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On Warped Product Semi-Slant Submanifolds of Nearly Trans-Sasakian Manifolds

Akram Ali[a], Ali H Alkhaldi[a], Rifaqat Ali[a]

[a]*Department of Mathematics, College of Science, King Khalid University, 61413 Abha, Saudi Arabia*

Abstract. In this paper, we study warped product semi-slant submanifold of type $M = N_T \times_f N_\theta$ with slant fiber, isometrically immersed in a nearly Trans-Sasakian manifold by finding necessary and sufficient conditions in terms of *Weingarten map*. A characterization theorem is proved as main result.

1. Introduction

The study on warped product submanifolds got momentum after B.-Y. Chen's papers on CR-warped product [13, 14]. A contact CR-warped product submanifold is the Riemannian product of invariant and anti-invariant submanifold. It was proved in [21] that there does not exist any contact CR-warped product of type $M = N_\perp \times_f N_T$, of nearly Trans-Sasakian manifolds in both cases when structure field tangent to either base manifold or fiber. Also, it was also found in the same paper, the non-trivial contact CR-warped product of the form $M = N_T \times_f N_\perp$, in a nearly Trans-Sasakian manifold such that N_T invariant tangent to Reeb vector field. Similarly, the Riemannian product of invariant and slant submanifolds with non constant warping function is called warped product semi-slant submanifold. The non-existence of the warped product semi-slant submanifold $M = N_\theta \times_f N_T$, isometrically immersed in a nearly Trans-Sasakian manifold with structure vector field is tangent N_θ and N_T has discussed in [22]. On the other hand, the existence case of the non-trivial warped product semi-slant submanifold of type $M = N_T \times_f N_\theta$, in a nearly Trans-Sasakian manifold has been proved in [22] with N_T is an invariant submanifold which is tangent to the Reeb vector field and constructed a geometric inequality for the extrinsic invariant in terms of warping function. Therefore, it is natural to see that the warped product semi-slant submanifold is a generalized version of contact CR-warped product submanifold in case of a nearly Trans-Sasakian manifold. Similar notions have been studied in the series of articles [1–8, 17, 21, 22, 24–29]. In this paper, we prove a characterization theorem involving the shape operator under which a semi-slant submanifold of a nearly Trans-Sasakian manifold reduces to a warped product.

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Email addresses: akramali133@gmail.com; akali@kku.edu.sa (Akram Ali), ahalkhaldi@kku.edu.sa (Ali H Alkhaldi), rifaqat.ali1@gmail.com (Rifaqat Ali)

2. Preliminaries

An almost contact manifold is an odd-dimensional manifold $\bar{M}$ which carries a field φ of endomorphisms of tangent space, vector field ξ, called *characteristic* or *Reeb vector field* and a 1-form η satisfying

$$\varphi^2 = -I + \eta \oplus \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity mapping. Now from definition it follows that $\varphi \circ \xi = 0$ and $\eta \circ \varphi = 0$, then the (1, 1) tensor field φ has constant rank $2n$ (cf. [9]). An almost contact manifold $(\bar{M}, \varphi, \eta, \xi)$ is said to be *normal* when the tensor field $N_\varphi = [\varphi, \varphi] + 2d\eta \oplus \xi$ vanishes identically, where $[\varphi, \varphi]$ is the *Nijenhuis* of φ. An almost contact metric structure (φ, ξ, η) is said to be a normal in the form of almost complex structure if almost complex structure J on a product manifold $\bar{M} \times R$ given by

$$J\Big(U, f\frac{d}{dt}\Big) = \Big(\varphi U - f\xi, \eta(U)\frac{d}{dt}\Big),$$

where f is a smooth function on $\bar{M} \times R$, has no torsion, i.e., J is integrable. Every almost contact manifold $(\bar{M}, \varphi, \eta, \xi)$ admits a Riemannian metric g which is satisfying

$$g(\varphi U, \varphi V) = g(U, V) - \eta(X)\eta(Y), \quad \eta(U) = g(U, \xi), \tag{2.2}$$

for all $U, V \in \Gamma(T\bar{M})$. This metric g is called *compatible metric* and the manifold $\bar{M}$ endowed with the structure (φ, η, ξ, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.1), we have $g(\varphi U, V) = -g(U, \varphi V)$. Hence, the second fundamental 2-form Φ is defined by $\Phi(U, V) = g(U, \varphi V)$. Almost contact manifold such that both η and Φ are closed is called *almost cosymplectic* manifold and those for which $d\eta = \Phi$ are called *contact metric* manifolds. Finally, a normal almost cosymplectic manifold is called *cosymplectic* manifold and a normal contact manifold is called *Sasakian* manifold. In term of the covariant derivative of φ the *cosymplectic* and the *Sasakian* manifolds conditions can be expressed respectively by

$$(\nabla_U \varphi)V = 0, \quad \textit{and} \quad (\nabla_U \varphi)V = g(U, V)\xi - \eta(V)U,$$

for all $U, V \in \Gamma(TM)$ (see [9]). It should be noted that both in *cosymplectic* and *Sasakian* manifolds ξ is killing vector field. On the other hand, the *Sasakian* and the *cosymplectic* manifolds represent the two external cases of the larger class of *quasi-Sasakian* manifolds. An almost contact metric structure (φ, η, ξ) is said to be *nearly Trans-Sasakian manifold (cf. [19])* i.e., if

$$(\bar{\nabla}_U \varphi)V + (\bar{\nabla}_V \varphi)U = \alpha\Big(2g(U, V)\xi - \eta(U)V - \eta(V)U\Big) - \beta\Big(\eta(V)\varphi U + \eta(U)\varphi V\Big), \tag{2.3}$$

for any U, V tangent to $\tilde{M}$, where $\bar{\nabla}$ is the Riemannian connection metric g on $\bar{M}$. If we replace $U = \xi$, $V = \xi$ in (2.3), we find that $(\bar{\nabla}_\xi \varphi)\xi = 0$ which is implies that $\varphi \bar{\nabla}_\xi \xi = 0$. Now applying φ and using (2.1), we get, $\bar{\nabla}_\xi \xi = 0$. Since from *Gauss formula* finally, we get $\nabla_\xi \xi = 0$ and $h(\xi, \xi) = 0$. For more classification (see [**? ?**]).

Note 2.1. *If* $\alpha = 0$ *and,* $\beta = 0$ *in* (2.3)*, then nearly Trans-Sasakian becomes nearly cosymplectic manifold, if* $\alpha = 1$ *and,* $\beta = 0$ *in* (2.3)*, thus its called nearly Sasakian manifold. Let* $\alpha = 0$ *and,* $\beta = 1$ *in* (2.3)*, then nearly Trans-Sasakian turn into nearly Kenmotsu manifold. Similarly nearly* α*-Sasakian manifold and nearly* $\beta-$*Kenmotsu manifold can be defined from the nearly Trans-Sasakian manifold by substituting* $\beta = 0$ *and* $\alpha = 0$ *in* (2.3)*, respectively.*

Now let M be a submanifold of $\tilde{M}$, then we will denote by ∇ is the induced Riemannian connection on M and g is the Riemannian metric on $\bar{M}$ as well as the metric induced on M. Let TM and $T^\perp M$ be the Lie algebra of vector fields tangent to M and normal to M, respectively and $\nabla^\perp$ the induced connection on $T^\perp M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the *Gauss* and *Weingarten* formulas are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.4}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{2.5}$$

for each $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\bar{M}$. They are related as:

$$g(h(U,V),N) = g(A_N U, V). \tag{2.6}$$

Now, for any $U \in \Gamma(TM)$, we defined as:

$$\varphi U = TU + FU, \tag{2.7}$$

where TU and FU are the tangential and normal components of φU, respectively. If M is invariant and anti-invariant, then FU and TU are identically zero, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\varphi N = tN + fN, \tag{2.8}$$

where tN (resp. fN) is the tangential (resp. normal) components of φN. From (2.2) and (2.7), it is easy to observe that for each $U, V \in \Gamma(TM)$ $g(TU,V) = -g(U,TV)$. Further, the covariant derivative of the endomorphism φ is defined as

$$(\bar{\nabla}_U\varphi)V = \bar{\nabla}_U\varphi V - \varphi\bar{\nabla}_U V, \ \ \forall U, V \in \Gamma(T\bar{M}). \tag{2.9}$$

,

Proposition 2.1. *On a nearly Trans-Sasakian manifold, the following condition is satisfied*

$$g(\bar{\nabla}_V\xi, U) + g(\bar{\nabla}_U\xi, V) = 2\beta g(\varphi X, \varphi Y),$$

for any vector fields U, V tangent to $\tilde{M}$, where $\tilde{M}$ is a nearly Trans-Sasakian manifold

Proof. Setting $U = \xi$ in (2.3), then we find

$$(\bar{\nabla}_\xi\varphi)V + (\bar{\nabla}_V\varphi)\xi = \alpha\{2g(\xi,V)\xi - V - \eta(V)\xi\} - \beta\varphi V.$$

Taking the inner product with φU in the above equation we get

$$g((\bar{\nabla}_\xi\varphi)V, \varphi U) = -g((\bar{\nabla}_V\varphi)\xi, \varphi U) - \alpha g(V, \varphi U) - \beta g(\varphi V, \varphi U). \tag{2.10}$$

Interchanging U and V in the above equation, we derive

$$g((\bar{\nabla}_\xi\varphi)U, \varphi V) = -g((\bar{\nabla}_U\varphi)\xi, \varphi V) - \alpha g(U, \varphi V) - \beta g(\varphi V, \varphi U). \tag{2.11}$$

Adding equation (2.10) and (2.11), we find

$$g((\bar{\nabla}_\xi\varphi)V, \varphi U) + g((\bar{\nabla}_\xi\varphi)U, \varphi V) = -g((\bar{\nabla}_V\varphi)\xi, \varphi U) - g((\bar{\nabla}_U\varphi)\xi, \varphi V) - 2\beta g(\varphi V, \varphi U).$$

As left hand side of the above equation should be zero from the fact that $\nabla_\xi\varphi = 0$, for almost contact metric manifold, hence

$$-g((\bar{\nabla}_V\varphi)\xi, \varphi U) - g((\bar{\nabla}_U\varphi)\xi, \varphi V) = 2\beta g(\varphi V, \varphi U).$$

The proof follows from the above equations and this complete the proof of the Proposition. □

We denote the tangential and normal parts of $(\tilde{\nabla}_U\varphi)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ such that

$$(\tilde{\nabla}_U\varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V$$

Then in a nearly Trans-Sasakian manifold, we have

$$\mathcal{P}_U V + \mathcal{P}_V U = \alpha\{2g(U,V)\xi - \eta(U)V - \eta(V)U\} - \beta\{\eta(V)TU + \eta(U)TV\}, \tag{2.12}$$
$$\mathcal{Q}_U V + \mathcal{Q}_V U = -\beta\{\eta(V)FU + \eta(U)FV\}, \tag{2.13}$$

for any U, V are tangent to $\tilde{M}$. It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$,

$$\left.\begin{array}{l} (i)\ \mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W, \\ (ii)\ \mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W, \\ (iii)\ \mathcal{P}_U(W+Z) = \mathcal{P}_U W + \mathcal{P}_U Z, \\ (iv)\ \mathcal{Q}_U(W+Z) = \mathcal{Q}_U W + \mathcal{Q}_U Z, \\ (v)\ g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W), \\ (vi)\ g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N), \\ (vii)\ \mathcal{P}_U \varphi V + \mathcal{Q}_U \varphi V = -\varphi(\mathcal{P}_U V + \mathcal{Q}_U V). \end{array}\right\} \tag{2.14}$$

Next we will give the definition of slant submanifold as follows:

Definition 2.1. *[11] For each non zero vector U tangent to M at p, such that U is not proportional to ξ_p, we denote by $0 \le \theta(U) \le \pi/2$, the angle between φU and T_pM is called the Wirtinger angle. If the angle $\theta(U)$ is constant for all $U \in T_PM - <\xi>$ and $p \in M$. Then M is said to be a slant submanifold and the angle θ is called slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.*

In a contact metric manifold, J. L Cabrerizo (cf. [11]) obtained the following theorem.

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{2.15}$$

Furthermore, in such a case, if θ is slant angle, then it satisfies that $\lambda = \cos^2\theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\bar{M}$, the following relations are consequences of the above theorem.

$$g(TX, TY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{2.16}$$
$$g(FX, FY) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\} \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$. Another characterization of consequence of the Theorem 2.1 is easily derived as follows:

Theorem 2.2. *Let M be a slant submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then*

$$(a)\ tFX = sin^2\theta(-X + \eta(X)\xi), \quad and \quad (b)\ fFX = -FTX, \tag{2.18}$$

for any $X \in \Gamma(TM)$.

3. Warped Product Semi-Slant Submanifolds

A natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant distribution was described by N. Papaghiuc (cf. [23]). These submanifolds are known as semi-slant submanifolds. The semi-slant submanifolds of almost contact metric manifolds were defined and studied by Cabererizo. They defined these submanfolds as:

Definition 3.1. *[12] A Riemannian submanifold M of an almost contact manifold $\tilde{M}$ is said to be a semi-slant submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that*

(i) $TM = \mathcal{D}^\theta \oplus \mathcal{D} \oplus \langle\xi\rangle$, *where* $<\xi>$ *is 1-dimensional distribution spanned by* ξ.

(ii) $\mathcal{D}$ *is invariant distribution under* φ *i.e.,* $\varphi\mathcal{D} \subseteq \mathcal{D}$.

(iii) $\mathcal{D}^\theta$ *is slant distribution with slant angle* $\theta \neq 0, \frac{\pi}{2}$.

If we denote the dimension of $\mathcal{D}_i$ by d_i for $i = 1, 2$, then it is clear that contact CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \pi/2$ and $d_1 = 0$, respectively. It is called proper semi-slant if slant angle is different from 0 and $\pi/2$. Moreover, if μ is an *invariant* subspace under φ of normal bundle $T^\perp M$, then in case of semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathcal{D}^\theta \oplus \mu$. A semi-slant submanifold is said to be a mixed totally geodesic if $h(X, Z) = 0$, for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D})$.

Let f be a positive differentiable function on N_1 of two Riemannian manifolds N_1 and N_2 endowed with two Riemannian metrics g_1 and g_2, respectively. Then warped product $M = N_1 \times_f N_2$ is the manifold $N_1 \times N_2$ equipped with the Riemannian metric $g = g_1 + f^2 g_2$. The function f is called warping function of the warped product. If for any $X, Y \in \Gamma(TN_1)$ and $Z, W \in \Gamma(TN_2)$, then

$$\nabla_Z X = \nabla_X Z = (X \ln f)Z, \tag{3.1}$$

where ∇ denote the Levi-Civitas connection on M. On the other hand, $\nabla \ln f$ is the gradient of $\ln f$ is defined as $g(\nabla \ln f, X) = X \ln f$. A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant.

There are two types of warped products between proper slant and invariant submanifolds. Now we study warped product semi-slant submanifolds and its characterization of type of $M = N_T \times_f N_\theta$. For first case, we recall the following result which was obtained by Mustafa, et.al (cf. [22]) for warped product semi-slant submanifolds of nearly Trans-Sasakian manifolds as:

Theorem 3.1. *[22] There do not exist warped product semi-slant submanifolds $M = N_\theta \times_f N_T$ in a nearly Trans-Sasakian manifold $\tilde{M}$, where N_θ and N_T are proper slant and invariant submanifolds of $\tilde{M}$, respectively.*

First, we give the following definition which based on the results S. Hiepk[20].

Definition 3.2. *Assume that M be a semi-slant submanifold of a nearly-Trans Sasakian manifold $\tilde{M}$, then we say that M is a locally warped product manifold semi-slant submanifold of $\tilde{M}$ if $\mathcal{D}$ defines a totally geodesic foliation on M and $\mathcal{D}^\theta$ defines a spherical foliation on M, that is each leaf of $\mathcal{D}^\theta$ is totally umbilical with parallel mean curvature vector field in M.*

Now we give following results for later use given in (cf. [22]) for warped product semi-slant submanifold of nearly-Trans Sasakian manifolds as:

Lemma 3.1. *[22] Assume that $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly Trans-Sasakian manifold $\bar{M}$. Then*

$$\xi \ln f = \beta, \tag{3.2}$$

$$g(h(X, Y), FZ) = 0, \tag{3.3}$$

$$g(h(Z, X), FZ) = -\Big((\varphi X\lambda) + \alpha\eta(X)\Big)\|Z\|^2, \tag{3.4}$$

$$g(h(X, Z), FTZ) = \frac{1}{3}\cos^2\theta\Big((X \ln f) - \beta\eta(X)\Big)\|Z\|^2, \tag{3.5}$$

for any $Z \in \Gamma(TN_\theta)$ and $X, Y \in \Gamma(TN_T)$.

Following relations are the particular case, of above lemma. By interchanging X by φX in (3.5), we find

$$g(h(\varphi X, Z), FTZ) = \frac{1}{3}\cos^2\theta(\varphi X \ln f)\|Z\|^2, \tag{3.6}$$

Now we prove our main result

Theorem 3.2. *Let a proper semi-slant submanifold M of nearly Trans-Sasakian manifold $\widehat{M}$ such that the normal component $Q_X U$ of $(\tilde{\nabla}_X \varphi)U$ lies in $\varphi-$invariant normal subbundle of M. Then M is locally a non-trivial warped product submanifold of type $M = N_T \times_f N_\theta$ if and only if the following condition is satisfied*

$$A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3}\Big(2 + \sin^2\theta\Big)(X\lambda)Z + \beta\eta(X)\Big(1 - \frac{1}{3}\cos^2\theta\Big)Z, \tag{3.7}$$

for any $U \in \Gamma(TM)$, $X \in \Gamma(\mathcal{D} \oplus \xi)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Moreover, for a differentiable function λ on M such that $Z\lambda = 0$, for any $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. Assume that $M = N_T \times_f N_\theta$ be a non-trivial warped product semi-slant submanifold of a nearly Trans-Sasakian manifold $\tilde{M}$ such that N_θ and N_T are proper slant and $\varphi-$invariant submanifolds of $\tilde{M}$, respectively. Thus from (3.3) and (2.6), we get $g(A_{FZ}X, Y) = 0$, for any $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. Since, N_T is an invariant submanifold, then rearranging X by φX, we find that $g(A_{FZ}\varphi X, Y) = 0$, which indicates that the components of linear operator $A_{FZ}\varphi X$ are not lying in TN_T. Similarly, rearranging Z by TZ in (3.3) and from (2.6), it is easily see that $g(A_{FTZ}X, Y) = 0$, which again shows that $A_{FZ}X$ has no components in TN_T. Hence, this means that $A_{FZ}\varphi X - A_{FTZ}X$ lies in TN_θ. Therefore, from the Lemmas 3.2-(3.5), we have

$$g(h(X, Z), FTZ) = \frac{1}{3}\cos^2\theta(X \ln f)\|Z\|^2 - \frac{1}{3}\cos^2\theta\beta\eta(X)\|Z\|^2. \tag{3.8}$$

On the other hand, replacing X by φX in the Eqs (3.3) and using (2.1)(i), we derive

$$g(h(Z, \varphi X), FZ) = (X \ln f)\|Z\|^2 - \eta(X)(\xi \ln f)\|Z\|^2 + \alpha\eta(\varphi X)\|Z\|^2.$$

Since, from (3.2) and the fact that $\eta(\varphi X) = g(\varphi X, \xi) = 0$, we obtain

$$g(h(Z, \varphi X), FZ) = (X \ln f)\|Z\|^2 - \beta\eta(X)\|Z\|^2, \tag{3.9}$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. We get the following relation by follows (3.8), (3.9) and (2.6) as:

$$g(A_{FZ}\varphi X - A_{FTZ}X, Z) = \left(1 - \frac{1}{3}\cos^2\theta\right)(X \ln f)\|Z\|^2 + \beta\eta(X)\left(\frac{1}{3}\cos^2\theta - 1\right)\|Z\|^2. \tag{3.10}$$

Thus $A_{FZ}\varphi X - A_{FTZ}X$ lies in TN_θ and from (3.10), we get the required result (3.7). Hence, the first part is proved completely.

Conversely Let M be a semi-slant submanifold of nearly Trans-Sasakian manifold with the condition (3.7) holds. For any $X, Y \in \Gamma(\mathcal{D} \oplus \xi)$ and $Z \in \Gamma(\mathcal{D}^\theta)$, we obtain

$$g(\nabla_X Y, Z) = g(\varphi\nabla_X Y, \varphi Z) + \eta(\widehat{\nabla}_X Y)\eta(Z) = g(\widehat{\nabla}_X \varphi Y, \varphi Z) - g((\widehat{\nabla}_X \varphi)Y, \varphi Z).$$

From the structure Eqs (2.3) and from the property of Riemannian connection, one obtains

$$g(\nabla_X Y, Z) = -g(\varphi Y, \tilde{\nabla}_X \varphi Z) - g(\mathcal{P}_X Y, TZ) + g(Q_X Y, FZ)$$

Taking the help of Eqs (2.8) and the property of covariant derivative of φ, we find

$$g(\nabla_X Y, Z) = g(\varphi\tilde{\nabla}_X TZ, Y) - g(\tilde{\nabla}_X FZ, \varphi Y) + g(Y, \mathcal{P}_X TZ) - g(Q_X Y, FZ).$$

Using (2.4), (2.5) and (2.12), we derive

$$\begin{aligned} g(\nabla_X Y, Z) =& g(\tilde{\nabla}_X T^2 Z, Y) + g(\tilde{\nabla}_X FTZ, Y) - g((\tilde{\nabla}_X \varphi)TZ, Y) \\ &+ g(FZ, h(X, \varphi Y)) + g(Y, \mathcal{P}_X TZ) - g(\mathcal{Q}_X Y, FZ). \end{aligned}$$

Following the Theorem 2.1 and (2.5), it is easy to see that

$$\sin^2\theta g(\nabla_X Y, Z) = -g(A_{FTZ}X - A_{FZ}\varphi X, Y) - g(\mathcal{P}_X TZ, Y) - g(\mathcal{Q}_X Y, FZ) + g(\mathcal{P}_X TZ, Y).$$

Therefore, from the hypothesis of the Theorem 3.2, we know that $\mathcal{Q}_X Y$ lies in μ and from the Eqs (3.7), we get

$$\sin^2\theta g(\nabla_X Y, Z) = \frac{1}{3}\left(2 + \sin^2\theta\right)(X\lambda)g(Z, Y) + \beta\eta(X)\left(\frac{1}{3}\cos^2\theta - 1\right)g(Z, Y),$$

which implies that

$$\sin^2\theta g(\nabla_X Y, Z) = 0. \tag{3.11}$$

But M is a proper semi-slant submanifold, i.e, $\sin^2\theta \neq 0$. From (3.11) we know that $g(\nabla_X Y, Z) = 0$, this means that $\nabla_X Y \in \Gamma(\mathcal{D}\oplus\xi)$, for every $X, Y \in \Gamma(\mathcal{D}\oplus\xi)$. Therefore, $\mathcal{D}\oplus\xi$ is integrable and its leaves are totally geodesic in M. Moreover, for any $Z, W \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D}\oplus\xi)$, we have

$$g([Z, W], X) = g(\varphi\tilde{\nabla}_W Z, \varphi X) + \eta(\tilde{\nabla}_W Z)\eta(X) - g(\varphi\tilde{\nabla}_Z W, \varphi X) - \eta(\tilde{\nabla}_Z W)\eta(X)$$

From the covariant derivative property, we get

$$\begin{aligned} g([Z, W], X) =& g(\tilde{\nabla}_W \varphi Z, \varphi X) - g((\tilde{\nabla}_W \varphi)Z, \varphi X) - g(\tilde{\nabla}_Z \varphi W, \varphi X) + g((\tilde{\nabla}_W \varphi)Z, \varphi X) \\ &+ \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X) \\ =& g(\tilde{\nabla}_W TZ, \varphi X) + g(\tilde{\nabla}_W FZ, \varphi X) - g(\mathcal{P}_Z W, \varphi X) - g(\tilde{\nabla}_Z TW, \varphi X) - g(\tilde{\nabla}_W FZ, \varphi X) \\ &+ g(\mathcal{P}_W Z, \varphi X) + \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X). \end{aligned}$$

It is implies from the Eqs (2.5), i.e.,

$$\begin{aligned} g([Z, W], X) =& - g(\varphi\tilde{\nabla}_W TZ, X) - g(A_{FZ}\varphi X, W) - g(\mathcal{P}_Z W, \varphi X) + g(\varphi\tilde{\nabla}_Z TW, X) \\ &+ g(A_{FW}\varphi X, Z) + g(\mathcal{P}_W Z, \varphi X) + \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X) \\ =& g((\tilde{\nabla}_W \varphi)TZ, X) - g(\tilde{\nabla}_W T^2 Z, X) - g(\tilde{\nabla}_W FTZ, X) \\ &- g(\mathcal{P}_Z W, \varphi X) - g((\tilde{\nabla}_Z \varphi)TW, X) + g(\tilde{\nabla}_Z T^2 W, X) \\ &+ g(\tilde{\nabla}_Z FTW, X) + g(\mathcal{P}_W Z, \varphi X) + g(A_{FW}\varphi X, Z) + \eta(\tilde{\nabla}_W Z)\eta(X) \\ &- \eta(\tilde{\nabla}_Z W)\eta(X) - g(A_{FZ}\varphi X, W). \end{aligned}$$

From Theorem 2.1 and Eq. (2.5) one derives

$$\begin{aligned} g([Z, W], X) =& g(\mathcal{P}_W TZ, X) + \cos^2\theta g(\tilde{\nabla}_W Z, X) + g(A_{FTZ}X, W) - (\mathcal{P}_Z W, \varphi X) - g(\mathcal{P}_Z TW, X) \\ &- \cos^2\theta g(\tilde{\nabla}_Z W, X) + g(A_{FW}\varphi X, Z) + g(\mathcal{P}_W Z, \varphi X) - g(A_{FTW}X, Z) \\ &+ \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X) - g(A_{FZ}\varphi X, W). \end{aligned} \tag{3.12}$$

Using the properties of $\mathcal{P} - \mathcal{Q}$ from (2.14), we find

$$\begin{aligned} g(\mathcal{P}_W TZ, X) - g(\mathcal{P}_Z TW, X) =& g(\mathcal{P}_W(\varphi Z - FZ), X) - g(\mathcal{P}_Z(\varphi W - FW), X) \\ =& g(\mathcal{P}_W \varphi Z, X) - g(\mathcal{P}_W FZ, X) - g(\mathcal{P}_Z \varphi W, X) + g(\mathcal{P}_Z FW, X) \\ =& - g(\varphi\mathcal{P}_W Z, X) + g(\mathcal{Q}_W X, FZ) + g(\varphi\mathcal{P}_W Z, X) - g(\mathcal{Q}_Z X, FW) \\ =& g(\mathcal{P}_W Z, \varphi X) + g(\mathcal{Q}_W X, FZ) - g(\mathcal{P}_Z W, \varphi X) - g(\mathcal{Q}_Z X, FW). \end{aligned}$$

Therefore, using the above relation in the Eqs (3.12), we get

$$\sin^2\theta g([Z,W],X) = g(A_{FTZ}X - A_{FZ}\varphi X, W) + 2g(\mathcal{P}_W Z, \varphi X) + g(\mathcal{Q}_W X, FZ) - g(A_{FTW}X - A_{FW}\varphi X, Z) \\ - 2g(\mathcal{P}_Z W, \varphi X) - g(\mathcal{Q}_Z X, FW) + \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X).$$

Using properties (2.12)-(2.13) and (2.17), we arrive at

$$\sin^2\theta g([Z,W],X) = g(A_{FTZ}X - A_{FZ}\varphi X, W) + 2g(\mathcal{P}_W Z + \mathcal{P}_Z W, \varphi X) - g(\mathcal{Q}_X W, FZ) \\ - \sin^2\theta\beta\eta(X)g(Z,W) - g(A_{FTW}X - A_{FW}\varphi X, Z) + g(\mathcal{Q}_X Z, FW) \\ + \sin^2\theta\beta\eta(X)g(Z,W) + \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X).$$

By assumption of the Theorem 3.2 that $\mathcal{Q}_X Z$ lies in μ and again using (2.12) we derive

$$\sin^2\theta g([Z,W],X) = g(A_{FTZ}X - A_{FZ}\varphi X, W) + \eta(\tilde{\nabla}_W Z)\eta(X) \\ - \eta(\tilde{\nabla}_Z W)\eta(X) - g(A_{FTW}X - A_{FW}\varphi X, Z). \tag{3.13}$$

Applying Eqs (3.7) in the Eqs (3.13), one obtains

$$\sin^2\theta g([Z,W],X) = -\frac{1}{3}\left(2 + \sin^2\theta\right)(X\lambda)g(Z,W) - \beta\eta(X)\left(\frac{1}{3}\cos^2\theta - 1\right)g(Z,W) \\ + \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X) + \frac{1}{3}\left(2 + \sin^2\theta\right)(X\lambda)g(Z,W) \\ + \beta\eta(X)\left(\frac{1}{3}\cos^2\theta - 1\right)g(Z,W),$$

which means that

$$\sin^2\theta g([Z,W],X) = \eta(\tilde{\nabla}_W Z)\eta(X) - \eta(\tilde{\nabla}_Z W)\eta(X).$$

Now interchanging X by φX in the above equation and using the fact that $\eta(\varphi X) = 0$, we derive

$$\sin^2\theta g([Z,W],\varphi X) = 0.$$

Since, M is a proper semi-slant submanifold, then from previous Eq, we deduce that the slant distribution $\mathcal{D}^\theta$ is integrable. Therefore, we can assume that N_θ be a leaf of $\mathcal{D}^\theta$ and h^θ be a the second fundamental form (extrinsic invariant) of N_θ into M. Then from Gauss formula (2.4), we have

$$g(h^\theta(Z,W),X) = g(\tilde{\nabla}_Z W, X) = g(\varphi\tilde{\nabla}_Z W, \varphi X) + \eta(\tilde{\nabla}_Z W)\eta(X) \\ = g(\tilde{\nabla}_Z \varphi W, \varphi X) - g((\tilde{\nabla}_Z \varphi)W, \varphi X) + \eta(\tilde{\nabla}_Z W)\eta(X).$$

From (2.8) and tangential components of $(\tilde{\nabla}_Z\varphi)W$, it is easily seen that

$$g(h^\theta(Z,W),X) = g(\widehat{\nabla}_Z TW, \varphi X) + g(\widehat{\nabla}_Z FW, \varphi X) - g(\mathcal{P}_Z W, \varphi X) + \eta(\tilde{\nabla}_Z W)\eta(X).$$

Using the covariant differentiation property of φ and (2.5), we obtain

$$g(h^\theta(Z,W),X) = g((\tilde{\nabla}_Z\varphi)TW, X) - g(\tilde{\nabla}_Z T^2 W, X) - g(\widehat{\nabla}_Z FTW, X) \\ - g(A_{FW}Z, \varphi X) + g(\varphi\mathcal{P}_Z W, X) + \eta(\tilde{\nabla}_Z W)\eta(X).$$

Then using the Theorem 2.1 and (2.5), we derive

$$g(h^\theta(Z,W),X) = g((\mathcal{P}_Z TW, X) + \cos^2\theta g(\nabla_Z W, X) + g(A_{FTW}Z, X) - g(A_{FW}Z, \varphi X) - g(\mathcal{P}_Z TW, X) \\ - g(\mathcal{P}_Z FW, X) + \eta(\tilde{\nabla}_Z W)\eta(X),$$

its implies that

$$\sin^2\theta g(h^\theta(Z,W),X) = g(A_{FTW}X - A_{FW}\varphi X, Z) + g(Q_Z X, FW) + \eta(\tilde{\nabla}_Z W)\eta(X).$$

Using Eq. (2.13) in the second term of the above Eqs. Then from (3.7) and (2.17), we arrive at

$$\sin^2\theta g(h^\theta(Z,W),X) = -\frac{1}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) - \beta\eta(X)\frac{1}{3}\cos^2\theta g(Z,W)$$
$$+\beta\eta(X)g(Z,W) - g(Q_X Z, FW) - \beta\eta(X)\sin^2\theta g(Z,W) + \eta(\tilde{\nabla}_Z W)\eta(X).$$

As we have assumed that $Q_X Z$ lies in μ, finally we get

$$\sin^2\theta g(h^\theta(Z,W),X) = -\frac{1}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) + \frac{2}{3}\cos^2\theta\beta\eta(X)g(Z,W) - \eta(X)g(\tilde{\nabla}_Z\xi, W). \tag{3.14}$$

Interchanging Z by W in (3.14), we find

$$\sin^2\theta g(h^\theta(Z,W),X) = -\frac{1}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) + \frac{2}{3}\cos^2\theta\beta\eta(X)g(Z,W) - \eta(X)g(\tilde{\nabla}_Z\xi, W). \tag{3.15}$$

From the symmetry of extrinsic invariant h^θ, then (3.15) and (3.14) implies that

$$2\sin^2\theta g(h^\theta(Z,W),X) = -\frac{2}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) + \frac{4}{3}\cos^2\theta\beta\eta(X)g(Z,W)$$
$$-\eta(X)\Big(g(\tilde{\nabla}_Z\xi, W) + g(\tilde{\nabla}_Z\xi, W)\Big). \tag{3.16}$$

Applying the Proposition 2.1 in last Eq. then easily get the following

$$2\sin^2\theta g(h^\theta(Z,W),X) = -\frac{2}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) + \frac{4}{3}\cos^2\theta\beta\eta(X)g(Z,W) - 2\beta\eta(X)g(Z,W),$$

which implies that

$$2\sin^2\theta g(h^\theta(Z,W),X) = -\frac{2}{3}\Big(2+\sin^2\theta\Big)(X\lambda)g(Z,W) + \Big(\frac{4}{3}\cos^2\theta - 2\Big)\beta\eta(X)g(Z,W) \tag{3.17}$$

Hence, replacing X by φX in the above relation (3.17) and using fact that $\eta(\varphi X) = g(\varphi X, \xi) = -g(X, \varphi\xi) = 0$, which gives

$$g(h^\theta(Z,W),\varphi X) = -\frac{1}{3}\Big(2+\csc^2\theta + \cot^2\theta\Big)(\varphi X\lambda)g(Z,W)$$

Finally, from the property of gradient of $\ln f$, simplification gives

$$g(h^\theta(Z,W),\varphi X) = -\frac{1}{3}\Big(2+\csc^2\theta + \cot^2\theta\Big)g(Z,W)g(\nabla\lambda, \varphi X)$$

It follows that

$$h^\theta(Z,W) = -g(Z,W)\frac{1}{3}\Big(2+\csc^2\theta + \cot^2\theta\Big)\nabla\lambda. \tag{3.18}$$

Therefore, Eq. (3.18) indicate that N_θ is totally umbilical submanifold into M with its mean curvature vector field $H^\theta = -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)\nabla\lambda$. Further, we will prove that the mean curvature H^θ is parallel along the normal connection ∇^θ of N_θ into M. For this object, we choose $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$, i.e.,

$$\begin{aligned} g(\nabla^\theta_Z H^\theta, X) &= -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(\nabla^\theta_Z grad\lambda, Y) = -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(\nabla_Z grad\lambda, Y)\\ &= -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(Zg(grad\lambda, X)) + \frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(grad\lambda, \nabla_Z X)\\ &= -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)(Z(X\lambda)) + \frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g([X,Z], grad\lambda)\\ &\quad -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(\nabla_X Z, grad\lambda)\\ &= -\frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)(X(Z\lambda)) - \frac{1}{3}\Big(2+\csc^2\theta+\cot^2\theta\Big)g(\nabla_X grad\lambda, Z) \end{aligned}$$

From the hypothesis of the Theorem 3.2 ($Z\lambda = 0$), for each $Z \in \Gamma(\mathcal{D}^\theta)$ and $\nabla grad\lambda$ lies in $\mathcal{D}$, thus last equation becomes

$$g(\nabla^\theta_Z H^\theta, X) = 0, \tag{3.19}$$

which means that $\nabla^\theta H^\theta \in \Gamma(\mathcal{D}^\theta)$. This implies that the mean curvature H^θ of N_θ is parallel. Hence, the condition of spherical is satisfied. Follows the **Definition** 3.2, thus M becomes the warped product manifold of N_θ and N_T, where N_θ and N_T are integral manifolds corresponding to $\mathcal{D}^\theta$ and $\mathcal{D}$, respectively. This complete the proof of the Theorem. □

Similarly, we gives another characterization theorem, .i.e,

Theorem 3.3. *Every proper semi-slant submanifold M of nearly Trans-Sasakian manifold $\widehat{M}$ such that the normal components of $(\widehat{\nabla}_X\varphi)U$ lies in invariant normal subbundle of M is locally a non-trivial warped product submanifold of type $M = N_T \times_f N_\theta$ such that N_θ proper slant and N_T $\varphi-$invariant submanifolds if and only if the following condition is satisfied*

$$A_{FTZ}\varphi X - A_{FZ}X = \frac{1}{3}\Big(2-\sin^2\theta\Big)(\varphi X\lambda)Z + \alpha\eta(X)Z, \tag{3.20}$$

for any $U \in \Gamma(TM)$, $X \in \Gamma(\mathcal{D}\oplus\xi)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Moreover, for a differentiable function λ on M such that $Z\lambda = 0$, for any $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. Directly part follows from (3.6) and (3.4). Moreover, converse part can be easily proved as the Theorem 3.2. □

The abbreviations of manifolds are: nearly Sasakian manifold, nearly Kenmotsu, nearly cosymplectic, $\alpha-$nearly Sasakian, $\beta-$nearly Kenmotsu which are classes of $(\alpha,\beta)-$nearly-Trans Sasakian manifold. The following table shows that the necessary and sufficient condition for the existence of warped product semi-slant submanifolds in almost contact manifolds with ξ tangent to the first factor which are directly generalizing from $(\alpha,\beta)-$nearly-Trans Sasakian manifold i.e.,

Case 3.1. *If we substitute $\alpha = 0$, and, $\beta = 0$ in Eqs.(2.3), we immediately get the following result from the Theorem 3.2, i.e.,*

Theorem 3.4. *A proper semi-slant submanifold M of a nearly cosymplectic manifold $\widehat{M}$ such that the normal components of $\left(\widehat{\nabla}_X\varphi\right)U$ lies in $\varphi-$invariant normal subbundle of M for any $X \in \Gamma(\mathcal{D})$ and $U \in \Gamma(TM)$. Then M is locally a non-trivial warped product submanifold of the type $M = N_T \times_f N_\theta$ such that N_θ is proper slant and N_T is $\varphi-$invariant submanifolds if and only if the following condition is satisfied*

$$A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3}\Big(2 + \sin^2\theta\Big)(X\lambda)Z, \tag{3.21}$$

for any $X \in \Gamma(\mathcal{D} \oplus \xi)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Moreover, for a differentiable function λ on M such that $Z\lambda = 0$, for any $Z \in \Gamma(\mathcal{D}^\theta)$.

Case 3.2. *Rearranging $\alpha = 1$ and $\beta = 0$ in Eqs.* (2.3)*, then nearly-Trans Sasakian manifold turn into nearly Sasakian manifold. Thus, we find the following Theorem which is a direct consequence of the Theorem 3.3, that is,*

Theorem 3.5. *Assume that M be a proper semi-slant submanifold of a nearly Sasakian manifold $\widehat{M}$ such that the normal components of $\left(\widehat{\nabla}_X\varphi\right)U$ lies in $\varphi-$invariant normal subbundle of M for any $X \in \Gamma(\mathcal{D})$ and $U \in \Gamma(TM)$. Then M is locally a non-trivial warped product submanifold of the type $M = N_T \times_f N_\theta$ such that N_θ is proper slant and N_T is $\varphi-$invariant submanifolds if and only if the following condition is satisfied*

$$A_{FTZ}\varphi X - A_{FZ}X = \left(\frac{1}{3}\cos^2\theta + 1\right)(\varphi X\lambda)Z + \eta(X)Z, \tag{3.22}$$

for any $X \in \Gamma(\mathcal{D} \oplus \xi)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Moreover, for a differentiable function λ on M such that $Z\lambda = 0$, for any $Z \in \Gamma(\mathcal{D}^\theta)$.

Equivalently, we give others necessary and sufficient conditions in the following table for a semi-slant submanifold to be a warped product semi-slant in numerous ambient manifolds, i.e.,

Generalizing the different types characterization results		
Manifolds Name and warped product of the form $M = N_T \times_f N_\theta$	Necessary and sufficient conditions with for a differentiable function λ on M such that $Z\lambda = 0$.	Cases to substitute in Eqs. (2.3)
Nearly Kenmotsu	$A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3}\big(2 + \sin^2\theta\big)(X\lambda)Z - \eta(X)\big(\frac{1}{3}\cos^2\theta - 1\big)Z.$	$\alpha = 0, \beta = 1.$
Nearly $\alpha-$Sasakian	$A_{FTZ}\varphi X - A_{FZ}X = \big(\frac{1}{3}\cos^2\theta + 1\big)(\varphi X\lambda)Z + \alpha\eta(X)Z.$	$\beta = 0.$
Nearly $\beta-$Kenmotsu	$A_{FTZ}X - A_{FZ}\varphi X = -\frac{1}{3}\big(2 + \sin^2\theta\big)(X\lambda)Z - \beta\eta(X)\big(\frac{1}{3}\sin^2\theta + 2\big)Z.$	$\alpha = 0.$

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