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Dynamical Behaviors in a Discrete Fractional-Order Predator-Prey System

Yao Shi[a], Qiang Ma[a,*], Xiaohua Ding[a]

[a]*Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, P.R. China*

Abstract. This paper is related to the dynamical behaviors of a discrete-time fractional-order predator-prey model. We have investigated existence of positive fixed points and parametric conditions for local asymptotic stability of positive fixed points of this model. Moreover, it is also proved that the system undergoes Flip bifurcation and Neimark-Sacker bifurcation for positive fixed point. Various chaos control strategies are implemented for controlling the chaos due to Flip and Neimark-Sacker bifurcations. Finally, numerical simulations are provided to verify theoretical results. These results of numerical simulations demonstrate chaotic behaviors over a broad range of parameters. The computation of the maximum Lyapunov exponents confirms the presence of chaotic behaviors in the model.

1. Introduction

Gupta and Chandra [1] had proposed the following predator-prey system

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{a_1 xy}{n + x} - \dfrac{qEx}{m_1 E + m_2 x}, \\ \dot{y} = sy\left(1 - \dfrac{a_2 y}{n + x}\right), \end{cases} \tag{1}$$

subjected to positive initial conditions $x(0) > 0, y(0) > 0$. Here x and y are the densities of the prey species and the predator species at time t, respectively; r and K are intrinsic growth rate and environmental carrying capacity for the prey population, respectively; a_1 is the maximum value of the per capita reduction rate of the prey; n measures the extent to which the environment provides protection for the predator and the prey (see [1–4]); s is intrinsic growth rate of the predator species; sa_2 is the maximum value of the per capita reduction rate of the predator; q is the catch-ability coefficient; E is the effort applied to harvest the prey species; and m_1, m_2 are suitable constants. All the model parameters are assumed to be only positive constants due to biological considerations. By considering the following non-dimensional scheme

$$\bar{t} = rt,\ \bar{x} = \frac{x}{K},\ \bar{y} = \frac{a_1 y}{K},\ \omega = \frac{1}{r},\ \beta = \frac{a_2}{a_1},\ k = \frac{n}{K},\ \sigma = \frac{qE}{rm_2 K},\ l = \frac{m_1 E}{m_2 K},\ \gamma = \frac{s}{r}$$

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Corresponding author: Qiang Ma
Email addresses: 16B912022@hit.edu.stu.cn (Yao Shi), hitmaqiang@hit.edu.cn (Qiang Ma), mathdxh@126.com (Xiaohua Ding)

and dropping the bars, then we obtain

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{\omega xy}{k+x} - \dfrac{\sigma x}{l+x}, \\ \dot{y} = \gamma y\left(1 - \dfrac{\beta y}{k+x}\right). \end{cases} \tag{2}$$

This is a system with Michaelis-Menten type prey harvesting and modified Leslie-Gower functional response. And this model is further studied by many authors [1, 5–8]. For example, The authors [1] have investigated that local bifurcations of the system consisting of saddle-node and Hopf bifurcations. Sufficient conditions for the existence and stability of Hopf bifurcation near positive constant equilibrium has been given to this model with diffusive effect by Li and Wang [5]. The existence of stationary distribution has been discussed in [6] by constructing a suitable Lyapunov function for this model with stochastic perturbation.

In recent decades, many authors [9–13] has pointed out that fractional-order models are more suitable than integer-order models in biology due to good memory and hereditary properties of fractional derivatives. Hence study and use the fractional-order differential equations (FDEs) help us have a better understanding of the biological system behaviors. In [7], the authors give the fractional-order predator-prey model

$$\begin{cases} D_t^{\alpha} x = x(1-x) - \dfrac{\omega xy}{k+x} - \dfrac{\sigma x}{l+x}, \\ D_t^{\alpha} y = \gamma y\left(1 - \dfrac{\beta y}{k+x}\right), \end{cases} \tag{3}$$

with the initial conditions

$$x(0) = x_0 > 0, \quad y(0) = y_0 > 0,$$

where $\alpha \in (0, 1)$. The rest of parameters is similar to that of model (2).

In the research of nonlinear dynamical systems, bifurcation and chaos have become an important topic and appears naturally in several important biological models [14–17]. Furthermore, Freedman [18] pointed out that the discrete time models governed by difference equations would indeed be more realistic than the continuous ones, when the population numbers are small or births and deaths all occur at discrete times, or with in certain intervals of time. Some dynamical systems generated by piecewise constant arguments were studied in [19–24]. They revealed far richer dynamics in discrete system compared with the continuous model. For example, in [23], authors proved that transcritical bifurcation, Flip bifurcation, Neimark-Sacker bifurcation and chaos are obtained in the discretized system. However, the dynamics of fractional-order counterpart is included only stable (unstable) equilibria. Moustafa [24] studied the globally stable of the positive fixed point for a fractional-order model of palm tree and its discretization. It is shown that the discretized system exhibits much richer dynamical behaviors than its corresponding fractional-order forms.

And motivated by the above-mentioned works, we are interested in applying the discretizations method of piecewise constant arguments to the fractional predator-prey dynamics model (3) yields

$$\begin{cases} x_{n+1} = x_n + \dfrac{h^{\alpha}}{\Gamma(\alpha+1)}\left[x_n - x_n^2 - \dfrac{\omega y_n x_n}{k+x_n} - \dfrac{\sigma x_n}{l+x_n}\right], \\ y_{n+1} = y_n + \dfrac{h^{\alpha}}{\Gamma(\alpha+1)}\left[\gamma y_n - \dfrac{\gamma\beta y_n^2}{k+x_n}\right]. \end{cases} \tag{4}$$

Where $h > 0$ represents the time interval of production.

In this paper, the discrete system (4) is further investigated in detail. And the rest of the paper is organized as follows. Sufficient conditions for the existence and stability of the fixed point of the discretized system (4) are investigated in Section 2. In Section 3, some conditions on the existence of Flip bifurcation and Neimark-Sacker bifurcation of system (4) are presented by using bifurcation theory [25]. Moreover, two different chaos control strategies are implemented for controlling the chaos due to Flip and Neimark-Sacker bifurcations in Section 4. Finally, numerical simulations verify the theoretical analysis results, including bifurcation diagrams, phase portraits and Lyapunov exponents. A brief summary is given in the last section.

2. Existence and Stability of the Fixed Points of the Discretized Fractional-Order Predator-Prey Model

2.1. Existence of the fixed points of the discretized system

In this section, we discuss the existence of the discrete predator-prey dynamics system (4). The fixed points of system (4) are solutions to the system

$$\begin{cases} x_* + \dfrac{\omega y_*}{k + x_*} + \dfrac{\sigma}{l + x_*} = 1, \\ \beta y_* = k + x_*. \end{cases} \tag{5}$$

Direct calculation yields that the system (4) has at most five non-negative fixed points,

$$E_0 = (0,0), \quad E_T = (0, y_T), \quad E_A = (x_A, 0), \quad E_B = (x_B, 0), \quad E_1 = (x_1, y_1), \quad E_2 = (x_2, y_2),$$

where

$$y_T = \frac{k}{\beta}, \quad x_A = \frac{1}{2}\left(1 - l - \sqrt{(1-l)^2 - 4(\sigma - l)}\right), \quad x_B = \frac{1}{2}\left(1 - l + \sqrt{(1-l)^2 - 4(\sigma - l)}\right),$$

$$x_1 = \frac{1}{2}\left(1 - l - \frac{\omega}{\beta} - \sqrt{\left(l - 1 + \frac{\omega}{\beta}\right)^2 - 4\left(\sigma - l + \frac{l\omega}{\beta}\right)}\right), \quad y_1 = \frac{k + x_1}{\beta}$$

and

$$x_2 = \frac{1}{2}\left(1 - l - \frac{\omega}{\beta} + \sqrt{\left(l - 1 + \frac{\omega}{\beta}\right)^2 - 4\left(\sigma - l + \frac{l\omega}{\beta}\right)}\right), \quad y_2 = \frac{k + x_2}{\beta}.$$

By simple calculation, we have the following result about the existence of positive fixed point of system (4).

Theorem 2.1. *The existence of boundary fixed points satisfies:*
(1). The point E_0 and E_T always exists.
(2). If $4\sigma \le (l+1)^2$, $l < \min\{1, \sigma\}$, then E_A and E_B exists.
(3). If $4\sigma \le (l+1)^2$, $\sigma < l$, then E_B exists.

Since the number of fixed points depends upon the quantity $\omega/\beta + \sigma/l - 1$, thus we come to the following results.

Theorem 2.2. *Suppose that $\omega/\beta + \sigma/l > 1$.*
(1). If $\omega/\beta + l < 1$ and $(\omega/\beta + l - 1)^2 > 4l(\omega/\beta + \sigma/l - 1)$, then there exists two interior fixed points E_1 and E_2.
(2). If $\omega/\beta + l < 1$ and $(\omega/\beta + l - 1)^2 = 4l(\omega/\beta + \sigma/l - 1)$, then the two interior fixed points E_1 and E_2 collide at $\bar{E} = (\bar{x}, \bar{y})$ where $\bar{x} = (1 - l - \omega/\beta)/2$.
(3). If $\omega/\beta + l > 1$ or $(\omega/\beta + l - 1)^2 < 4l(\omega/\beta + \sigma/l - 1)$, then no interior fixed point exists.

Theorem 2.3. *Suppose that $\omega/\beta + \sigma/l < 1$. Then only one interior fixed point exists, which is denoted by $E_3 = (x_3, y_3) \equiv (x_2, y_2)$.*

Theorem 2.4. *Suppose that $\omega/\beta + \sigma/l = 1$.*
(1). If $\omega/\beta + l < 1$ then $4l(\omega/\beta + \sigma/l - 1) = 0$ and $x_2 = 2(1 - l - \omega/\beta) > 0$, which means only one interior fixed point (x_2, y_2) exists.
(2). If $\omega/\beta + l > 1$ then no interior fixed point exists.

2.2. Stability of the fixed points of the discretized system

In the following, we study the asymptotic stability of the fixed points of the system (4). The Jacobian matrix of system (4) evaluated at (x, y) is given by

$$J(x,y) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - 2x - \frac{\omega k y}{(k+x)^2} - \frac{l\sigma}{(l+x)^2}\right) & -\frac{h^\alpha}{\Gamma(\alpha+1)}\frac{\omega x}{k+x} \\ \frac{h^\alpha}{\Gamma(\alpha+1)}\frac{\gamma\beta y^2}{(k+x)^2} & 1 + \frac{h^\alpha \gamma}{\Gamma(\alpha+1)}\left(1 - \frac{2\beta y}{k+x}\right) \end{pmatrix}. \tag{6}$$

The characteristic equation of the Jacobian matrix (6) can be written as

$$F(\lambda) = \lambda^2 - \lambda \mathrm{Tr} + \mathrm{Det}, \tag{7}$$

where Tr is the trace and Det is the determinant of the Jacobian matrix (6).

In order to study stability analysis of the fixed points of system (4), we need the following lemma.

Lemma 2.5. *[25] Suppose that $F(1) > 0$ in (7), λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. Then,*
(1). $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $\mathrm{Det} < 1$,
(2). $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
(3). $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $\mathrm{Det} > 1$,
(4). $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $\mathrm{Tr} \neq 0, -2$,
(5). λ_1 and λ_2 are conjugate complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $\mathrm{Tr}^2 - 4\mathrm{Det} < 0$ *and* $\mathrm{Det} = 1$.

Let λ_1 and λ_2 be the two roots of the characteristic equation of Jacobian matrix $J[E^*(x^*, y^*)]$, which are called eigenvalues of the fixed point (x^*, y^*), then we have the following definitions.

Lemma 2.6. *[25]*
(1) A fixed point E^* is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable.
(2) A fixed point E^* is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable.
(3) A fixed point E^* is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).
(4) A fixed point E^* is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Using the lemma 2.6, we can obtain the following results.

Theorem 2.7. *For system (4), the following statements hold true*
(1). The fixed point E_0 is always unstable.
(2). The fixed points E_A and E_B are always unstable.
(3). The fixed point E_T is locally asymptotically stable if and only if $\omega/\beta + \sigma/l > 1$ and

$$h < \min\left\{\left(\frac{2\Gamma(1+\alpha)}{\frac{\omega}{\beta} + \frac{\sigma}{l} - 1}\right)^{\frac{1}{\alpha}}, \left(\frac{2\Gamma(1+\alpha)}{\gamma}\right)^{\frac{1}{\alpha}}\right\}.$$

Proof. (1). The Jacobian matrix (4) evaluated at the fixed point E_0 is

$$J(E_0) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{\sigma}{l}\right) & 0 \\ 0 & 1 + \frac{h^\alpha \gamma}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{8}$$

The eigenvalues of $J(E_0)$ are

$$\lambda_1 = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{\sigma}{l}\right), \quad \lambda_2 = 1 + \frac{h^\alpha \gamma}{\Gamma(\alpha+1)}.$$

Since $0 < \alpha \leqslant 1$, $h > 0$ and $\gamma > 0$, then $\lambda_2 > 1$ which implies that fixed point E_0 is unstable.
(2). The Jacobian matrix (4) evaluated at the fixed point $E_{A,B}$ is

$$J(E_{A,B}) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - 2x_{A,B} - \frac{l\sigma}{(l + x_{A,B})^2}\right) & -\frac{h^\alpha}{\Gamma(\alpha+1)}\frac{\omega x_{A,B}}{k + x_{A,B}} \\ 0 & 1 + \frac{h^\alpha \gamma}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{9}$$

The eigenvalues of $J(E_{A,B})$ are

$$\lambda_1 = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - 2x - \frac{l\sigma}{(l+x)^2}\right), \quad \lambda_2 = 1 + \frac{h^\alpha \gamma}{\Gamma(\alpha+1)}.$$

And that λ_2 is larger than one. Therefore, $E_{A,B}$ are unstable according to the lemma 2.6.
(3). The Jacobian matrix (4) evaluated at the fixed point E_T is

$$J(E_T) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{\omega}{\beta} - \frac{\sigma}{l}\right) & 0 \\ \frac{h^\alpha}{\Gamma(\alpha+1)}\frac{\gamma}{\beta} & 1 - \frac{h^\alpha \gamma}{\Gamma(\alpha+1)} \end{pmatrix}, \tag{10}$$

which the eigenvalues are

$$\lambda_1 = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{\omega}{\beta} - \frac{\sigma}{l}\right), \quad \lambda_2 = 1 - \frac{h^\alpha \gamma}{\Gamma(\alpha+1)}.$$

Applying the stability conditions $|\lambda_1| < 1$ and $|\lambda_2| < 1$, implies that the fixed point E_1 is locally asymptotically stable if $\omega/\beta + \sigma/l > 1$ and

$$h < \min\left\{\left(\frac{2\Gamma(1+\alpha)}{\frac{\omega}{\beta} + \frac{\sigma}{l} - 1}\right)^{\frac{1}{\alpha}}, \left(\frac{2\Gamma(1+\alpha)}{\gamma}\right)^{\frac{1}{\alpha}}\right\}.$$

Otherwise it is unstable point. □

Finally, let us discuss the stability of the fixed points $E_{1,2}$. The Jacobian matrix (4) evaluated at these fixed points is

$$J(E_{1,2}) = \begin{pmatrix} 1 + Aa_{11} & Aa_{12} \\ Aa_{21} & 1 + Aa_{22} \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} A &= \frac{h^\alpha}{\Gamma(\alpha+1)}, \quad a_{11} = 1 - 2x_{1,2} - \frac{\omega k y_{1,2}}{(k + x_{1,2})^2} - \frac{l\sigma}{(l + x_{1,2})^2}, \\ a_{12} &= -\frac{\omega x_{1,2}}{k + x_{1,2}}, \quad a_{21} = \frac{\gamma\beta y_{1,2}^2}{(k + x_{1,2})^2}, \quad a_{22} = \gamma\left(1 - \frac{2\beta y_{1,2}}{k + x_{1,2}}\right), \end{aligned} \tag{12}$$

the variables $x_{1,2}$ and $y_{1,2}$ are defined in equation (5). The characteristic equation of $J(E_{1,2})$ is given by

$$F(\lambda) = \lambda^2 - \lambda\,(AM + 2) + \left(A^2N + AM + 1\right), \tag{13}$$

where $M = a_{11} + a_{22}$ and $N = a_{11}a_{22} - a_{12}a_{21}$.

Theorem 2.8. *The equilibrium point E_1 is always unstable.*

Proof. Now using the value of E_1, we get

$$\begin{aligned} F(1) &= A^2 N \\ &= -A^2 \frac{\gamma\beta x_1 y_1}{k + x_1}\left(-1 + \frac{\sigma}{(l + x_1)^2}\right) \\ &= -A^2 \frac{\gamma x_1 y_1}{(k + x_1)(l + x_1)} \sqrt{\left(\frac{\omega}{\beta} + l - 1\right)^2 - 4l\left(\frac{\omega}{\beta} + \frac{\sigma}{l} - 1\right)} < 0, \end{aligned}$$

which implies that at least one characteristic root is greater than one, thus the fixed point E_1 is unstable. □

Theorem 2.9. *The fixed point E_2 of system (4) has at least four different topological types for all permissible values of parameters:*

(i) E_2 is asymptotically stable (sink) if one of the following conditions holds:
(i.1) $M^2 < 4N$ and $0 < h < h_2$,
(i.2) $M^2 \geq 4N$ and $0 < h < h_1$,
(ii) E_2 is unstable (saddle) if one of the following conditions holds:
(ii.1) $M^2 \geq 4N$ and $h_1 < h < h_3$,
(iii) E_2 is unstable (source) if one of the following conditions holds:
(iii.1) $M^2 < 4N$ and $h > h_2$,
(iii.2) $M^2 \geq 4N$ and $h > h_3$,
(iv) E_2 is non-hyperbolic if one of the following conditions holds:
(iv.1) $M^2 > 4N$ and $h = h_1$ or h_3,
(iv.2) $M^2 < 4N$ and $h = h_2$,

$$h_1 = \left(\Gamma(1+\alpha)\frac{-M - \sqrt{M^2 - 4N}}{N}\right)^{\frac{1}{\alpha}}, \quad h_2 = \left(\frac{-\Gamma(1+\alpha)M}{N}\right)^{\frac{1}{\alpha}}, \quad h_3 = \left(\Gamma(1+\alpha)\frac{-M + \sqrt{M^2 - 4N}}{N}\right)^{\frac{1}{\alpha}}.$$

Proof. By applying lemma 2.6 and lemma 2.5, we can easily get the stability conditions (i)-(iii). For (iv), if $M^2 > 4N$, then equation (13) has two real roots. And if $F(-1) = 0$, i.e.,

$$F(-1) = 1 + (AM + 2) + (A^2 N + AM + 1) = A^2 N + 2AM + 4 = 0.$$

By simply calculation, we can get $h = h_1$ or h_3. On the other hand, the eigenvalues $\lambda_{1,2}$ are complex roots if $(AM + 2)^2 - 4(A^2 N + AM + 1) < 0$, which leads to

$$M^2 < 4N.$$

Let $h = h_2$, we get

$$\lambda_{1,2} = \frac{AM + 2}{2} \pm \frac{A\sqrt{4N - M^2}}{2} i,$$

then equation (13) has two conjugate eigenvalue and the modulus of each of them equals to one. □

3. Bifurcations of the Discretized Fractional-Order Predator-Prey Model

In this section, we will analyze the Flip bifurcation and Neimark-Sacker bifurcation behaviors of the positive fixed point E_2 of model (4).

3.1. Flip bifurcation of the discretized fractional-order predator-prey model

First, we discuss Flip bifurcation by choosing h as the bifurcation parameter. We can see that E_2 undergoes Flip bifurcation when one of the eigenvalues of Jacobian matrix at a fixed point is -1 and another eigenvalues is neither 1 nor -1.

The Jacobian matrix J of system (4) at the positive fixed point E_2 is shown as Equation (11). The characteristic equation of Jacobian matrix J is be written as (13), i.e.,

$$F(\lambda) = \lambda^2 - \lambda\,(AM + 2) + \left(A^2 N + AM + 1\right). \tag{14}$$

By theorem 2.9, we known that, if $M^2 > 4N$ and $h^* = h_1$ or h_3, then the eigenvalues of the fixed point E_2 are

$$\lambda_1 = -1, \quad \lambda_2 = AM + 3.$$

Meanwhile, the occurrence of Flip bifurcation requires $|\lambda_2| \neq 1$, thus $h^* \neq h_4$ and $h^* \neq h_5$, where

$$h_4 = \left(\frac{-2\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}, \quad h_5 = \left(\frac{-4\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}. \tag{15}$$

Summarize the above analysis into the following theorem.

Theorem 3.1. *The fixed point E_2 loses its stability, via a Flip bifurcation when $M^2 \geq 4N$ and $h = h_1$ or $h = h_3$ and $h \neq h_4, h_5$, where*

$$h_1 = \left(\Gamma(1+\alpha)\frac{-M - \sqrt{M^2 - 4N}}{N}\right)^{\frac{1}{\alpha}}, \quad h_3 = \left(\Gamma(1+\alpha)\frac{-M + \sqrt{M^2 - 4N}}{N}\right)^{\frac{1}{\alpha}},$$

$$h_4 = \left(\frac{-2\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}, \quad h_5 = \left(\frac{-4\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}.$$

3.2. Nerimark-Sacker bifurcation of the discretized fractional-order predator-prey model

We next give the conditions of existence of Neimark-Sacker bifurcation by using the bifurcation theorem [25], where h is chosen as a bifurcation parameter. Neimark-Sacker bifurcation occurs when two eigenvalues of the Jacobian matrix at a fixed point are a pair of complex conjugate numbers with module one. The characteristic equation of Jacobian matrix J is be written as (13), i.e.,

$$F(\lambda) = \lambda^2 - \lambda\,(AM + 2) + \left(A^2 N + AM + 1\right).$$

By theorem 2.9, the eigenvalues $\lambda_{1,2}$ are complex conjugate eigenvalue and the modulus of each of them equals to one for $M^2 < 4N$ and $h = h_2$. Under these conditions, there are

$$\lambda_{1,2} = \frac{AM + 2}{2} \pm \frac{A\sqrt{4N - M^2}}{2} i.$$

Moreover, the occurrence of Neimark-Sacker bifurcation also requires the following conditions,

$$d = \left.\frac{d|\lambda(h)|^2}{dh}\right|_{h=h_2} = \frac{\alpha M^2}{N} h_2^{-1} > 0, \tag{16}$$

and

$$(\lambda(h_2))^{\theta} \neq 1, \quad \theta = 1, 2, 3, 4.$$

In addition, $\lambda^2 \neq 0$ and $\lambda^4 \neq 0$ equals to

$$(2 + MA)^2 - 2 \neq 0$$

and

$$(2 + MA)^2 - 4 \neq 0.$$

Obviously, (16) is right unconditionally and conditions $M^2 < 4N$ ensure $\lambda^2 \neq 0$ and $\lambda^4 \neq 0$ are right unconditionally. on the other hands, if $2 + MA \neq 0, -1$, i.e.,

$$AM \neq -2, -3, \tag{17}$$

then $\lambda^3 \neq 0$. So, we obtain $\lambda_{1,2}^k \neq 1$, $k = 1, 2, 3, 4$, when $AM \neq -2, -3$ holds. Analyzing above and Neimark-Sacker bifurcation conditions given in [25], we write the theorem as below:

Theorem 3.2. *The system (4) undergoes a Neimark-Sacker bifurcation at fixed point E_2, if the conditions $M^2 < 4N$, $h = h_2$ and $h \neq h_4, h_6$ hold, where*

$$h_2 = \left(-\frac{M\Gamma(1+\alpha)}{N}\right)^{\frac{1}{\alpha}}, \quad h_4 = \left(-\frac{2\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}, \quad h_6 = \left(-\frac{3\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}.$$

4. Chaos Control

In this section, we study two control strategies in order to move the unstable fractional periodic orbits or the fractional chaotic orbits towards the stable one. Firstly, we apply the linear feedback control method [26] to system (4). For this, we assume that the fractional-order controller of (4) is defined by

$$\begin{cases} x_{n+1} = x_n + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left[x_n - x_n^2 - \dfrac{\omega y_n x_n}{k + x_n} - \dfrac{\sigma x_n}{l + x_n}\right] + S_n, \\ y_{n+1} = y_n + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left[\gamma y_n - \dfrac{\gamma\beta y_n^2}{k + x_n}\right], \end{cases} \tag{18}$$

where $S_n = -p_1(x_n - x^*) - p_2(y_n - y^*)$ is feedback controlling force, $p_{1,2}$ stands for the feedback gains, and (x^*, y^*) be unique positive fixed point of system (4). The Jacobian matrix of system (18) evaluated at unique positive fixed point (x^*, y^*) is given as

$$J_1(x^*, y^*) = \begin{pmatrix} 1 + Aa_{11} - p_1 & Aa_{12} - p_2 \\ Aa_{21} & 1 + Aa_{22} \end{pmatrix}, \tag{19}$$

the variables A, a_{11}, a_{12}, a_{21}, a_{22} are defined in equation (12). The corresponding characteristic equation of the Jacobian matrix $J_1(x^*, y^*)$ as follow

$$\lambda^2 - \lambda\,(AM + 2 - p_1) + \left(A^2N + AM + 1 + Aa_{21}p_2 - Aa_{22}p_1 - p_1\right) = 0, \tag{20}$$

where $M = a_{11} + a_{22}$ and $N = a_{11}a_{22} - a_{12}a_{21}$. Let λ_1 and λ_2 are the eigenvalues of the characteristic equation (20), then we have

$$\lambda_1\lambda_2 = (1 + Aa_{22})(1 + Aa_{11} - p_1) - Aa_{21}(Aa_{12} - p_2). \tag{21}$$

The lines of marginal stability are determined by solving equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These restrictions guarantee that the eigenvalues λ_1 and λ_2 have absolute value less than 1. Suppose that $\lambda_1\lambda_2 = 1$, then equation (21) implies that

$$l_1: \quad (Aa_{22} + 1)p_1 - Aa_{21}p_2 = A^2N + AM. \tag{22}$$

We then suppose that $\lambda_1 = 1$ or -1, then from the equation (20) we obtain

$$l_2: \quad a_{22}p_1 - a_{21}p_2 = AN, \tag{23}$$

$$l_3: \quad (Aa_{22} + 2)p_1 - Aa_{21}p_2 = A^2N + 2AM + 4. \tag{24}$$

The stable eigenvalues lie within the triangular region bounded by the lines l_1, l_2 and l_3.

Next, in order to control the chaos produced by Neimark-Sacker bifurcation in system (4), we introduce hybrid control strategy [27]. Assuming that system (4) undergoes Neimark-Sacker bifurcation at fixed point (x^*, y^*), then corresponding fractional-order controlled system can be written as

$$\begin{cases} x_{n+1} = \rho x_n + \dfrac{\rho h^\alpha}{\Gamma(\alpha+1)}\left[x_n - x_n^2 - \dfrac{\omega y_n x_n}{k + x_n} - \dfrac{\sigma x_n}{l + x_n}\right] + (1-\rho)x_n, \\ y_{n+1} = \rho y_n + \dfrac{\rho h^\alpha}{\Gamma(\alpha+1)}\left[\gamma y_n - \dfrac{\gamma\beta y_n^2}{k + x_n}\right] + (1-\rho)y_n, \end{cases} \tag{25}$$

where $0 < \rho < 1$ and controlled strategy in (25) is a combination of both parameter perturbation and feedback control. Moreover, by suitable choice of controlled parameter ρ, the Neimark-Sacker bifurcation of the fixed point (x^*, y^*) of controlled system (25) can be advanced (delayed) or even completely eliminated. The Jacobian matrix of controlled system (25) evaluated at positive fixed point (x^*, y^*) is given by

$$J_2(x^*, y^*) = \begin{pmatrix} 1 + A\rho a_{11} & A\rho a_{12} \\ A\rho a_{21} & 1 + A\rho a_{22} \end{pmatrix}, \tag{26}$$

where the variables $A, a_{11}, a_{12}, a_{21}, a_{22}$ are defined in equation (12). Then, positive equilibrium (x^*, y^*) of the controlled system (25) is locally asymptotically stable if roots of the characteristic polynomial of (26) lie in an open unit disk.

5. Numerical Experiments

In this section, by considering some special cases of system (4), we can confirm above theoretical analysis, and find some new interesting complex dynamics behaviors. Moreover, linear feedback technique and hybrid control strategy for chaos control are also supported by numerical simulations.

Example 5.1. *We take* $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$ *and* $1.9 \le h \le 2.9$ *in system (4), here we discuss Flip bifurcation. By calculation, we know that the system (4) have unique positive fixed point* $E_2 = (x_2, y_2) = (0.2476, 0.3250)$. *Moreover, we verify the conditions of Theorem 3.1 as follows:* $M = -1.3393$, $N = 0.2849$, $M^2 - 4N = 0.6544 > 0$, $h_1 = 1.9900$, $h_3 = 11.4357$, $h_4 = 1.5104$, $h_5 = 3.5923$. *And the characteristic polynomial evaluated at* E_2 *with* $h_1 = 1.9900$ *is given by*

$$F(\lambda) = \lambda^2 + 0.4938\lambda - 0.5062 = 0. \tag{27}$$

The roots of (27) are $\lambda_1 = -1$ *and* $\lambda_2 = 0.5062$. *Hence, according to Theorem 3.1, the conditions of Flip bifurcation are obtained near the fixed point* E_2 *at the bifurcation critical value* h_1.

Furthermore, the predator population and prey population undergo Flip bifurcation diagram are given in Figures 1(a) and 1(b) and corresponding maximum Lyapunov exponents (MLEs) are shown in Figure 1(c). Meanwhile the phase portraits of system (4) for different values of h are shown in Figure 2.

From Figure 1(a), Figure 1(b) and Figure 2, we can observe that the fixed point E_2 is stable for $h < h_1$, lose its stability at $h = h_1$ and then a cascade of period 2, 4, 8, 16 orbits emerge, finally follow irregular chaotic orbits with some uncertain period-windows. For example, Figure 2(f) illustrate that orbits of period 6 for $h = 2.79$, and chaotic attractors for $h = 2.79$ and $h = 2.83$ in Figure 2(g) and Figure 2(h). In general, the positive MLEs is considered to be one of the characteristics implying the existence of chaos. From Figure 1(c), we can see that the MLEs is negative when $h < h_1$, so system (4) is stable at this region. MLE equals to 0 when $h = h_1$, so system (4) is unstable at fixed point E_2. And as the increasing of h, MLEs greater than 0 confirm the existence of the chaotic sets.

Example 5.2. *We take*

$$\omega = 0.14,\ \beta = 2.5,\ l = 2,\ \sigma = 0.01,\ k = 0.1,\ \gamma = 599/422,\ \alpha = 0.8,\ h \in [1.78, 1.95],$$

then we get

$$M = -2.3083,\ N = 1.3336,\ M^2 - 4N = -0.0059 < 0,\ h_2 = 1.8166,\ h_4 = 0.7649,\ h_6 = 1.2697.$$

The coefficients of system (4) satisfy Theorem 3.2. By calculation, we known that at $h = h_2 = 1.8166$ *the system (4) have unique positive fixed point* $E_2 = (x_2, y_2) = (0.9406, 0.4162)$. *The characteristic polynomial evaluated at* E_2 *is given by*

$$F(\lambda) = \lambda^2 + 1.9956\lambda + 1 = 0. \tag{28}$$

The roots of (28) are $\lambda_{1,2} = -0.9978 \pm 0.066296002896102265691683258835366i$ *with* $|\lambda_{1,2}| = 1$ *and transversality condition* $d = 1.7596 > 0$. *Hence, according to Theorem 3.2, the conditions of Neimark-Sacker bifurcation are obtained near the positive fixed point* E_2 *at the bifurcation critical value* h_2.

The Neimark-Sacker bifurcation diagrams in (h, x) plane, (h, y) plane and in (h, x, y) space are shown in Figure 3(a), 3(b) and 3(c), respectively. Moreover, the corresponding MLEs of Figure 3(a) is shown in Figure 3(d). And the phase portraits of system (4) for different values of h are shown in Figure 4.

From Figure 3 and Figure 4, it is easy to observe that unique positive fixed point of system (4) is locally asymptotically stable for $h < h_2 = 1.8166$, lose its stability at $h = h_2$ and a stable invariant cycle bifurcates from the fixed point E_2 for $h > h_2$. We also see that quasi-periodic orbits on the invariant cycle arise for $h > h_2$, some period orbits emerge in the period-windows, for example, period-13 orbits for $h = 1.87$ in Figure 4(c), period-11 orbits for $h = 1.896$ in Figure 4(d), period-9 orbits for $h = 1.92$ in Figure 4(e), and period-18 orbits for $h = 1.925$ in Figure 4(f). The orbits approach to chaos with the increasing of h. Figure 4 (a)-(h) display how a smooth invariant circle bifurcates from the fixed point E_2 when $h > h_2$, then the stable circle disappears and period-13, period-11, period-9, period-18 orbits, quasi-period orbits and chaotic orbits appear. The MLEs confirm the existence of the chaotic sets in Figure 3(d).

Example 5.3. *Next, we take* $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$ *and* $h = 1.9900$. *In this case, the unique positive fixed point* $E_2 = (x_2, y_2) = (0.2476, 0.3250)$ *of system (4) is unstable. The plot of* x_n *is shown in Figure 6(a), and the plot of* y_n *is shown in Figure 6(b) for system (4). Then, the stable triangular region bounded by the marginal lines* l_1, l_2 *and* l_3 *for the controlled system (18) is shown in Figure 5. In order to make the fixed point* E_2 *locally asymptotically stable, we use the linear feedback control strategy. For this, we consider the corresponding controlled system (18) in which the feedback controlling force is taken as* $S_n = -p_1(x_n - 0.2476) - p_2(y_n - 0.3250)$ *with feedback gains* $p_1 = -2.171$ *and* $p_2 = 3.00$. *The plot of* x_n *is shown in Figure 7(a) and the plot of* y_n *is shown in Figure 7(b) for system (18). The results of our theoretical analysis are confirmed.*

Example 5.4. *Finally, let* $\omega = 0.14$, $\beta = 2.5$, $l = 2$, $\sigma = 0.01$, $k = 0.1$, $\gamma = 599/422$, $\alpha = 0.8$ *and with initial values* $(x_0, y_0) = (0.9, 0.4)$, *then the second example shows that system (4) undergoes Neimark-Sacker bifurcation as h varies in* $[1.78, 1.95]$. *Moreover, Figure 8 shows that a closed invariant circle appears at* $h = 1.817$ *enclosing this unstable positive fixed point* $E_2 = (x_2, y_2) = (0.9406, 0.4162)$. *For these parametric values, the controlled system (25) can be written as*

$$\begin{cases} x_{n+1} = x_n + \dfrac{\rho h^\alpha}{\Gamma(\alpha+1)}\left[x_n - x_n^2 - \dfrac{\omega y_n x_n}{k + x_n} - \dfrac{\sigma x_n}{l + x_n}\right], \\ y_{n+1} = y_n + \dfrac{\rho h^\alpha}{\Gamma(\alpha+1)}\left[\gamma y_n - \dfrac{\gamma\beta y_n^2}{k + x_n}\right], \end{cases} \tag{29}$$

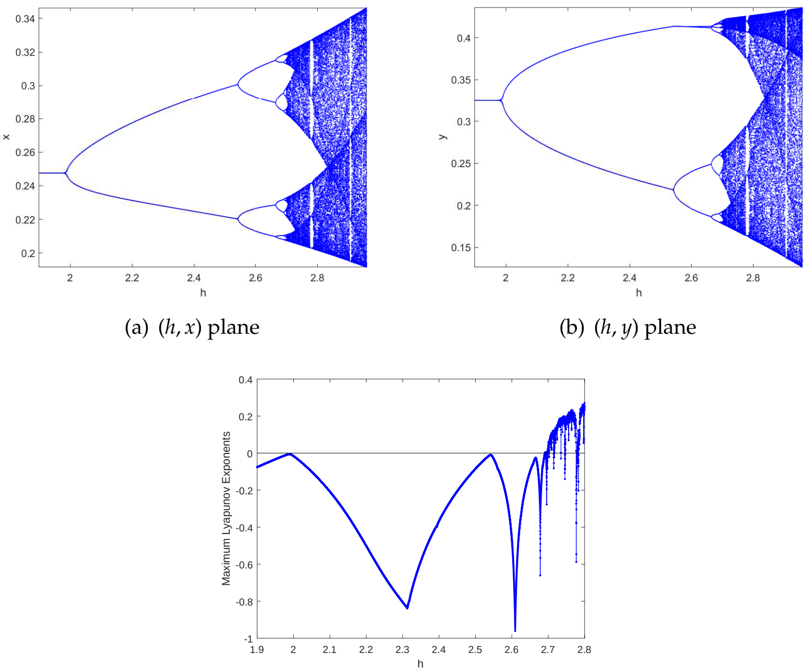
where $\omega = 0.14$, $\beta = 2.5$, $l = 2$, $\sigma = 0.01$, $k = 0.1$, $\gamma = 599/422$, $\alpha = 0.8$, $h = 1.817$ *and* $0 < \rho < 1$. *Then Jacobian matrix of controlled system (29) evaluated at* E_2 *is given by*

$$\begin{pmatrix} 1 - 1.53888\rho & -0.21908\rho \\ 0.98276\rho & 1 - 2.45689\rho \end{pmatrix}. \tag{30}$$

The characteristic polynomial of (30) is given by

$$\lambda^2 - (2 - 3.99577\rho)\lambda + 3.99617\rho^2 - 3.99577\rho + 1 = 0. \tag{31}$$

Then, the roots of (31) lie in the unit open disk if and only if $0 < \rho < 0.99990$. *Moreover, the plots for* x_n, y_n *of the controlled system (29) are shown in Figure 9 with* $\rho = 0.9935$. *From Figure 9(a), 9(b) and 9(c), it is clear that the positive fixed point* E_2 *is stable.*

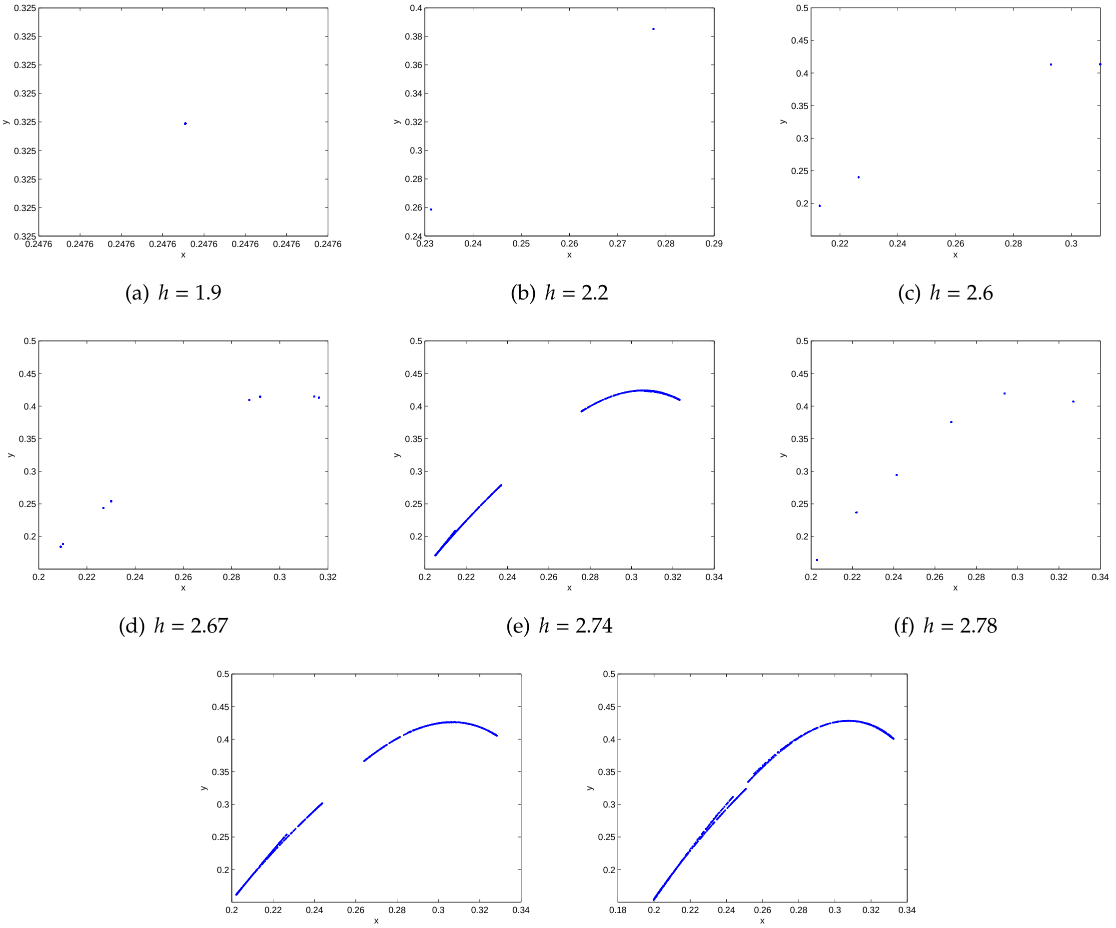


(a) (h, x) plane

(b) (h, y) plane

(c) Maximum Lyapunov exponents

Figure 1: Flip bifurcation diagram and maximum Lyapunov exponents in the $(h$, x) and $(h$, y) plane for $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$. The initial values is $(0.24, 0.32)$.

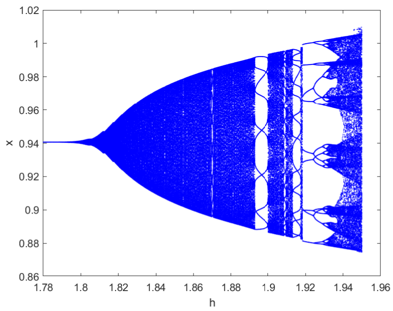


(a) $h = 1.9$ (b) $h = 2.2$ (c) $h = 2.6$

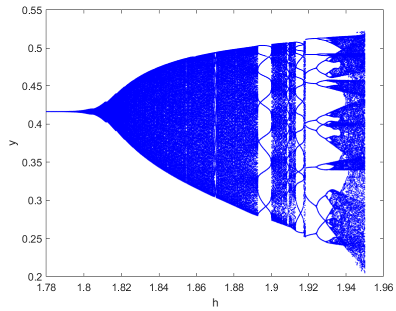
(d) $h = 2.67$ (e) $h = 2.74$ (f) $h = 2.78$

(g) $h = 2.79$ (h) $h = 2.83$

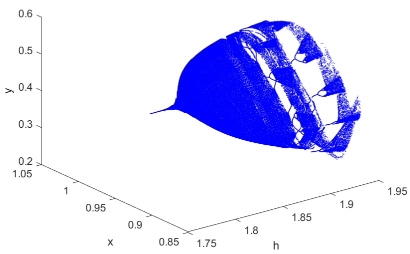
Figure 2: Phase portrait of period-1, period-2, period-4, period-6, period-8 and chaos. Here $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$. The initial values is $(0.24, 0.32)$.



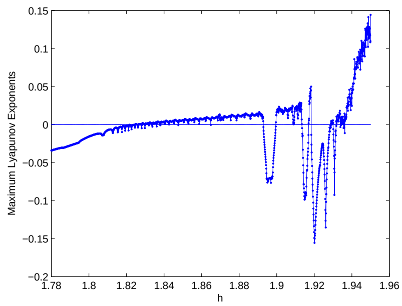
(a) (h, x) plane



(b) (h, y) plane

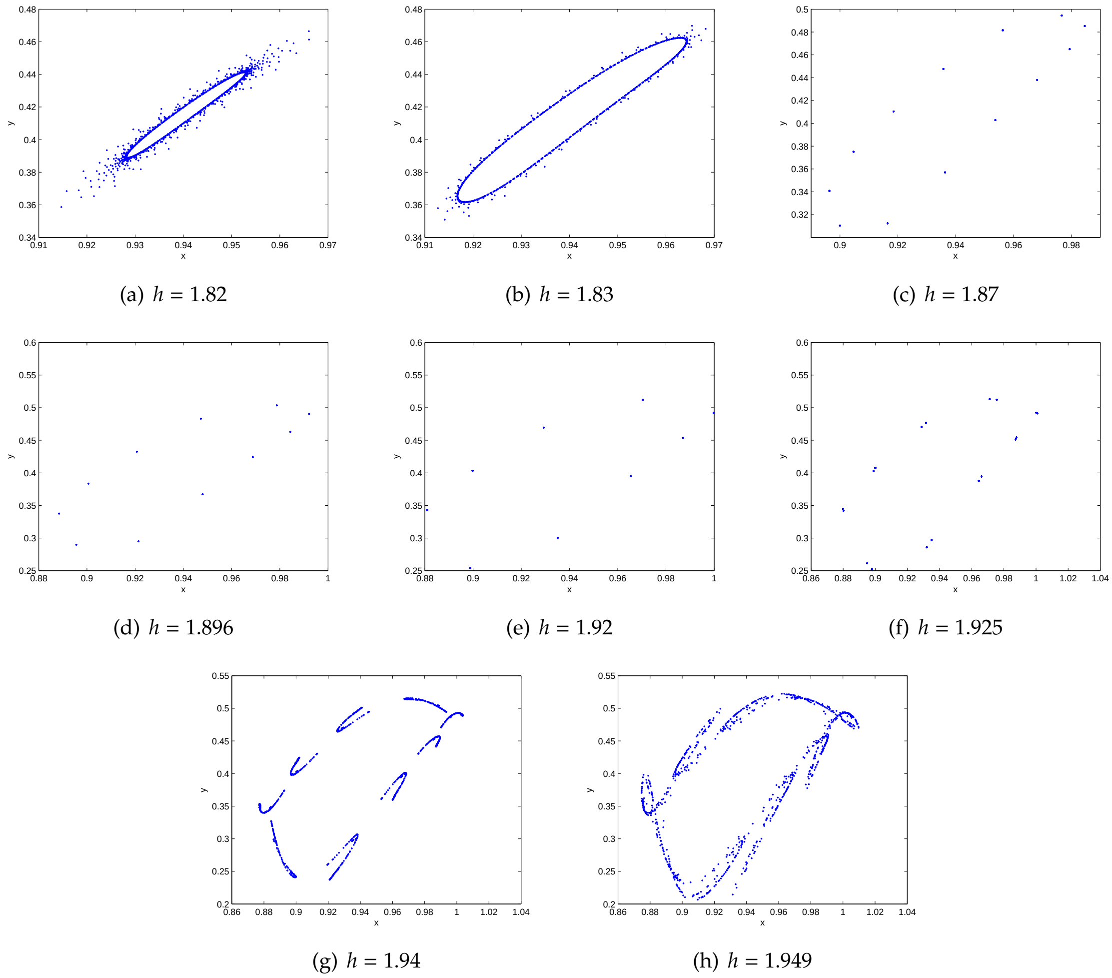


(c) (h, x, y) plane



(d) Maximum Lyapunov exponents

Figure 3: Neimark-Sacker bifurcation diagram and Maximum Lyapunov exponents in the (h, x), (h, y) and (h, x, y) plane for $\omega = 0.14$, $\beta = 2.5$, $l = 2$, $\sigma = 0.01$, $k = 0.1$, $\gamma = 599/422$, $\alpha = 0.8$. The initial values is $(0.9, 0.4)$.



(a) $h = 1.82$ (b) $h = 1.83$ (c) $h = 1.87$

(d) $h = 1.896$ (e) $h = 1.92$ (f) $h = 1.925$

(g) $h = 1.94$ (h) $h = 1.949$

Figure 4: Phase portrait of period-1, period-2, period-4, period-6, period-8 and chaos. Here $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 599/422$, $\alpha = 0.8$. The initial values is $(0.24, 0.32)$.

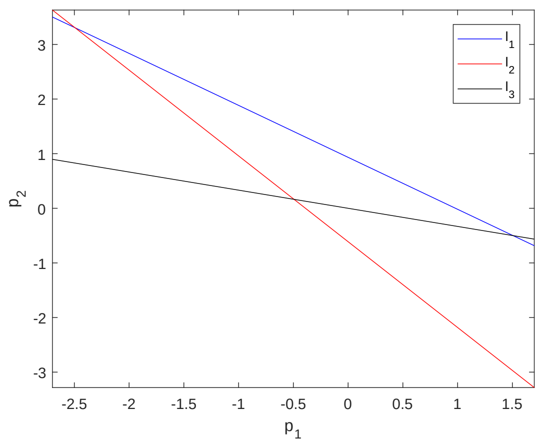
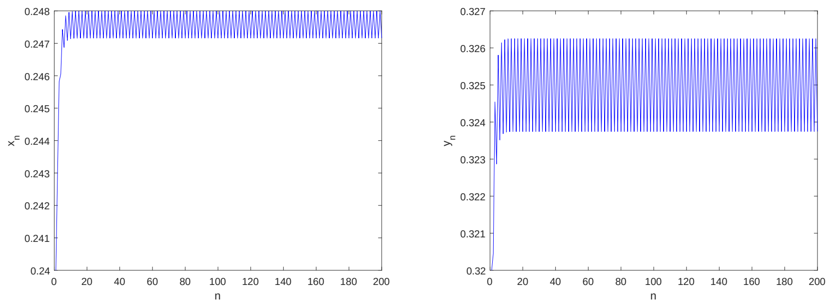
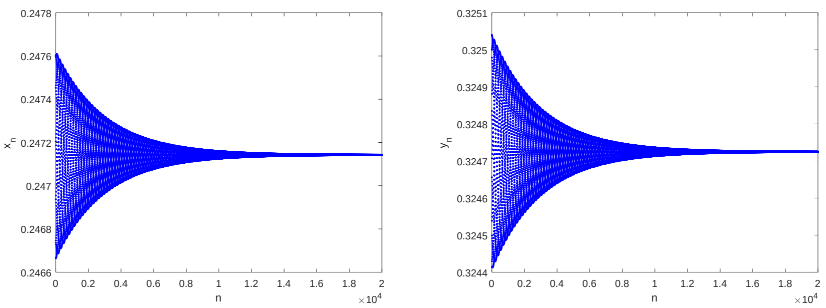


Figure 5: Stability region for the controlled system (18).



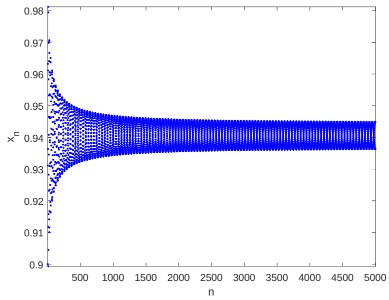
(a) Plot of x_n for system (4) (b) Plot of y_n for system (4)

Figure 6: Plots for system (4) with $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$, $h = 1.9900$ and initial conditions $(0.24, 0.32)$.

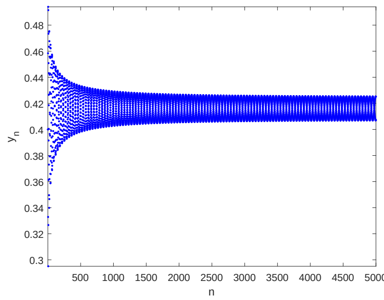


(a) Plot of x_n for controlled system (18) (b) Plot of y_n for controlled system (18)

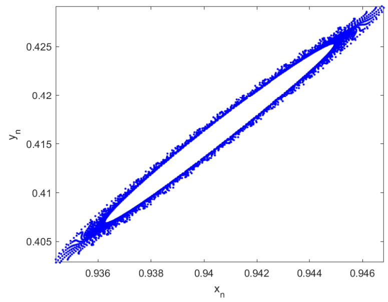
Figure 7: Plots for controlled system (18) with $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$, $h = 1.9900$, $p_1 = -2.171$, $p_2 = 3.00$ and initial conditions $(0.24, 0.32)$.



(a) Plot of x_n for system (4)

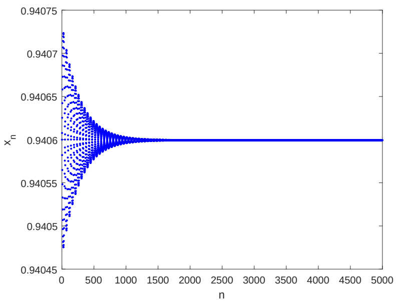


(b) Plot of y_n for system (4)

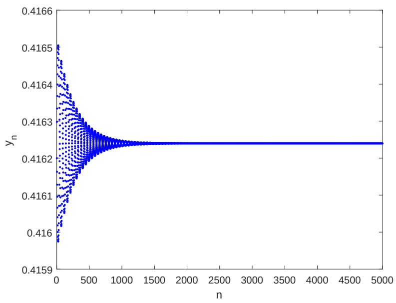


(c) Phase portrait for system (4)

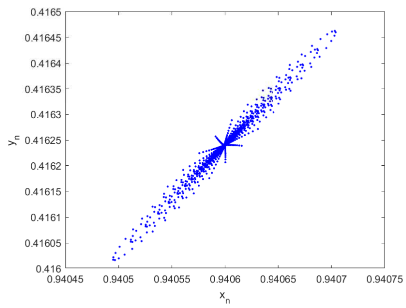
Figure 8: Plots for system (4) with $\omega = 0.14$, $\beta = 2.5$, $l = 2$, $\sigma = 0.01$, $k = 0.1$, $\gamma = 599/422$, $\alpha = 0.8$, $h = 1.817$ and initial conditions $(0.9, 0.4)$.



(a) Plot of x_n for controlled system (25)



(b) Plot of y_n for controlled system (25)



(c) Phase portrait for controlled system (25)

Figure 9: Plots for controlled system (25) with $\omega = 0.751$, $\beta = 1.571$, $l = 1.52$, $\sigma = 0.485$, $k = 0.263$, $\gamma = 1.362$, $\alpha = 0.8$, $h = 1.817$, $\rho = 0.9935$ and initial conditions $(0.9, 0.4)$.

6. Conclusions

In this paper, we have investigated some nonlinear dynamics behaviors of the fractional-order discretized predator-prey model, in which the modification is based on Leslie-Gower functional response and Mechaelis-Menten type prey harvesting. Sufficient conditions for existence of the solution of the fractional-order discrete predator-prey system (4) are have analyzed. Also, we have investigated the local stability of all the fixed points of the fractional-order system (4). We have deduced that system (4) undergoes Flip bifurcation and Neimark-Sacker bifurcation for a small range of bifurcation parameter h. Finally, two control strategies are successfully implemented to control the chaos due to emergence of Flip and Neimark-Sacker bifurcations. The analytical results have also been supported with various numerical verifications.

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