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Shift Commutator Algebras and Multipliers

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Abstract. We determine the precise structure of all multipliers on the commutator algebra associated to the shift operator on a Hilbert space. The problem has its own interest by its connection with the theory of Toeplitz and Laurent operators.

1. Preliminar

If $\mathcal{U}$ is a vector space $\mathcal{L}(\mathcal{U})$ will denote the class of linear endomorphisms of $\mathcal{U}$. Further, if $\mathcal{U}$ is an algebra let $\mathcal{M}(\mathcal{U})$ be the set of multipliers of $\mathcal{U}$. Precisely, an element $M \in \mathcal{L}(\mathcal{U})$ is called a multiplier if $M(a)b = aM(b)$ for all $a, b \in \mathcal{U}$. The notion of multiplier was introduced by S. Helgason in 1956 [5]. If $\mathcal{U}$ is a Banach algebra then any multiplier on $\mathcal{U}$ is bounded and $\mathcal{M}(\mathcal{U})$ is a closed subalgebra of $\mathcal{B}(\mathcal{U})$ [6]. Our aim in this paper is to determine the multipliers of the commutator of the algebra generated by the unilateral or the bilateral shift on an underlying Hilbert space. Whence $\mathcal{U}$ will be Banach subalgebra of operators or a C^*-algebra of operators respectively. In the first case, the elements of $\mathcal{U}$ will be represented by (finite or infinite) lower triangular (scalar or operator) matrices with constant diagonals, while the second case (in infinite dimension) allows infinite matrices with constant diagonals. Thus, there is a close connection with the theory of Toeplitz and Laurent operators which are of special interest to researches of Asymptotic Linear Algebra and Functional Analysis [2].

As usual, if $r \in \mathbb{N}$ by $\mathbb{M}_r(\mathbb{C})$ we shall denote the set of $r \times r$ matrices over $\mathbb{C}$ endowed with the Hilbert-Schmidt norm defined for $\left(z_{i,j}\right)_{i,j=1}^{r} \in \mathbb{M}_r(\mathbb{C})$ as

$$\left\| \left(z_{i,j}\right)_{i,j=1}^{r} \right\|_2 \triangleq \left[\sum_{i,j=1}^{r} \left|z_{i,j}\right|^2 \right]^{1/2} .$$

As a consequence of the Cauchy-Schwartz inequality $\mathbb{M}_r(\mathbb{C})$ is a Banach algebra. If $n \in \mathbb{N}$ and $z \in \mathbb{M}_r(\mathbb{C})^n$ let $\|z\|_1 \triangleq \sum_{k=1}^{n} \|z_k\|_2$. Then $\left(\mathbb{M}_r(\mathbb{C})^n, \|\circ\|_1\right)$ becomes an nr^2-dimensional Banach space on $\mathbb{C}$. Indeed, $\mathbb{M}_r(\mathbb{C})^n$ is also an associative Banach algebra if for $z, w \in \mathbb{M}_r(\mathbb{C})^n$ we define $z \cdot w \in \mathbb{M}_r(\mathbb{C})^n$ as

$$z \cdot w \triangleq (z_1 w_1, z_1 w_2 + z_2 w_1, ..., z_1 w_n + ... + z_n w_1) .$$

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So, $\mathbb{M}_r(\mathbb{C})^n$ is unitary and it is abelian if and only if $r = 1$. If $L \in \mathcal{L}(\mathbb{M}_r(\mathbb{C})^n)$ we will write $\|L\| \triangleq \sup_{\|z\|_1=1} \|L(z)\|_1$. Certainly, $(\mathcal{L}(\mathbb{M}_r(\mathbb{C})^n), \|\circ\|)$ is a Banach space. More generally, let $\mathcal{H}$ be a separable complex Hilbert space. Then, let the unilateral shift $S^+ \in \mathcal{B}(\oplus_1^\infty \mathcal{H})$ and the bilateral shift $S \in \mathcal{B}(\oplus_{-\infty}^{+\infty}\mathcal{H})$ be given as

$$S^+(f_1, f_2, ...) \triangleq (0, f_1, f_2, ...) \quad if \quad (f_1, f_2, ...) \in \oplus_1^\infty \mathcal{H},$$
$$S(\{f_m\}_{m\in\mathbb{Z}}) = \{f_{m-1}\}_{m\in\mathbb{Z}} \quad if \quad \{f_m\}_{m\in\mathbb{Z}} \in \oplus_{-\infty}^{+\infty}\mathcal{H}.$$

Clearly, $\oplus_1^\infty \mathcal{H}$ can be identified with a Hilbert subspace of $\oplus_{-\infty}^{+\infty}\mathcal{H}$. If $i, j \in \mathbb{Z}$ let $\pi_i : \oplus_{-\infty}^{+\infty}\mathcal{H} \to \mathcal{H}$ be the natural projection of $\oplus_{-\infty}^{+\infty}\mathcal{H}$ onto $\mathcal{H}$ and and let $\iota_j : \mathcal{H} \hookrightarrow \oplus_{-\infty}^{+\infty}\mathcal{H}$ be the natural injection of $\mathcal{H}$ into $\oplus_{-\infty}^{+\infty}\mathcal{H}$. By $\{S^+\}^c$ and $\{S\}^c$ we will denote the commutators of $\{S^+\}$ and $\{S\}$ respectively.

The following is a well known consequence of the Putnam-Fuglede theorem: If μ is a compactly supported measure on $\mathbb{C}$ then $\left\{N_\mu\right\}^c \approx \left\{M_\phi : \phi \in L^\infty(\mu)\right\}$, where $N_\mu, M_\phi \in \mathcal{B}\left(L^2(\mu)\right)$ are given as $N_\mu(f) = zf$, $M_\phi = \phi f$ for $f \in L^2(\mu)$, $\phi \in L^\infty(\mu)$ (cf. [3], Corollary 6.9; [4], [8]). Precisely, operators like N_μ behaves similarly to shift ones and they constitute a central tool to the multiplicity theory of normal operators. Particularly, our interest in multipliers was motivated in previous studies concerning about the structure of derivations on nonamenable nuclear Banach algebras, X-Hadamard and B-derivations [1], [7].

The article is organized in two sections: Section 2 in the context of finite dimensionality, and Section 3 for the infinite dimensional case. Then $\mathcal{M}(\{S^+\}^c)$ is characterized in the Corollary 2.7 of the Section 2 and in Prop. 3.1 of the Section 3, while the multipliers on $\{S\}^c$ will be described in Corollary 3.7.

2. Multipliers on finite matrix algebras

Proposition 2.1. *Let $M \in \mathcal{L}(\mathbb{M}_r(\mathbb{C})^n)$. Then $M \in \mathcal{M}(\mathbb{M}_r(\mathbb{C})^n)$ if and only if there is a unique $a \in Z[\mathbb{M}_r(\mathbb{C})^n]$ so that $M(z) = a \cdot z$.*

Proof. If $M \in \mathcal{M}(\mathbb{M}_r(\mathbb{C})^n)$ we put $a \triangleq M(e)$, where $e = (1_{r\times r}, 0, ..., 0)$ and $1_{r\times r}$ is the unit matrix of $\mathbb{M}_r(\mathbb{C})$. Then e is the unit of $\mathbb{M}_r(\mathbb{C})^n$ and if $z \in \mathbb{M}_r(\mathbb{C})^n$ we see that

$$a \cdot z = M(e) \cdot z = e \cdot M(z) = M(z) = M(z) \cdot e = z \cdot M(e) = z \cdot a,$$

i.e. $a \in Z[\mathbb{M}_r(\mathbb{C})^n]$. By the associativity of $\mathbb{M}_r(\mathbb{C})^n$ it is immediate that the condition is also sufficient. □

Remark 2.2. *It is readily seen that*

$$Z[\mathbb{M}_r(\mathbb{C})^n] = Z[\mathbb{M}_r(\mathbb{C})]^n = (\mathbb{C} \cdot 1_{r\times r})^n.$$

Corollary 2.3. *If $M \in \mathcal{M}(\mathbb{M}_r(\mathbb{C})^n)$ there are unique $a^1, ..., a^n \in \mathbb{C}$ so that $M(e) = \left(a^1 1_{r\times r}, ..., a^n 1_{r\times r}\right)$ and*

$$M_v(z) = \sum_{j=1}^{v} a^{v-j+1} z_j, 1 \le v \le n, \ z \in \mathbb{M}_r(\mathbb{C})^n.$$

Corollary 2.4. *Let $m : \mathbb{C}^n \to \mathcal{M}(\mathbb{M}_r(\mathbb{C})^n)$ so that*

$$m(a^1, ..., a^n)(z) \triangleq \left(a^1 1_{r\times r}, ..., a^n 1_{r\times r}\right) \cdot z, \ z \in \mathbb{M}_r(\mathbb{C})^n.$$

Then m defines an isomorfism of Banach algebras and $\|m(a)\| = r^{1/2}\|a\|_1$ for all $a \in \mathbb{C}^n$.

Proposition 2.5. *Let $\mathcal{H} = \oplus_1^n \mathbb{C}^r$, and let $S^+ \in \mathcal{L}(\mathcal{H})$ be the shift operator*

$$S^+(f_1, ..., f_n) = (0, f_1, ..., f_{n-1}), \ (f_1, ..., f_n) \in \mathcal{H}.$$

If $T \in \mathcal{L}(\mathcal{H})$, T commutates with S^+ if and only if there exists a matrix

$$\left\|T_{i,j}\right\|_{1\le i,j\le n} \in \mathbb{M}_n\left(\mathcal{L}\left(\mathbb{C}^r\right)\right)$$

so that

$$T_{i,j} = \begin{cases} 0 & if \quad 1 \le i < j \le n, \\ T_{i+1,j+1} & if \quad 1 \le j \le i < n \end{cases}$$

and given $f \in \mathcal{H}$ is

$$T(f) = \left(T_{1,1}f_1, T_{2,1}f_1 + T_{1,1}f_2, ..., T_{n,1}f_1 + T_{n-1,1}f_2 + ... + T_{1,1}f_n\right).$$

Proposition 2.6. *Let $\lambda : \mathbb{M}_r(\mathbb{C})^n \to \{S^+\}^c$ be given as*

$$\lambda(z)(f) \triangleq \left(z_1 f_1, z_2 f_1 + z_1 f_2, ..., z_n f_1 + z_{n-1} f_2 + ... + z_1 f_n\right),$$

where $z \in \mathbb{M}_r(\mathbb{C})^n$ and $f \in \mathcal{H}$. Then λ is an isomorphism of algebras.

Corollary 2.7. *With the notation of Prop. 2.5, any multiplier $\mu \in \mathcal{M}(\{S^+\}^c)$ is uniquely determined by a set of scalars $a^1, ..., a^n \in \mathbb{C}$ so that if $T \in \{S^+\}^c$, $1 \le k \le n$ and $f \in \mathcal{H}$ then*

$$\mu(T)_k(f) = \sum_{j=1}^{k}\sum_{i=1}^{j} a^{j-i+1} T_{i,1}(f_{k-j+1}). \tag{1}$$

3. Multipliers on infinite matrix algebras

Proposition 3.1. *Any multiplier $\mu \in \mathcal{M}(\{S^+\}^c)$ is uniquely determined by a sequence of scalars $a = \{a^n\}_{n\in\mathbb{N}}$ so that*

$$\sup_{f\in\left[\oplus_1^\infty \mathcal{H}\right]_1} \left[\sum_{k=1}^{\infty}\left\|\sum_{j=1}^{k} a^{k-j+1} f_j\right\|^2\right]^{1/2} < +\infty \tag{2}$$

and the following equalities

$$\mu(T)_k(f) = \sum_{j=1}^{k} a^{k-j+1}\sum_{i=1}^{j} T_{j-i+1,1}(f_i) = \sum_{j=1}^{k} T_{k-j+1,1}\left(\sum_{i=1}^{j} a^{j-i+1} f_i\right) \tag{3}$$

hold if $T \in \{S^+\}^c$, $k \in \mathbb{N}$ and $f \in \oplus_1^\infty \mathcal{H}$.

Proof. It is straightforward to see that any $T \in \mathcal{B}\left(\oplus_1^\infty \mathcal{H}\right)$ is performed by the infinite matrix $\left\|T_{i,j}\right\|_{i,j\in\mathbb{N}}$ with coefficients in $\mathcal{B}(\mathcal{H})$, where

$$T_{i,j} \triangleq \pi_i \mid_{\oplus_1^\infty \mathcal{H}} \circ T \circ \iota_j \mid^{\oplus_1^\infty \mathcal{H}},$$

and $T = s\text{-}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \iota_i \mid^{\oplus_1^\infty \mathcal{H}} \circ T_{i,j} \circ \pi_j \mid_{\oplus_1^\infty \mathcal{H}}$ (cf. [3], §6, p. 276). Moreover, if $T \in \{S^+\}^c$ then $T_{i,j} = T_{i+1,j+1}$ if $1 \le j \le i$ and $T_{i,j} = 0$ if $j > i \ge 1$. Hence, if $\mu \in \mathcal{M}(\{S^+\}^c)$ then $\mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right) \in Z(\{S^+\}^c)$ and it is easy to deduce the existence of a sequence of scalars $a = \{a^n\}_{n=1}^\infty$ so that $\mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right)_{j,1} = a^j Id_{\mathcal{H}}$ if $j \in \mathbb{N}$. Now, given $T \in \{S^+\}^c$,

$k \in \mathbb{N}$ and $f \in \oplus_1^\infty \mathcal{H}$ we write

$$\begin{aligned}
\mu(T)_k(f) &= \left(Id_{\oplus_1^\infty \mathcal{H}} \circ \mu(T)\right)_k(f) \\
&= \left(\mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right) \circ T\right)_k(f) \\
&= \mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right)_k(Tf) \\
&= \sum_{j=1}^{k} \mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right)_{k-j+1,1}(Tf)_j \\
&= \sum_{j=1}^{k} a^{k-j+1} \sum_{i=1}^{j} T_{j-i+1,1}(f_i).
\end{aligned}$$

The second equality of (3) is always true, and it follows similarly because

$$\mu(T) = \mu(T) \circ Id_{\oplus_1^\infty \mathcal{H}} = T \circ \mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right).$$

Indeed,

$$\|\mu\| = \left\|\mu\left(Id_{\oplus_1^\infty \mathcal{H}}\right)\right\| = \sup_{f \in \left[\oplus_1^\infty \mathcal{H}\right]_1} \left[\sum_{k=1}^{\infty} \left\|\sum_{j=1}^{k} a^{k-j+1} f_j\right\|^2\right]^{1/2} < +\infty.$$

On the other hand, let $a = \{a^n\}_{n \in \mathbb{N}}$ be a sequence of scalars so that (2) holds. Hence, if $f \in \oplus_1^\infty \mathcal{H}$ then $\mu_a(f) \triangleq \left\{\sum_{j=1}^{k} a^{k-j+1} f_j\right\}_{k \in \mathbb{N}}$ is a well defined element of $\oplus_1^\infty \mathcal{H}$ and μ_a becomes a continuous linear operator on $\oplus_1^\infty \mathcal{H}$. Further, by (3) we deduce that $\mu_a \in Z(\{S^+\}^c)$. Therefore, if we set $\mu(T) = \mu_a \circ T$ for $T \in \{S^+\}$ then $\mu \in \mathcal{M}(\{S^+\}^c)$ and our claim follows. $\square$

Remark 3.2. *If a multiplier μ on $\{S^+\}^c$ is implemented by a sequence a it is easy to see that $a \in l^2(\mathbb{N})$ and $\|a\|_2 \leq \|\mu\|$. In general, this inequality is strict. For instance, let $\mathcal{H} = \mathbb{C}$ and consider $a = \left\{2^{-1/2}, 2^{-1/2}, 0, 0, ...\right\}$. If μ_a is the multiplier performed by a on the subalgebra $\{S^+\}^c$ of $\mathcal{B}\left(l^2(\mathbb{N})\right)$ then*

$$\|\mu_a\| = \left\|\mu_a\left(Id_{l^2(\mathbb{N})}\right)\right\| = \sup_{f \in \left[l^2(\mathbb{N})\right]_1} \left(1 + \operatorname{Re} \sum_{n=1}^{\infty} f_n \overline{f_{n+1}}\right)^{1/2}.$$

Consequently, taking $f = \left\{2^{-n/2}\right\}_{n \geq 1}$ we conclude that $\|\mu_a\| > \|a\|_2 = 1$.

Lemma 3.3. *Let $A \in \mathcal{B}(\oplus_{-\infty}^{+\infty} \mathcal{H})$. Then $A \in \{S\}^c$ if and only if there is a unique bounded sequence $\{A_m\}_{m \in \mathbb{Z}}$ in $\mathcal{B}(\mathcal{H})$ so that $Af = \left\{\sum_{m \in \mathbb{Z}} A_{q-m} f_m\right\}_{q \in \mathbb{Z}}$ and the extended number*

$$\eta \triangleq \eta(\{A_m\}_{m \in \mathbb{Z}}) \triangleq \sup_{f \in \left[\oplus_{-\infty}^{+\infty} \mathcal{H}\right]_1} \left(\sum_{q \in \mathbb{Z}} \left\|\sum_{m \in \mathbb{Z}} A_{q-m} f_m\right\|^2\right)^{1/2} \tag{4}$$

is finite.

Proof. It is readily seen that $Id_{\oplus_{-\infty}^{+\infty} \mathcal{H}} = s - \sum_{m \in \mathbb{Z}} \iota_m \pi_m$. So, if $A \in \mathcal{B}(\oplus_{-\infty}^{+\infty} \mathcal{H})$ then

$$A = s - \sum_{m \in \mathbb{Z}} A \iota_m \pi_m = s - \sum_{m \in \mathbb{Z}} s - \sum_{n \in \mathbb{Z}} \iota_n \pi_n A \iota_m \pi_m. \tag{5}$$

Hence, if $A_{n,m} \triangleq \pi_n A \iota_m$ for $n, m \in \mathbb{Z}$ then $A_{n,m} \in \mathcal{B}(\mathcal{H})$ and $\left\|A_{n,m}\right\| \leq \|A\|$. If $q \in \mathbb{Z}$ we see that $\pi_q A = s - \sum_{m\in\mathbb{Z}} A_{q,m}\pi_m$. Now, let us suppose that $A \in \{S\}^c$. Then if $f \in \oplus_{-\infty}^{+\infty}\mathcal{H}$ we obtain

$$\sum_{m\in\mathbb{Z}} A_{q,m}\left(f_{m-1}\right) = \pi_q AS(f) = \pi_q SA(f) = \pi_{q-1}A(f) = \sum_{m\in\mathbb{Z}} A_{q-1,m-1} f\left(_{m-1}\right).$$

It is immediate that $A_{q,m} = A_{q-1,m-1}$ for all $m, q \in \mathbb{Z}$ and we can write $A_{q,m} \triangleq A_{q-m}$. In particular, $\eta = \|A\| < +\infty$ and the condition is necessary. Now, for $q \in \mathbb{Z}$ and $f \in \mathcal{H}^{(\mathbb{Z})}$ let $A^q(f) \triangleq \sum_{m\in\mathbb{Z}} A_{q-m} f_m$. By (4) we have $\left\|A^q(f)\right\| \leq \eta \left\|f\right\|$ and so A^q extends to an element $A^q \in \mathcal{B}\left(\oplus_{-\infty}^{+\infty}\mathcal{H};\mathcal{H}\right)$. If we set $A(f) \triangleq \{A^q(f)\}_{q\in\mathbb{Z}}$ by (4) we see that $A \in \mathcal{B}(\oplus_{-\infty}^{+\infty}\mathcal{H})$. Finally, if $f \in \oplus_{-\infty}^{+\infty}\mathcal{H}$ and $q \in \mathbb{Z}$ then

$$\pi_q AS(f) = \sum_{m\in\mathbb{Z}} A_{q-m}(f_{m-1}) = \sum_{m\in\mathbb{Z}} A_{(q-1)-m}(f_m) = A^{q-1}(f) = \pi_q SA(f),$$

i.e. $A \in \{S\}^c$. □

Corollary 3.4. *Let $\mathfrak{T}(\mathcal{H})$ be the set of bounded sequences $\left\{A_q\right\}_{q\in\mathbb{Z}}$ of bounded linear operators on $\mathcal{H}$ so that the extended number $\eta\left(\left\{A_q\right\}_{q\in\mathbb{Z}}\right)$ in (4) is finite. Then $(\mathfrak{T}(\mathcal{H}), \eta)$ has a natural norm space structure. Moreover, it is a Banach algebra isometrically isomorphic with $\{S\}^c$.*

Proof. Clearly $(\mathfrak{T}(\mathcal{H}), \eta)$ is a norm space. As in Prop. 2.6, we consider the following assignment induced by the Lemma 3.3:

$$\Lambda : \mathcal{T}(\mathcal{H}) \to \{S\}^c,\ \Lambda\left(\left\{A_q\right\}_{q\in\mathbb{Z}}\right) \triangleq A.$$

In particular, $\Lambda\left(\{\delta_{m,0} Id_{\mathcal{H}}\}_{m\in\mathbb{Z}}\right) = Id_{\mathcal{H}}$ and $\Lambda\left(\{\delta_{m,1} Id_{\mathcal{H}}\}_{m\in\mathbb{Z}}\right) = S$. Indeed, $\eta\left(\Lambda\left(\left\{A_q\right\}_{q\in\mathbb{Z}}\right)\right) = \|A\|$, i.e. Λ is an isometry. If $\alpha, \beta \in \mathcal{T}(\mathcal{H})$ we set

$$\alpha \cdot \beta \triangleq \Lambda^{-1}\left(\Lambda(\alpha) \circ \Lambda(\beta)\right).$$

Therefore, if $\alpha = \left\{A_q\right\}_{q\in\mathbb{Z}}$ and $\beta = \{B_r\}_{r\in\mathbb{Z}}$ then

$$\alpha \cdot \beta = \left\{s - \sum_{m\in\mathbb{Z}} A_{q-m} B_m\right\}_{q\in\mathbb{Z}}.$$

□

Remark 3.5. *Let $\mathbb{T} : |z| = 1$ be the complex unitary circumference endowed with the normalized Lebesgue measure. If $a \in L^1(\mathbb{T})$ and $m \in \mathbb{Z}$ the m-th Fourier coefficient of a is given as*

$$\widehat{a}(m) = \frac{1}{2\pi i}\int_{\mathbb{T}} a(z)\overline{z}^m \frac{dz}{z}.$$

Let M^p be the collection of all $a \in L^1(\mathbb{T})$ so that $\widehat{a} * z \in l^p(\mathbb{Z})$ whenever $z \in \mathbb{C}^{(\mathbb{Z})}$ and $\sup_{z\in\mathbb{C}^{(\mathbb{Z})}:\|z\|=1} \left\|\widehat{a} * z\right\|_p < +\infty$, where $\widehat{a} * z = \{\sum_{m\in\mathbb{Z}} \widehat{a}(q-m) z_m\}_{q\in\mathbb{Z}}$. Given $a \in M^p$ the operator $\mathbb{C}^{(\mathbb{Z})} \to l^p(\mathbb{Z})$, $z \to a * z$ extends to an operator $L(a) \in \mathcal{B}(l^p(\mathbb{C}))$. Usually, $L(a)$ is known as the Laurent operator generated by a. On the other hand, given $A \in \mathcal{B}(l^p(\mathbb{C}))$ there is an $a \in M^p$ so that $A = L(a)$ and $\widehat{a}(n-m) = A_n(e_m)$ for all $n, m \in \mathbb{Z}$ (cf. [2], Prop. 2.4). Indeed, the following generalization of a well known Toeplitz theorem holds: Let $1 \leq p < +\infty$ and let $A \in \mathcal{B}(L^p(\mathbb{T}))$ so that there is a sequence of complex numbers $\{a_m\}_{m\in\mathbb{Z}}$ so that $\langle Az^m, z^n\rangle = a_{n-m}$. Thus there is an $a \in L^\infty(\mathbb{T})$ so that $Af = af$ for all $f \in L^p(\mathbb{T})$, $\{a_m\}_{m\in\mathbb{Z}}$ becomes to be the Fourier coefficient sequence of a and $\|A\| = \|a\|_\infty$ (cf. [9] and Prop. 2.2 of [2]). As besides there is an isometric isomorphism $l^2(\mathbb{Z}) \approx L^2(\mathbb{T})$ then $M^2 = L^\infty(\mathbb{T})$.

Proposition 3.6. *If $A \in \mathcal{B}(\oplus_{-\infty}^{+\infty}\mathcal{H})$, $A \in Z(\{S\}^c)$ if and only if there exists $a \in L^\infty(\mathbb{T})$ so that*

$$\{\widehat{a}(m)\, Id_{\mathcal{H}}\}_{m\in\mathbb{Z}} \in \mathfrak{T}(\mathcal{H}) \tag{6}$$

and

$$Af = \left\{\sum_{m\in\mathbb{Z}} \widehat{a}(q-m) f_m\right\}_{q\in\mathbb{Z}} \quad \text{if } f \in \oplus_{-\infty}^{+\infty}\mathcal{H}. \tag{7}$$

Proof. (⇒) First, $\Lambda^{-1}(A) \subseteq Z(\mathcal{B}(\mathcal{H}))$. For, if $b \in \mathcal{B}(\mathcal{H})$ then

$$\eta\left(\{\delta_{m,0} b\}_{m\in\mathbb{Z}}\right) \le \|b\| < +\infty.$$

So, $B \triangleq \Lambda\left(\{\delta_{m,0} b\}_{m\in\mathbb{Z}}\right)$ is well defined in $\{S\}^c$. If $f \in \oplus_{-\infty}^{+\infty}\mathcal{H}$ and $q \in \mathbb{Z}$ we have

$$\sum_{m\in\mathbb{Z}} A_m\left(b\left(f_{q-m}\right)\right) = \pi_q(ABf) = \pi_q(BAf) = \sum_{m\in\mathbb{Z}} b(A_m f_{q-m}).$$

Thus $A_m \circ b = b \circ A_m$ for all $m \in \mathbb{Z}$ and the claim holds. We can write $\Lambda^{-1}(A) = \{a_m Id_{\mathcal{H}}\}_{m\in\mathbb{Z}}$ for some unique complex sequence $\{a_m\}_{m\in\mathbb{Z}}$. Now, let $f_0 \in [\mathcal{H}]_1$ and $z \in l^2(\mathbb{Z})$. Then $\{z_m f_0\}_{m\in\mathbb{Z}} \in \oplus_{-\infty}^{+\infty}\mathcal{H}$ and

$$\begin{aligned} \left(\sum_{q\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}} a_{q-m} z_m\right|^2\right)^{1/2} &= \left\|A\left(\{z_m f_0\}_{m\in\mathbb{Z}}\right)\right\| \\ &\le \|A\| \left\|\{z_m f_0\}_{m\in\mathbb{Z}}\right\| \\ &= \|A\|\,\|z\|_2, \end{aligned}$$

i.e. $\{a_m\}_{m\in\mathbb{Z}}$ induces an element of $\mathcal{B}\left(l^2(\mathbb{Z})\right)$. The claim now follows by Remark 3.5.

(⇐) Let $a \in M^2$ be given so that (6) holds and let $A \in \mathcal{B}(\oplus_{-\infty}^{+\infty}\mathcal{H})$ be defined by (7). If $B \in \{S\}^c$ and $\Lambda^{-1}(B) = \{B_m\}_{m\in\mathbb{Z}}$, given $f \in \mathcal{H}^{(\mathbb{Z})}$ we set $F \triangleq \{m \in \mathbb{Z} : f_m \neq 0\}$. Thus F is a finite set and given $q \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_q(ABf) &= \sum_{p\in\mathbb{Z}} \widehat{a}(q-p) \sum_{m\in F} B_{p-m}(f_m) \\ &= \sum_{m\in F}\left(\sum_{p\in\mathbb{Z}} \widehat{a}(q-p)\, B_{p-m}\right)(f_m) \\ &= \sum_{m\in F}\left(\sum_{r\in\mathbb{Z}} \widehat{a}(r-m)\, B_{q-r}\right)(f_m) \\ &= \sum_{r\in\mathbb{Z}} B_{q-r} \sum_{m\in F} \widehat{a}(r-m)\, f_m \\ &= \pi_q(BAf). \end{aligned}$$

Thus $AB = BA$ on $\mathcal{H}^{(\mathbb{Z})}$. Since $\mathcal{H}^{(\mathbb{Z})}$ is dense in $\oplus_{-\infty}^{+\infty}\mathcal{H}$ and A and B are bounded operators we conclude that $AB = BA$, i.e. $A \in \mathcal{Z}(\{S\}^c)$.

□

Corollary 3.7. *If $\mu \in \mathcal{M}(\{S\}^c)$ there is $a \in L^\infty(\mathbb{T})$ so that (6) holds and*

$$\left(\Lambda^{-1} \circ \mu \circ \Lambda\right)(\{A_m\}_{m\in\mathbb{Z}}) = \left\{s - \sum_{q\in\mathbb{Z}} \widehat{a}(m-q) A_q\right\}_{m\in\mathbb{Z}} \quad \text{if } \{A_m\}_{m\in\mathbb{Z}} \in \mathfrak{T}(\mathcal{H}).$$

References

[1] A. L. Barrenechea, Peña, C. C.: On the structure of derivations on certain nonamenable nuclear Banach algebras. New York J. of Maths., Vol. 15 (2009),199-210. Zbl pre05561324.
[2] A. Böttcher, B. Silbermann, Analysis of Toeplitz operators. 2nd Edition. Springer-Verlag. Germany, (2006). ISBN-10: 364206891X.
[3] J. B. Conway, A course in functional analysis. 2nd. Edition. Springer-Verlag. New York Inc., (1990). ISBN: 0-387-97245-5.
[4] B. Fuglede, A commutativity theorem for normal operators, Proc. Nat. Acad. Sci., **36** (1950), 35-40.
[5] S. Helgason, Multipliers of Banach algebras, Ann. of Math. (2) **64** (1956), 240-254. MR18-494.
[6] B. Johnson, An introduction to the theory of centralizers, Proc. London Math. Soc. (3) **1** (1964), 299-320. MR28♮2450.
[7] A. P. Madrid, C. C. Peña, On X-Hadamard and B-derivations, General Maths., Vol. 16, no. 1 (2008), 41-50. Zbl 1234.47029.
[8] C. R. Putnam, On normal operators in Hilbert space, Amer. J. Math., **73** (1951), 357-362.
[9] O. Toeplitz, Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen, Math. Annalen **70** (1911), 351-376. JFM 42.0366.01.