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A Note on Base-Paracompact and Monotone Base-Covering Properties

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Abstract. In the first part of this note we show that if X is a paracompact Hausdorff space and there is a locally compact closed subspace Y of X such that for every $x \in X \setminus Y$ there exists an open neighborhood O_x of x in X such that $\overline{O_x}$ is base-paracompact, then the space X is base-paracompact. In the second part of this note we introduced notions of monotonically base-paracompact (base-metacompact, base-Lindelöf) and discuss some of their properties.

1. Introduction

For two collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of a space X, we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subset V$. For a subspace U of a space X and a collection $\mathcal{V}$ of subsets of a space X, we write $U \prec \mathcal{V}$ to mean that there is some $V \in \mathcal{V}$ with $U \subset V$. For a topological space X, $w(X)$ denotes the weight of X.

A topological space X is *base-paracompact* [13] *(base-metacompact* [9]*)* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite (point-finite) refinement by members of $\mathcal{B}$. In [13] and [9], some properties of base-paracompact spaces and base-metacompact spaces are investigated. In [5], it is proved that every paracompact generalized ordered topological space (ab. GO-space) is base-paracompact.

A topological space $(X, \mathcal{T})$ is *monotonically (countably) metacompact* if each (countable) open cover $\mathcal{U}$ of the space X has a point-finite open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are (countable) open covers of the space X and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$[12]. Popvassilev showed that ω_1 and $\omega_1 + 1$ are not monotonically countably metacompact[12]. In [1], it is proved that any metacompact Moore space is monotonically metacompact and any monotonically metacompact GO-space is hereditarily paracompact. In [11], it is proved that a monotonically normal space that is monotonically countably metacompact (monotonically meta-Lindelöf) must be hereditarily paracompact. In 2013, Chase and Gruenhage proved that compact monotonically metacompact Hausdorff spaces are metrizable [3].

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In the first part of this note we show that if X is a paracompact Hausdorff space and there is a locally compact closed subspace Y of X such that for every $x \in X \setminus Y$ there exists an open neighborhood O_x of x in X such that $\overline{O_x}$ is base-paracompact, then the space X is base-paracompact. In the second part of this note we introduced notions of monotonically base-paracompact (base-metacompact, base-Lindelöf) and discuss some of their properties.

A topological space X is called *monotonically base-paracompact (monotonically base-metacompact, monotonically base-Lindelöf)* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for each open cover $\mathcal{U}$ of X there is a locally finite (point-finite, countable) open refinement $r(\mathcal{U})$ by members of $\mathcal{B}$ such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$. In this case, the operator r is called *a monotone base-paracompact (monotone base-metacompact, monotone base-Lindelöf) operator* for the space X. We point out that there exists a paracompact scattered space which is not monotonically base-paracompact. We prove that any topological space with a regular (point-regular) base is monotonically base-paracompact (monotonically base-metacompact). As a corollary, we get that every metric space is monotonically base-paracompact and every developable metacompact space is monotonically base-metacompact. In [2], it is proved that any separable GO-space is hereditarily monotonically Lindelöf. We show that any separable GO-space is hereditarily monotonically base-Lindelöf.

A subspace M of a topological space X is called *base-paracompact (base-metacompact) relative to* X if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every family $\mathcal{U}$ of open subsets of X with $M \subset \bigcup \mathcal{U}$ there is a subfamily $r(\mathcal{U})$ of $\mathcal{B}$ which is locally finite (point-finite) in X such that $r(\mathcal{U}) \prec \mathcal{U}$ and $M \subset \bigcup r(\mathcal{U})$. The notion of base-paracompact relative to a space X is introduced in [13]. A subspace M of a topological space X is called *monotonically base-paracompact (base-metacompact) relative to* X if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every family $\mathcal{U}$ of open subsets of X with $M \subset \bigcup \mathcal{U}$ there is a subfamily $r(\mathcal{U})$ of $\mathcal{B}$ which is locally finite (point-finite) in X such that $r(\mathcal{U}) \prec \mathcal{U}$, $M \subset \bigcup r(\mathcal{U})$ and if families $\mathcal{U}$ and $\mathcal{V}$ of open subsets of X satisfying that $\mathcal{U} \prec \mathcal{V}$ and $M \subset \bigcup \mathcal{U}$ then $r(\mathcal{U}) \prec r(\mathcal{V})$. If M is monotonically base-paracompact (monotonically base-metacompact) relative to X, then M is also called a *monotonically base-paracompact (monotonically base-metacompact) set relative to* X. We prove that if X is the countable union of closed monotonically base-metacompact sets relative to X, then X is monotonically base-metacompact. As a corollary, we show that every F_σ-set A of a monotonically base-metacompact space X satisfying that $w(A) = w(X)$ is monotonically base-metacompact.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [4].

2. Main Results

In [5], it is pointed out that every paracompact GO-space is base-paracompact. It is an open problem that whether every paracompact space is base-paracompact [13].

Definition 1. A subspace M of a topological space X is called *base-paracompact (base-metacompact) in* X if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ and for every open cover $\mathcal{U}$ of X there is a subfamily $\mathcal{U}'$ of $\mathcal{B}$ such that $\mathcal{U}' \prec \mathcal{U}$, $\mathcal{U}'$ is locally finite (point-finite) in X and $M \subset \bigcup \mathcal{U}'$. If M is base-paracompact (base-metacompact) in X, then M is also called a *base-paracompact (base-metacompact) set in* X.

Clearly, if a subspace M of a topological space X is base-paracompact (base-metacompact) relative to X, then M is base-paracompact (base-metacompact) in X. Every subspace M of a compact topological space X is base-paracompact in X. $\omega_1 + 1$ with the order topology is compact, but the subspace ω_1 is not paracompact. The subspace ω_1 of $\omega_1 + 1$ is base-paracompact in $\omega_1 + 1$, but it is not base-paracompact relative to $\omega_1 + 1$. Thus a subset of a topological space X which is base-paracompact in X need not be a paracompact subspace of X and a base-paracompact set in a topological space X need not be a base-paracompact set relative to X.

Proposition 2. *If M is a closed subspace of a topological space X, then M is base-paracompact (base-metacompact) relative to X if and only if M is base-paracompact (base-metacompact) in X.*

Lemma 3. *Let X be a paracompact Hausdorff space and let F be a closed subspace of X. If for each $x \in F$ there is an open neighborhood V_x of x in X such that $\overline{V_x}$ is base-paracompact, then F is base-paracompact in X.*

Proof. Let $\mathcal{B}^*$ be a base for X such that $|\mathcal{B}^*| = w(X)$. For each $x \in F$, there is an open neighborhood V_x of x in X such that $\overline{V_x}$ is base-paracompact. Thus there is a base $\mathcal{B}_x$ for $\overline{V_x}$ such that $\overline{V_x}$ is base-paracompact. So $|\mathcal{B}_x| = w(\overline{V_x})$ for each $x \in F$. The family $\{V_x : x \in F\} \cup \{X \setminus F\}$ is an open cover of X. Since X is paracompact Hausdorff, the space X is regular. The open cover $\{V_x : x \in F\} \cup \{X \setminus F\}$ of X has a locally finite open refinement $\mathcal{V}'_1$ such that for each $V_1 \in \mathcal{V}'_1$ with $V_1 \cap F \neq \emptyset$, there is some $x \in F$ such that $V_1 \subset \overline{V_1} \subset V_x$. Denote $\mathcal{V}_1 = \{V \in \mathcal{V}'_1 : V \cap F \neq \emptyset\}$. The family $\mathcal{V}_1 \cup \{X \setminus F\}$ is an open cover of X. Since X is a paracompact regular space, the open cover $\mathcal{V}_1 \cup \{X \setminus F\}$ of X has a locally finite open refinement $\mathcal{V}'_2$ such that for each $V_2 \in \mathcal{V}'_2$ with $V_2 \cap F \neq \emptyset$ there is some $V_1 \in \mathcal{V}_1$ such that $\overline{V_2} \subset V_1$. Denote $\mathcal{V}_2 = \{W \in \mathcal{V}'_2 : W \cap F \neq \emptyset\}$. For each $W \in \mathcal{V}_2$ there is some $A_W \in \mathcal{V}_1$ such that $W \subset \overline{W} \subset A_W$ and there is some $x_W \in F$ such that $A_W \subset \overline{A_W} \subset V_{x_W}$. Thus $W \subset \overline{W} \subset A_W \subset \overline{A_W} \subset V_{x_W}$. For each $x \in F$ $|\mathcal{B}_x| \leq w(X)$. Since $\mathcal{V}_2$ is locally finite in X, for each $x \in X$ there is some open neighborhood O_x of x such that $O_x \in \mathcal{B}^*$ and $|\{W \in \mathcal{V}_2 : O_x \cap W \neq \emptyset\}| < \omega$. Thus $|\mathcal{V}_2| \leq w(X)$. For each $W \in \mathcal{V}_2$ we denote $\mathcal{B}_W = \{O \in \mathcal{B}_{x_W} : O \subset A_W\}$. So $|\mathcal{B}_W| \leq w(X)$ for each $W \in \mathcal{V}_2$. Let $\mathcal{B} = \mathcal{B}^* \cup (\bigcup\{\mathcal{B}_W : W \in \mathcal{V}_2\})$. We can see that $\mathcal{B}$ is a base for X and $|\mathcal{B}| = w(X)$.

Let $\mathcal{U}$ be any open cover of X. For each $W \in \mathcal{V}_2$ the family $\mathcal{C}_W = \{U \cap A_W : U \in \mathcal{U}\} \cup \{\overline{V_{x_W}} \setminus \overline{W}\}$ is an open cover of $\overline{V_{x_W}}$. The subspace $\overline{V_{x_W}}$ of X is base-paracompact, so there is a family $\mathcal{U}^*_W \subset \mathcal{B}_{x_W}$ such that $\mathcal{U}^*_W$ is locally finite in $\overline{V_{x_W}}$ such that $\overline{V_{x_W}} = \bigcup \mathcal{U}^*_W$ and $\mathcal{U}^*_W \prec \mathcal{C}_W$. Let $\mathcal{U}_W = \{V \in \mathcal{U}^*_W : V \cap \overline{W} \neq \emptyset\}$. So $\bigcup \mathcal{U}_W \subset A_W$. Thus $\mathcal{U}_W \subset \mathcal{B}_W \subset \mathcal{B}$ and $\mathcal{U}_W$ is locally finite in X. If $\mathcal{V} = \bigcup\{\mathcal{U}_W : W \in \mathcal{V}_2\}$, then $\mathcal{V} \prec \mathcal{U}$. Since $\{A_W : W \in \mathcal{V}_2\}$ is locally finite in X and $\mathcal{U}_W$ is locally finite in X such that $\bigcup \mathcal{U}_W \subset A_W$, the family $\mathcal{V}$ is locally finite in X. We can see that $\mathcal{V} \subset \mathcal{B}$. Thus F is base-paracompact in X. □

Theorem 4. *Let X be a paracompact Hausdorff space. If there is a locally compact closed subspace Y of X such that for every $x \in X \setminus Y$ there exists an open neighborhood O_x of x in X such that $\overline{O_x}$ is base-paracompact, then X is base-paracompact.*

Proof. Let Y be a locally compact closed subspace of X such that for every $x \in X \setminus Y$ there exists an open neighborhood O_x of x in X such that $\overline{O_x}$ is base-paracompact. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| = w(X)$. It is well known that a paracompact Hausdorff space is regular. Thus the space X is regular. For each $x \in Y$ there is an open neighborhood V_x of x in X such that $\overline{V_x} \cap Y$ is compact. For each $x \in X \setminus Y$ there is an open neighborhood V_x of x in X such that $x \in V_x \subset \overline{V_x} \subset X \setminus Y$. Since X is paracompact regular space, the open cover $\{V_x : x \in X\}$ of X has a locally finite open refinement $\mathcal{V}_1$ such that for each $A \in \mathcal{V}_1$ there is some $x_A \in X$ such that $A \subset \overline{A} \subset V_{x_A}$. We can see that $|\mathcal{V}_1| \leq w(X)$. Since X is a paracompact regular space, the open cover $\mathcal{V}_1$ of X has an open refinement $\mathcal{V}_2$ which is locally finite in X, and for each $W \in \mathcal{V}_2$ there is some $A_W \in \mathcal{V}_1$ and some $x_{A_W} \in X$ such that $W \subset \overline{W} \subset A_W \subset V_{x_{A_W}} \subset \overline{V_{x_{A_W}}}$. We can see that $|\mathcal{V}_2| \leq w(X)$. Denote $\mathcal{V}_{21} = \{W \in \mathcal{V}_2 : \overline{W} \cap Y \neq \emptyset\}$ and $\mathcal{V}_{22} = \{W \in \mathcal{V}_2 : \overline{W} \cap Y = \emptyset\}$. Since $\mathcal{V}_2$ is locally finite in X, we have $|\mathcal{V}_2| \leq w(X)$. Thus $|\mathcal{V}_{21}| \leq w(X)$ and $|\mathcal{V}_{22}| \leq w(X)$. If $W \in \mathcal{V}_{21}$, then $\overline{W} \subset A_W \subset V_{x_{A_W}} \subset \overline{V_{x_{A_W}}}$. Since $\overline{W} \cap Y \neq \emptyset$, the point $x_{A_W} \in Y$. So $\overline{V_{x_{A_W}}} \cap Y$ is compact if $W \in \mathcal{V}_{21}$.

For each $W \in \mathcal{V}_{21}$ we denote $\mathcal{C}_W = \{\overline{W} \setminus \bigcup \mathcal{B}_W : \mathcal{B}_W \subset \mathcal{B}, \overline{W} \cap Y \subset \bigcup \mathcal{B}_W \subset A_W \text{ and } |\mathcal{B}_W| < \omega\}$. Since $|\mathcal{B}| = w(X)$, we have $|\mathcal{C}_W| \leq w(X)$. Let $\mathcal{C}_{21} = \bigcup\{\mathcal{C}_W : W \in \mathcal{V}_{21}\}$. So $|\mathcal{C}_{21}| \leq w(X)$. Let $\mathcal{C}_{22} = \{\overline{W} : W \in \mathcal{V}_{22}\}$. So $|\mathcal{C}_{22}| \leq w(X)$. Denote $\mathcal{C} = \mathcal{C}_{21} \cup \mathcal{C}_{22}$. Let C be an arbitrary element of $\mathcal{C}$. Then the set $C \cap Y = \emptyset$. Thus for every $x \in C$ there exists an open neighborhood O_x of x in X such that $\overline{O_x}$ is base-paracompact. So the closed subspace C of X is base-paracompact in X by Lemma 3. Thus C is base-paracompact in X for each $C \in \mathcal{C}$. So for each $C \in \mathcal{C}$ there is a base $\mathcal{B}^*_C$ for X such that $|\mathcal{B}^*_C| = w(X)$ and C is base-paracompact in X with respect to the base $\mathcal{B}^*_C$. If $\mathcal{B}' = \mathcal{B} \cup (\bigcup\{\mathcal{B}^*_C : C \in \mathcal{C}\})$, then $\mathcal{B}'$ is a base for X and $|\mathcal{B}'| = w(X)$.

Let $\mathcal{U}$ be any open cover of X. For each $W \in \mathcal{V}_{21}$ the set $\overline{W} \cap Y$ is compact. Thus there is a finite family $\mathcal{B}_W \subset \mathcal{B}$ such that $\overline{W} \cap Y \subset \bigcup \mathcal{B}_W \subset A_W$ and satisfying that for each $B \in \mathcal{B}_W$ there is some $U_B \in \mathcal{U}$ such that $B \subset U_B$. If $C_W = \overline{W} \setminus \bigcup \mathcal{B}_W$, then $C_W \in \mathcal{C}_W$ and hence $C_W \in \mathcal{C}_{21}$. If $\mathcal{U}_W = \{U \cap A_W : U \in \mathcal{U}\} \cup \{X \setminus C_W\}$, then $\mathcal{U}_W$ is an open cover of X. The set C_W is base-paracompact in X by Lemma 3. Thus there is a family $\mathcal{V}^*_{C_W} \subset \mathcal{B}^*_{C_W}$ which is locally finite in X and $\mathcal{V}^*_{C_W} \prec \mathcal{U}_W$. Denote $\mathcal{V}_{C_W} = \{V \in \mathcal{V}^*_{C_W} : V \cap C_W \neq \emptyset\}$. Thus $\mathcal{V}_{C_W} \subset \mathcal{B}^*_{C_W}$, $\mathcal{V}_{C_W}$ is locally finite in X, $\bigcup \mathcal{V}_{C_W} \subset A_W$, and every element of $\mathcal{V}_{C_W}$ is contained in some member of $\mathcal{U}$. If $\mathcal{V}_W = \mathcal{V}_{C_W} \cup \mathcal{B}_W$, then $\mathcal{V}_W \subset \mathcal{B}'$ and $\mathcal{V}_W$ is locally finite in X such that $\bigcup \mathcal{V}_W \subset A_W$ and $\overline{W} \subset \bigcup \mathcal{V}_W$.

For each $W \in \mathcal{V}_{22}$ the set $\overline{W} \cap Y = \emptyset$. So $\overline{W}$ is base-paracompact in X by Lemma 3. Denote $\mathcal{U}_W = \{U \cap A_W : U \in \mathcal{U}\} \cup \{X \setminus \overline{W}\}$ for each $W \in \mathcal{V}_{22}$. Thus $\mathcal{U}_W$ is an open cover of X. Since $\overline{W}$ is base-paracompact in X, there is a locally finite family $\mathcal{V}^*_W \subset \mathcal{B}^*_W$ such that $\mathcal{V}^*_W \prec \mathcal{U}_W$ and $\overline{W} \subset \bigcup \mathcal{V}^*_W$. If $\mathcal{V}_W = \{V \in \mathcal{V}^*_W : V \cap \overline{W} \neq \emptyset\}$, then $\mathcal{V}_W$ is locally finite in X, $\bigcup \mathcal{V}_W \subset A_W$, $\mathcal{V}_W \subset \mathcal{B}^*_W$ and every element of $\mathcal{V}_{C_W}$ is contained in some member of $\mathcal{U}$.

If $\mathcal{V} = \bigcup\{\mathcal{V}_W : W \in \mathcal{V}_2\}$, then $\mathcal{V} \subset \mathcal{B}'$ is a locally finite open refinement of $\mathcal{U}$. Thus X is a base-paracompact space. □

In what follows, we discuss some properties of monotone base-covering properties which are stronger than base-covering properties and monotone covering properties, respectively.

Obviously,

$$\begin{array}{ccc} \text{monotonically paracompact} & \Rightarrow & \text{monotonically metacompact} \\ \Uparrow & & \Uparrow \\ \text{monotonically base-paracompact} & \Rightarrow & \text{monotonically base-metacompact} \\ \Downarrow & & \Downarrow \\ \text{base-paracompact} & \Rightarrow & \text{base-metacompact.} \end{array}$$

In [12], it is proved that $\omega_1 + 1$ is not monotonically metacompact. Thus $\omega_1 + 1$ is not monotonically base-metacompact. So there exists a paracompact scattered space which is not monotonically base-paracompact. Every paracompact GO-space is base-paracompact [5, Theorem 3.1], the space $\omega_1 + 1$ is base-paracompact. Thus there is a base-paracompact space which is not monotonically base-paracompact and there is a base-metacompact space which is not monotonically base-metacompact.

In what follows, we discuss some basic properties of monotonically base-paracompact spaces and monotonicaliy base-metacompact spaces.

Recall that a base $\mathcal{B}$ for a topological space X is *point-regular* if for every point $x \in X$ and any neighborhood U of x the set of all members of $\mathcal{B}$ that contain x and meet $X \setminus U$ is finite, and a base $\mathcal{B}$ for a topological space X is *regular* if for every point $x \in X$ and any neighborhood U of x there exists a neighborhood $V \subset U$ of the point x such that the set of all members of $\mathcal{B}$ that meets both V and $X \setminus U$ is finite [4]. Clearly, every regular base of a topological space X is point-regular.

Lemma 5. ([4, Theorem 1.1.15]) *Let κ be a cardinal. If $w(X) \leq \kappa$, then for every base $\mathcal{B}$ for X there exists a base $\mathcal{B}_0$ for X such that $|\mathcal{B}_0| \leq \kappa$ and $\mathcal{B}_0 \subset \mathcal{B}$.*

For a family $\mathcal{A}$ of subsets of a topological space X we denote by $\mathcal{A}^m$ the subfamily of $\mathcal{A}$ consisting of all *maximal elements* (i.e., of sets $A \in \mathcal{A}$ such that if $A \subset A'$ and $A' \in \mathcal{A}$, then $A = A'$).

Lemma 6. ([4, Theorem 5.4.3]) *If $\mathcal{B}$ is a point-regular (regular) base for a space X, then the family $\mathcal{B}^m \subset \mathcal{B}$ is a point-finite (locally finite) cover of X.*

Theorem 7. *Let X be a topological space. If X has a regular base, then X is monotonically base-paracompact.*

Proof. Let $\mathcal{B}'$ be a regular base for X. Thus there is a regular base $\mathcal{B} \subset \mathcal{B}'$ for X such that $|\mathcal{B}| = w(X)$ by Lemma 5. Let $\mathcal{U}$ be any open cover of X. Put $r'(\mathcal{U}) = \{B \in \mathcal{B} : B \subset U$ for some $U \in \mathcal{U}\}$. Thus $r'(\mathcal{U})$ is a regular base for X. Define $r(\mathcal{U}) = r'(\mathcal{U})^m$. Thus $r(\mathcal{U})$ is a locally finite open refinement of $\mathcal{U}$ by members of $\mathcal{B}$ by Lemma 6. If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and $\mathcal{U} \prec \mathcal{V}$, then $r'(\mathcal{U}) \subset r'(\mathcal{V})$. For each $S \in r(\mathcal{U})$, the set $S \in r'(\mathcal{U})$. Since $r'(\mathcal{U}) \subset r'(\mathcal{V})$, there exists some $T' \in r'(\mathcal{V})$ such that $S \subset T'$. Since $r(\mathcal{V})$ is a collection of maximal elements from $r'(\mathcal{V})$, there exists some $T \in r(\mathcal{V})$ such that $T' \subset T$. Therefore there is some $T \in r(\mathcal{V})$ such that $S \subset T$. So $r(\mathcal{U}) \prec r(\mathcal{V})$. Thus X is monotonically base-paracompact. □

Lemma 8. ([4, Theorem 5.4.6]) *A topological space is metrizable if and only if it is a T_1-space and has a regular base.*

By Theorem 7 and Lemma 8, we have:

Theorem 9. *Every metric space is monotonically base-paracompact.*

Corollary 10. ([13, Theorem 3.3]) *Every metric space is base-paracompact.*

Recall that a base $\mathcal{B}$ for a topological space X is said to be *non-Archimedean* if $B_1, B_2 \in \mathcal{B}$ and $B_1 \cap B_2 \neq \emptyset$, then either $B_1 \subset B_2$ or $B_2 \subset B_1$. A topological space is called *non-Archimedean* if it has a non-Archimedean base. In [10], it is proved that every non-Archimedean space has a base which is a tree by reverse inclusion.

Theorem 11. *Every non-Archimedean space is monotonically base-paracompact.*

Proof. Let X be a non-Archimedean space and let $\mathcal{U}$ be any open cover of X. Thus there exists a base $\mathcal{B}$ which is a tree by reverse inclusion and $|\mathcal{B}| = w(X)$. Let $\mathcal{B}(\mathcal{U}) = \{B \in \mathcal{B} : B \subset U \text{ for some } U \in \mathcal{U}\}$. Denote $r(\mathcal{U}) = \mathcal{B}(\mathcal{U})^m$.

(1) For any $x \in X$, there exists $U \in \mathcal{U}$ and $B \in \mathcal{B}$ such that $x \in B \subset U$. Hence there exists $B_x \in r(\mathcal{U})$ such that $x \in B \subset B_x$. Thus $r(\mathcal{U})$ is a pairwise disjoint open refinement of the cover $\mathcal{U}$. So $r(\mathcal{U})$ is a locally finite refinement of $\mathcal{U}$ by members of $\mathcal{B}$.

(2) Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of the space X such that $\mathcal{V} \prec \mathcal{U}$. For any $W \in r(\mathcal{V})$ there is $V_W \in \mathcal{V}$ such that $W \subset V_W$. Since $\mathcal{V} \prec \mathcal{U}$, there exists $U_W \in \mathcal{U}$ such that $V_W \subset U_W$. So $W \in \mathcal{B}(\mathcal{U})$. Thus there exists $B_W \in r(\mathcal{U})$ such that $W \subset B_W$. Then $r(\mathcal{V}) \prec r(\mathcal{U})$. So X is a monotonically base-paracompact space. □

By a similar proof with Theorem 7, we have the following conclusion.

Theorem 12. *Let X be a topological space. If X has a point-regular base, then X is monotonically base-metacompact.*

Lemma 13. ([4, Theorem 5.4.7]) *For every Hausdorff space X the following conditions are equivalent:*
(1) The space X has a point-regular base.
(2) The space X is metacompact and has a development.

By Theorem 12 and Lemma 13, we have:

Theorem 14. *Every developable metacompact space is monotonically base-metacompact.*

Thus we have the following corollaries.

Corollary 15. ([1, Theorem 3.1]) *Every metacompact Moore space is monotonically metacompact.*

Corollary 16. ([9, Theorem 1.5]) *Every developable metacompact space is base-metacompact.*

A topological space X is *monotonically Lindelöf* [2] if for each open cover $\mathcal{U}$ of X there is a countable open cover $r(\mathcal{U})$ of X such that $r(\mathcal{U})$ refines $\mathcal{U}$ and has the property that if an open cover $\mathcal{U}$ of X refines an open cover $\mathcal{V}$ of X then $r(\mathcal{U})$ refines $r(\mathcal{V})$. The function r is called a *monotone Lindelöf operator for X*.

Proposition 17. *Every second-countable space is monotonically base-Lindelöf.*

Proof. Let X be a second-countable space and let $\mathcal{B}$ be a countable base for X. For any open cover $\mathcal{U}$ of X, put $r(\mathcal{U}) = \{B \in \mathcal{B} : \text{there is some } U \in \mathcal{U} \text{ such that } B \subset U\}$. Thus r is a monotone base-Lindelöf operator for X. □

Corollary 18. *Any separable metric space is hereditarily monotonically base-Lindelöf.*

In [2], it is proved that any separable GO-space is hereditarily monotonically Lindelöf. By a similar proof, we can show that any separable GO-space is hereditarily monotonically base-Lindelöf. In [7], it is proved that if X is a linearly ordered topological space (LOTS), then X is separable if and only if X is hereditarily separable. In [6], it is pointed out that X is a GO-space if and only if X is a subspace of a LOTS. Recall that a LOTS Y is a *linearly ordered dense extension* of a GO-space $X = (X, \tau, <)$ if Y contains X as a dense subspace and the ordering of Y extends the ordering $<$ of X [8]. Every GO-space has a linearly ordered dense extension [8]. Thus a separable GO-space is hereditarily separable.

Theorem 19. *Any separable GO-space is hereditarily monotonically base-Lindelöf.*

Proof. Let X be a separable GO-space. Since any subspace of X is a separable GO-space, it is sufficient to show that X is monotonically base-Lindelöf. Let E be a countable dense subset of X. Let $I = \{x : x$ is an isolated point of $X\}$. Since X is separable, $|I| \leq \omega$. Let $R = \{x \in X \setminus I : [x, \rightarrow)$ is open $\}$ and $L = \{x \in X \setminus I : (\leftarrow, x]$ is open $\}$. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| = w(X)$. Since every open subset of a GO-space can be uniquely represented as the union of some maximal convex open sets, we can assume every element of $\mathcal{B}$ is a convex open subset of X. Let $\mathcal{B}' = \{\{x\} : x \in I\} \cup \{(e_1, e_2) : e_1, e_2 \in E\} \cup \{[x, e) : x \in R, e \in E\} \cup \{(e, x] : x \in L, e \in E\}$. Since E is countable, $|\{(e_1, e_2) : e_1, e_2 \in E\}| \leq \omega \leq w(X)$. For any $x \in R$, the set $[x, \rightarrow)$ is open. Thus there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, \rightarrow)$. If $y \in R$ and $y \neq x$, then $[y, \rightarrow)$ is open. Thus there is some $B_y \in \mathcal{B}$ such that $y \in B_y \subset [y, \rightarrow)$. Since $x \neq y$, we have $B_x \neq B_y$. So $|R| \leq |\mathcal{B}| = w(X)$. Analogously, we have $|L| \leq w(X)$. Since E is countable, $|\{[x, e) : x \in R, e \in E\}| \leq w(X)$ and $|\{(e, x] : x \in L, e \in E\}| \leq w(X)$. Thus $\mathcal{B}'$ is a base for X such that $|\mathcal{B}'| = w(X)$.

Let $\mathcal{U}$ be any open cover of X. Let $r_1(\mathcal{U}) = \{\{x\} : x \in I\}$ and let $r_2(\mathcal{U}) = \{(e_1, e_2) : e_1, e_2 \in E$ and $(e_1, e_2) \prec \mathcal{U}\}$. Thus $r_1(\mathcal{U}) \cup r_2(\mathcal{U}) \subset \mathcal{B}'$, $|r_1(\mathcal{U}) \cup r_2(\mathcal{U})| \leq \omega$ and $r_1(\mathcal{U}) \cup r_2(\mathcal{U}) \prec \mathcal{U}$. Let $r_3(\mathcal{U}) = \{[x, e) : x \in R, e \in E, [x, e) \prec \mathcal{U}$ and $(d, e) \nprec \mathcal{U}$ for any $d \in E$ with $d < x\}$. Clearly, $r_3(\mathcal{U}) \prec \mathcal{U}, r_3(\mathcal{U}) \subset \mathcal{B}'$. Let $r_4(\mathcal{U}) = \{(e, x] : x \in R, e \in E, (e, x] \prec \mathcal{U}$ and $(e, d) \nprec \mathcal{U}$ for any $d \in E$ with $d > x\}$. Clearly, $r_4(\mathcal{U}) \prec \mathcal{U}$ and $r_4(\mathcal{U}) \subset \mathcal{B}'$. Let $r(\mathcal{U}) = r_1(\mathcal{U}) \cup r_2(\mathcal{U}) \cup r_3(\mathcal{U}) \cup r_4(\mathcal{U})$.

Firstly, we prove that $r(\mathcal{U})$ is countable. Since $r_1(\mathcal{U}) \cup r_2(\mathcal{U})$ is countable, we only need to show that $r_3(\mathcal{U}) \cup r_4(\mathcal{U})$ is countable. Now we prove that $r_3(\mathcal{U})$ is countable. Since E is countable, it is sufficient to show that the set $R(\mathcal{U}) = \{x \in R :$ there exists $e \in E$ such that $[x, e) \prec \mathcal{U}$ and $(d, e) \nprec \mathcal{U}$ for any $d \in E$ with $d < x\}$ is countable. For each $e \in E$, let $W(e) = \{x \in R : [x, e) \prec \mathcal{U}$ and $(d, e) \nprec \mathcal{U}$ for any $d \in E$ with $d < x\}$. Thus $R(\mathcal{U}) = \bigcup\{W(e) : e \in E\}$. If we show that $|W(e)| \leq 2$ for each $e \in E$, then $R(\mathcal{U})$ is countable. Suppose that there exist three distinct points x_1, x_2, x_3 in some set $W(e)$. We may assume $x_1 < x_2 < x_3$. Then $x_1 < x_2 < x_3 < e$ and $[x_i, e) \prec \mathcal{U}$ for $i = 1, 2, 3$. Since $(x_1, x_3) \neq \emptyset$, there is some $d' \in E$ such that $d' \in (x_1, x_3)$. Hence $(d', e) \prec \mathcal{U}$, where $d' \in E$. So $x_3 \notin W(e)$. A contradiction. Thus $|W(e)| \leq 2$ for each $e \in E$. So $r_3(\mathcal{U})$ is countable. Similarly, $r_4(\mathcal{U})$ is also countable. Thus $r(\mathcal{U})$ is countable.

We show that $r(\mathcal{U})$ covers X. For any $x \in X$, we show that $x \in \bigcup r(\mathcal{U})$. If $x \in I$, then $x \in \bigcup r_1(\mathcal{U})$. If $x \in X \setminus (I \cup R \cup L)$, then choose some $U \in \mathcal{U}$ such that $x \in U$. Thus there are points $e_1, e_2 \in E$ such that $x \in (e_1, e_2) \subset U$. So $(e_1, e_2) \in r_2(U)$ and hence $x \in \bigcup r(\mathcal{U})$. Now we consider the case of $x \in R \cup L$. Assume $x \in R \setminus ((\bigcup r_1(\mathcal{U})) \cup (\bigcup r_2(\mathcal{U})))$. Then for any $e \in E$ and for any $d \in E$ with $d < x < e$, we have $(d, e) \nprec \mathcal{U}$. Choose some $U \in \mathcal{U}$ such that $x \in U$. Since $x \in R \subset X \setminus I$, there is some $e \in E$ such that $x < e$ and $[x, e) \subset U$. Since $x \notin \bigcup r_2(\mathcal{U})$, the set $[x, e) \in r_3(\mathcal{U})$. Similarly, we have $x \in \bigcup r_4(\mathcal{U})$ if $x \in L \setminus ((\bigcup r_1(\mathcal{U})) \cup (\bigcup r_2(\mathcal{U})))$. So $r(\mathcal{U})$ covers X.

Finally, let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X such that $\mathcal{U} \prec \mathcal{V}$. It is obvious that $r_i(\mathcal{U}) \prec r_i(\mathcal{V})$ for $i = 1, 2$. If $x \in R$ and $[x, e) \in r_3(\mathcal{U})$, then $[x, e) \prec \mathcal{U}$ and $(d, e) \nprec \mathcal{U}$ for any $d \in E$ with $d < x$. If there exists some $d < x$ such that $(d, e) \prec \mathcal{V}$, then $(d, e) \prec r(\mathcal{V})$ and $[x, e) \subset (d, e)$. Now we assume that $(d, e) \nprec \mathcal{V}$ for any $d \in E$ with $d < x$. Thus $[x, e) \in r_3(\mathcal{V})$. Hence $r_3(\mathcal{U}) \prec r(\mathcal{V})$. Similarly, $r_4(\mathcal{U}) \prec r(\mathcal{V})$. Therefore $r(\mathcal{U}) \prec r(\mathcal{V})$.

So X is hereditarily monotonically base-Lindelöf. □

In what follows we discuss some properties of a monotonically base-paracompact (monotonically base-metacompact) set relative to a topological space X and discuss some basic properties on monotonically base-paracompact (monotonically base-metacompact) spaces.

Theorem 20. *If X is a monotonically base-paracompact (monotonically base-metacompact) space, then every closed subspace of the space X is monotonically base-paracompact (monotonically base-metacompact) relative to X.*

Proof. Let F be a closed subspace of X. Let $\mathcal{B}$ be a base for X which witnesses monotone base-paracompactness (monotone base-metacompactness) for the space X and let r be a monotone base-paracompact (monotone base-metacompact) operator for X. For any family $\mathcal{U}$ of open subsets of X with $F \subset \bigcup\mathcal{U}$, the family $\mathcal{U}' = \mathcal{U} \cup \{X \setminus F\}$ is an open cover of X. Thus $r(\mathcal{U}') \subset \mathcal{B}$ is a locally finite (point-finite) refinement of the open cover $\mathcal{U}'$. Let $r_F(\mathcal{U}) = \{B \in r(\mathcal{U}'): B \cap F \neq \emptyset\}$. Clearly, $r_F(\mathcal{U})$ is locally finite (point-finite) in X, $r_F(\mathcal{U}) \prec \mathcal{U}$ and $F \subset \bigcup r_F(\mathcal{U})$. If families $\mathcal{U}$ and $\mathcal{V}$ of open subsets of X satisfying that $\mathcal{U} \prec \mathcal{V}$ and $F \subset \bigcup\mathcal{U}$, then $\mathcal{U}' \prec \mathcal{V}'$. Since r is a monotone base-paracompact (monotone base-metacompact) operator for the space X, we have $r(\mathcal{U}') \prec r(\mathcal{V}')$. So $r_F(\mathcal{U}) \prec r_F(\mathcal{V})$. Therefore F is monotonically base-paracompact (monotonically base-metacompact) relative to X. □

Theorem 21. *Let X be a monotonically base-paracompact (monotonically base-metacompact) space. If F is a closed subspace of X with $w(F) = w(X)$, then F is monotonically base-paracompact (monotonically base-metacompact).*

Proof. Let $\mathcal{B}$ be a base which witnesses monotone base-paracompactness (monotone base-metacompactness) for the space X and let r be a monotone base-paracompact (monotone base-metacompact) operator for the space X. Put $\mathcal{B}_F = \{B \cap F : B \in \mathcal{B}\}$. Thus $\mathcal{B}_F$ is a base for F and $|\mathcal{B}_F| \leq |\mathcal{B}|$. Since $w(F) = w(X)$, we have $|\mathcal{B}_F| = w(F) = |\mathcal{B}|$. Let $\mathcal{U}$ be any open cover of F.

Let $\mathcal{U}' = \{U \cup (X \setminus F) : U \in \mathcal{U}\}$. Thus the open cover $\mathcal{U}'$ of X has a locally finite (point-finite) refinement $r(\mathcal{U}')$ by members of $\mathcal{B}$. Let $r_F(\mathcal{U}) = \{B \cap F : B \in r(\mathcal{U}')$ and $B \cap F \neq \emptyset\}$. Clearly, $r_F(\mathcal{U})$ is a locally finite (point-finite) refinement of $\mathcal{U}$ and $r_F(\mathcal{U}) \subset \mathcal{B}_F$. If an open cover $\mathcal{U}$ of F refines an open cover $\mathcal{W}$ of F, then for any $U \in \mathcal{U}$, there is some $W \in \mathcal{W}$ such that $U \subset W$. Hence $\mathcal{U}' \prec \mathcal{W}'$. Thus $r(\mathcal{U}') \prec r(\mathcal{W}')$. So $r_F(\mathcal{U}) \prec r_F(\mathcal{W})$. Therefore F is monotonically base-paracompact (monotonically base-metacompact). □

Theorem 22. *If X is the countable union of closed monotonically base-metacompact sets relative to X, then X is monotonically base-metacompact.*

Proof. Let $X = \bigcup_{i\in\omega} X_i$, where each X_i is closed and monotonically base-metacompact set relative to X. For each $i \in \omega$, there exists a base $\mathcal{B}_i$ for X such that X_i is monotone base-metacompact relative to X. Thus $|\mathcal{B}_i| = w(X)$ for each $i \in \omega$. Let $\mathcal{B} = \{B \setminus \bigcup_{j<i} X_j : B \in \mathcal{B}_i, i \in \omega\} \cup (\bigcup_{i\in\omega} \mathcal{B}_i)$. Clearly, $\mathcal{B}$ is a base for X with $|\mathcal{B}| = w(X)$ and $\mathcal{B}$ witnesses monotone base-metacompactness relative to X for each X_i. Let $\mathcal{U}$ be an open cover for X. Put $\mathcal{U}_n = \{U \in \mathcal{U} : U \cap X_n \neq \emptyset\}$ for each $n \in \omega$. Thus $\mathcal{U}_n$ is a family of open subsets of X with $X_n \subset \bigcup\mathcal{U}_n$ for each $n \in \omega$. So $r_n(\mathcal{U}_n)$ is point-finite in X such that $r_n(\mathcal{U}_n) \prec \mathcal{U}_n$, $r_n(\mathcal{U}_n) \subset \mathcal{B}_n$ and $X_n \subset \bigcup r_n(\mathcal{U}_n)$, where r_n is a monotone base-metacompact operator relative to X for the subspace X_n of X. Let $r(\mathcal{U}_0) = r_0(\mathcal{U}_0)$ and $r(\mathcal{U}_n) = \{B \setminus \bigcup_{j<n} X_j : B \in r_n(\mathcal{U}_n)\}$ for each $n > 0$. Thus $r(\mathcal{U}_n) \subset \mathcal{B}$ for each $n \in \omega$. Denote $r(\mathcal{U}) = \bigcup\{r(\mathcal{U}_n) : n \in \omega\}$. Thus $r(\mathcal{U}) \subset \mathcal{B}$.

Claim The operator r is a monotone base-metacompact operator for the space X.

Proof of Claim. (1) For any $x \in X$, there exists a minimal number $m_x < \omega$ such that $x \in X_{m_x}$. If $m_x = 0$, then $x \in B$ for some $B \in r_0(\mathcal{U}_0)$, which implies that $x \in B \in r(\mathcal{U}_0) \subset r(\mathcal{U})$. If $m_x > 0$, then $x \in B$ for some $B \in r_{m_x}(\mathcal{U}_{m_x})$. Thus $x \in B \setminus \bigcup_{j<m_x} X_j \in r(\mathcal{U}_{m_x}) \subset r(\mathcal{U})$. So $r(\mathcal{U})$ covers X.

(2) For any $x \in X$, there exists a minimal number $m_x < \omega$ such that $x \in X_{m_x}$. If $n > m_x$, then $x \notin B \setminus \bigcup_{j<n} X_j$ for each $B \in r_n(\mathcal{U}_n)$. Thus $x \notin \bigcup r(\mathcal{U}_n)$ if $n > m_x$. Since $r_i(\mathcal{U}_i)$ is point-finite in X for each $i \leq m_x$, the point x is in only finitely many members of $r(\mathcal{U}_i)$. Hence x is in only finitely many members of $r(\mathcal{U})$. For each $V \in r(\mathcal{U})$, there exists some $m_V \in \omega$ such that $V \in r(\mathcal{U}_{m_V})$. Thus there is some $W_V \in r_{m_V}(\mathcal{U}_{m_V})$ such that $V = W_V \setminus \bigcup_{i<m_V} X_i$. Since $r_{m_V}(\mathcal{U}_{m_V}) \prec \mathcal{U}_{m_V}$, there is some $U_V \in \mathcal{U}_{m_V}$ such that $W_V \subset U_V$. Thus $V \subset W_V \subset U_V$ and $U_V \in \mathcal{U}$. So $r(\mathcal{U})$ is a point-finite open refinement of $\mathcal{U}$ by members of $\mathcal{B}$.

(3) If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and $\mathcal{U} \prec \mathcal{V}$, then $\mathcal{U}_n \prec \mathcal{V}_n$ for each $n \in \omega$. Thus $r_n(\mathcal{U}_n) \prec r_n(\mathcal{V}_n)$. So $r(\mathcal{U}) \prec r(\mathcal{V})$.

Thus X is monotonically base-metacompact. □

Corollary 23. *Let X be a monotonically base-metacompact space. If $M \subset X$ is an F_σ-set of X with $w(M) = w(X)$, then M is monotonically base-metacompact.*

Proof. Since $M \subset X$ is an F_σ-set, we let $M = \bigcup_{n\in\omega} M_n$, where M_n is closed for each $n \in \omega$. By Theorem 20 each M_n is monotonically base-metacompact relative to X. Since $w(M) = w(X)$, M_n is monotonically base-metacompact relative to M. Thus M is monotonically base-metacompact by Theorem 22. □

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