \end{aligned}$$

Consequently,

$$\frac{\|T^{\circledast} - A^{\circledast}\|}{\|A^{\circledast}\|} \leq \|E - I\| + \|I - E^{\dagger}\| + \frac{\|A^{\circledast}A\|}{\tau}(\|E - I\| + \|I - F^{\dagger}\|)(\|E\| + \|E^{\dagger}\|).$$

□

Corollary 4.16. *Let $A, E, F \in B(H)$, $T = EAF$ and suppose $A^{\circledast}$, $T^{\circledast}$ exist. Assume E, F are invertible in $B(H)$. Set*

$$\tau = 1 - \|A^{\circledast}A\|(\|I - E\| + \|I - F^{-1}\|).$$

If $\tau > 0$, then

(1) $T^{\circledast} = P_{R(T),N(T)}F^{-1}A^{\circledast}E^{-1}Q_{R(T),R(T)^{\perp}} = (I + \theta_F)A^{\circledast}(I + \theta_E)$,

(2) $\|T^{\circledast}\| \leq \dfrac{\|A^{\circledast}A\|\|F^{-1}\|\|A^{\circledast}\|\|E^{-1}\|}{\tau}$,

(3) $\dfrac{\|T^{\circledast} - A^{\circledast}\|}{\|A^{\circledast}\|} \leq \|E - I\| + \|I - E^{-1}\| + \dfrac{\|A^{\circledast}A\|}{\tau}(\|E - I\| + \|I - F^{-1}\|)(\|E\| + \|E^{-1}\|)$.

Here,

$$\theta_F = (I - P_{R(T),N(T)})(E - I) - P_{R(T),N(T)}(I - F^{-1}),$$
$$\theta_E = (E^* - I)(I - Q_{R(T),R(T)^{\perp}}) - (I - E^{-1})Q_{R(T),R(T)^{\perp}}.$$

Similar to the proof of Theorem 4.15, associate with Remark 4.13, we have the following results if $R(A)$ is closed and $A^{\#}$ does not necessary exist.

Theorem 4.17. *Let $A, E, F \in B(H)$ with $R(A), R(E), R(F)$ closed and let $T = EAF$ with $T^{\circledast}$ exist. Assume that $E^{\dagger}EA = A$ and $AFF^{\dagger} = A$. Set*

$$\tau = 1 - (\|I - E\| + \|I - F^{\dagger}\|).$$

If $\tau > 0$, then

(1) $T^{\circledast} = P_{R(T),N(T)}F^{\dagger}A^{\dagger}E^{\dagger}Q_{R(T),R(T)^{\perp}} = (I + \beta_F)A^{\dagger}(I + \theta_E)$,

(2) $\|T^{\circledast}\| \leq \dfrac{\|F^{\dagger}\|\|A^{\dagger}\|\|E^{\dagger}\|}{\tau}$

(3) $\dfrac{\|T^{\circledast} - A^{\dagger}\|}{\|A^{\dagger}\|} \leq \|E - I\| + \|E^{\dagger} - I\| + (\dfrac{\|F\| + \|F^{\dagger}\|}{\tau} + \|F - I\|)(\|E\| + \|E^{\dagger}\|)$.

Here,

$$\beta_F = P_{R(T),N(T)}F^{\dagger} + (I - P^*_{R(T),N(T)})F^* - I,$$
$$\theta_E = (E^* - I)(I - Q_{R(T),R(T)^{\perp}}) - (I - E^{\dagger})Q_{R(T),R(T)^{\perp}}.$$

Proof. Since $N(T) \subseteq N(E^{\dagger}EAF) = N(AF) \subseteq N(T)$, then $N(T) = N(AF)$. Thus,

$$A^{\dagger}AF(I - P_{R(T),N(T)}) = 0.$$

Noting that

$$\begin{aligned}(P_{R(T),N(T)}F^{\dagger}A^{\dagger})^* &= (A^{\dagger})^*(F^{\dagger})^*P^*_{R(T),N(T)} \\ &= (A^{\dagger})^*\{(F^{\dagger})^*P^*_{R(T),N(T)} + A^{\dagger}AF(I - P_{R(T),N(T)})\} \\ &= (A^{\dagger})^*\{(F^{\dagger})^*P^*_{R(T),N(T)} + F(I - P_{R(T),N(T)})\},\end{aligned}$$

we have

$$P_{R(T),N(T)}F^{\dagger}A^{\dagger} = \{P_{R(T),N(T)}F^{\dagger} + (I - P^*_{R(T),N(T)})F^*\}A^{\dagger}.$$

Put $\beta_F = P_{R(T),N(T)}F^\dagger + (I - P^*_{R(T),N(T)})F^* - I$. Then, we have, from the proof of Theorem 4.15,

$$T^{\circledast} = P_{R(T),N(T)}F^\dagger A^\dagger E^\dagger Q_{R(T),R(T)^\perp} = (I + \beta_F)A^\dagger(I + \theta_E),$$

where $\theta_E = (E^* - I)(I - Q_{R(T),R(T)^\perp}) - (I - E^\dagger)Q_{R(T),R(T)^\perp}$.

It is easy to check

$$T^{\circledast} - A^\dagger = A^\dagger\theta_E + \beta_F A^\dagger + \beta_F A^\dagger\theta_E.$$

Since

$$\begin{aligned}||\beta_F|| &= || - P^*_{R(T),N(T)}F^* + P_{R(T),N(T)}F^\dagger + F^* - I||\\ &\leq ||P_{R(T),N(T)}||(||F|| + ||F^\dagger||) + ||F - I||\\ &\leq \frac{||F|| + ||F^\dagger||}{\tau} + ||F - I||,\end{aligned}$$

then,

$$\begin{aligned}||T^{\circledast} - A^\dagger|| &\leq ||A^\dagger\theta_E|| + ||\beta_F A^\dagger(I + \theta_E)||\\ &\leq ||A^\dagger||(||E - I|| + ||E^\dagger - I||) + ||A^\dagger||(\frac{||F|| + ||F^\dagger||}{\tau} + ||F - I||)(||E|| + ||E^\dagger||).\end{aligned}$$

Thus,

$$\frac{||T^{\circledast} - A^\dagger||}{||A^\dagger||} \leq ||E - I|| + ||E^\dagger - I|| + (\frac{||F|| + ||F^\dagger||}{\tau} + ||F - I||)(||E|| + ||E^\dagger||).$$

□

Corollary 4.18. *Let $A, E, F \in B(H)$, $T = EAF$ with $R(A)$ closed and $T^{\circledast}$ exists. Assume E, F are invertible in $B(H)$. Set*

$$\tau = 1 - (||I - E|| + ||I - F^{-1}||).$$

If $\tau > 0$, then

(1) $T^{\circledast} = P_{R(T),N(T)}F^{-1}A^\dagger E^{-1}Q_{R(T),R(T)^\perp} = (I + \beta_F)A^\dagger(I + \theta_E)$,

(2) $||T^{\circledast}|| \leq \dfrac{||F^{-1}||||A^\dagger||||E^{-1}||}{\tau}$,

(3) $\dfrac{||T^{\circledast} - A^\dagger||}{||A^\dagger||} \leq ||E - I|| + ||E^{-1} - I||) + (\dfrac{||F|| + ||F^{-1}||}{\tau} + ||F - I||)(||E|| + ||E^{-1}||)$.

Here,

$$\begin{aligned}\beta_F &= P_{R(T),N(T)}F^{-1} + (I - P^*_{R(T),N(T)})F^* - I,\\ \theta_E &= (E^* - I)(I - Q_{R(T),R(T)^\perp}) - (I - E^{-1})Q_{R(T),R(T)^\perp}.\end{aligned}$$

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