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On Filters in BZ-Algebras

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Abstract. The concept of weak BCC algebra (shortly: BZ-algebra) is a much-researched logic-algebraic entity. Observing and analyzing the substructures of this algebraic structure is important since some of them can be related to the congruences in such algebras. In this article we introduce and discuss the concept of BZ-filters in BZ-algebras in a slightly different way than is present in the literature. Also, we establish connection between BZ-ideals and BZ-filters. In addition, we consider several additional conditions imposed on BZ-filters and establish links between them.

1. Introduction

BCC-algebras, introduced by Y. Komori (see [6, 7]), are an algebraic model of BIK^+-logic, i.e., implicational logic. Many authors have tried to construct some generalizations of this and similar algebras. One such an algebraic system have the same partial order as BCC-algebras and BCK-algebras but has no minimal element. Such obtained system is called a BZ-algebra [2, 10] or a weak BCC-algebra [4, 9]. From the mathematical point of view the last name is more corrected but more popular is the first ([5]).

Many mathematicians studied such algebras as BCI-algebras, B-algebras, difference algebras, implication algebras, G-algebras, Hilbert algebras, d-algebras and many others. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras and, in fact, are generalization or a special case of weak BCC-algebras. So, results obtained for weak BCC-algebras are in some sense fundamental for these algebras, especially for BCC/BCH/BCI/BCK-algebras.

A very important role in the theory of such algebras plays ideals. Many types of ideals in these algebras have been studied with various relations between them [2, 3, 5, 9–11].

In this article our intention is introduction of the concept of BZ-filters in BZ-algebras. In addition, we design some additional conditions that a BZ filter can satisfy and analyze their interconnections.

2. Preliminaries

Definition 2.1. A non-empty set A with a binary operation $'\cdot'$ and a distinguished element 0 is called a BZ-algebra (or a weak BCC-algebra) if the following axioms

(BZ-1) $(\forall x, y, z \in A)(((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = 0)$,

(BZ-2) $(\forall x \in A)(x \cdot 0 = x)$,

(BZ-3) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot z = 0) \implies x = y)$

are satisfied.

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Similarly as in BCI-algebras in any BZ-algebra A we can introduce a natural partial order $' \leqslant '$ putting

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

It is not difficult to see that in BZ-algebras the following hold
(i) $(\forall x \in A)(x \cdot x = 0)$,
(ii) $(\forall x, y, z \in A)(x \leqslant y \implies (x \cdot z \leqslant y \cdot z \ \wedge \ z \cdot y \leqslant z \cdot x))$.

Definition 2.2. ([1]) Determine $\varphi : A \longrightarrow A$ by

$$(\forall x \in A)(\varphi(x) = 0 \cdot x).$$

Some authors this mapping call "Dudek's map" as a sign of respect for its designer.
The following lemma is true.

Lemma 2.3. ([2, Lemma 2.2]) *In any BZ-algebra we have:*
(a) $(\forall x, y \in A)(\varphi(x \cdot y) \leqslant y \cdot x)$,
(b) $(\forall x \in A)(\varphi^2(x) \leqslant x)$,
(c) $(\forall x, y \in A)(\varphi(x) \cdot (y \cdot x) = \varphi(y))$,
(d) $(\forall x, y \in A)(\varphi(x \cdot y) \cdot \varphi(x) = \varphi^2(y))$,
(e) $(\forall x \in A)(\varphi^3(x) = \varphi(x))$,
(f) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi^2(x) \cdot \varphi^2(y))$,
(g) $(\forall x, y \in A)(x \leqslant y \implies \varphi(x) = \varphi(y))$,
(h) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi(y \cdot x))$,
(i) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot \varphi(x) \leqslant x \cdot ((y \cdot z) \cdot \varphi(z)))$.

3. Some Types of Ideals in BZ-Algebras

Definition 3.1. ([2]) A non-empty subset J of a BZ-algebra A is a BZ-ideal if
(1) $0 \in J$,
(2) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$.

In the following statement is given a fundamental property of BZ-ideal.

Lemma 3.2. *Let J be a BZ-ideal of a BZ-algebra A. Then:*
(3) $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ y \in J) \implies x \in J)$,
(4) $(\forall x, y \in A)((\varphi(y) \in J \ \wedge \ x \in J) \implies x \cdot y \in J)$.

Proof. If put $z = 0$ in (2) we get (3).
If we put $y = x$ and $z = y$ in (2) have

$$(\forall x, y \in A)(((x \cdot x) \cdot y = 0 \cdot y = \varphi(y) \in J) \ \wedge \ y \in J) \implies x \cdot y \in J).$$

So, the statement (4) is proven. □

Corollary 3.3. *Let J be a BZ-ideal in a BZ-algebra A. Then:*
(5) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \in J) \implies x \in J)$.

In what follows, we remind the reader on some types of BZ-ideals.

Definition 3.4. ([2, Definition 3.4]; [9, Definition 68]) An ideal BZ-ideal J of BZ-algebra A is called:
- *closed* if $\varphi(J) \subseteq J$;
- *(∗)-BZ-ideal* if $(\forall x, y \in A)((x \in J \ \wedge \ y \in A \setminus J) \implies x \cdot y \in J)$;
- *anti-grouped* if $(\forall x \in A)(\varphi^2(x) \in J \implies x \in J)$;
- *strong* if $(\forall xy \in A)((x \in J \ \wedge \ y \in A \setminus J) \implies x \cdot y \in A \setminus J)$;
- *regular* if $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ x \in J) \implies y \in J)$;
- *associative* if $(\forall x, y \in A)(x \cdot \varphi(y) \in J \implies y \cdot x \in J)$;
- T-*ideal* if $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$.

For any ideal J of a BZ-algebra A we can define a binary relation $' \prec '$ on A putting:

$$(\forall x, y \in A)(x \prec y \iff x \cdot y \in J).$$

Such defined relation is a quasi-order relation on A left-compatible and right anti-compatible with the internal operation in A. Then the relation $' \sim '$ defined in A by $\sim = \prec \cap \prec^{-1}$ is a congruence on A. This can be proven in an analogous way as it was done in the article [8]. The set $A/\sim = \{[x]_\sim : x \in A\}$ is a BZ-algebra with respect to the internal operation $[x]_\sim \circ [y]_\sim = [x \cdot y]_\sim$ (for any $x, y \in A$) ([2]). In this algebra the relation $' \leq '$ defined by $[x]_\sim \leq [y]_\sim \iff x \prec y$ is an order relation on $A/\sim$.

4. The Concept of BZ-Filters in BZ-Algebras

First, we introduce the concept of BZ-filters in a BZ-algebra by the following definition looking at the way we made it into the article [8].

Definition 4.1. A subset F of a BZ-algebra A is a BZ-filter of A if:
(6) $\neg(0 \in F)$,
(7) $(\forall x, y, z \in A)((\neg((x \cdot y) \cdot z \in F) \wedge x \cdot z \in F) \implies y \in F)$.

The BZ-filter defined on this way has the following properties.

Theorem 4.2. *Let A be BZ-algebra and F a BZ-filter of A. Then:*
(8) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge x \in F) \implies y \in F)$,
(9) $(\forall x, y \in A)((x \cdot y \in F \wedge \neg(\varphi(y) \in F)) \implies x \in F)$.

Proof. Putting $z = 0$ in (7) we obtain (8).
If we put $y = x$ and $z = y$ in (7), we have

$$(\neg((x \cdot x) \cdot y = 0 \cdot y = \varphi(y) \in F) \wedge x \cdot y \in F) \implies x \in F.$$

Therefore, (9) is proved. □

Corollary 4.3. *Let F be a BZ-filter of a BZ-algebra A. Then:*
(10) $(\forall x, y \in F)((x \leqslant y \wedge x \in F) \implies y \in F)$

Proof. Let $x, y \in A$ be arbitrary elements such that $x \leqslant y$ and $x \in F$. Thus $\neg(x \cdot y = 0 \in F)$ and $x \in F$. Then by (8) we have $y \in F$. □

Theorem 4.4. *If F is a BZ-filter of BZ-algebra A, then the set $J = A \backslash F$ is a BZ-ideal. Opposite, if J is a BZ-ideal of BZ-algebra A, then the set $F = A \backslash J$ is a BZ-filter of A.*

Proof. It is clear that $0 \in J$. Let $x, y, z \in A$ be arbitrary elements such that $(x \cdot y) \cdot z \in J$ and $y \in J$. Then we have $\neg((x \cdot y) \cdot z \in F)$ and $\neg(y \in F)$. If we suppose that $x \cdot z \in F$ by (7) we will have $y \in F$. So, must to be $\neg(x \cdot z \in F)$.

Opposite, let J be a BZ-ideal of A. It is that $\neg(0 \in A \backslash J)$. Let $x, y, z \in A$ be arbitrary elements such that $\neg((x \cdot y) \cdot z \in A \backslash J)$ and $x \cdot z \in A \backslash J$. Then $y \in A \backslash J$. Indeed. If it were $y \in J$ then $x \cdot z \in J$ follows from $(x \cdot y) \cdot z \in J$ and the $y \in J$ which is in a contradiction with the assumption $x \cdot z \in A \backslash J$. □

Theorem 4.5. *The family $\mathfrak{F}_A$ of all BZ-filters in BZ-algebra A forms a completely lattice.*

Proof. Let $\{F_{i \in I}\}$ be a family of BZ-filters in BZ-algebra A. It is clear that $\neg(0 \in \bigcap_{i \in I} F_i)$ and $\neg(0 \in \bigcup_{i \in I} F_i)$.

(a) Let $x, y, z \in A$ arbitrary elements such that $\neg((x \cdot y) \cdot z \in \bigcup_{i \in I} F_i)$ and $x \cdot z \in \bigcup_{i \in I} F_i$. Thus $\neg((x \cdot y) \cdot z \in F_i)$ for all $i \in I$ and there exists an index $j \in I$ such that $x \cdot z \in F_j$. Then $y \in F_j \subseteq \bigcup_{i \in I} F_i$.

(b) Let $\mathfrak{X}$ be the family of all BCC-filters which contained in the intersection $\bigcap_{i \in I} F_i$. The union $\bigcup \mathfrak{X}$ is the minimal BCC-filter contained in the intersection $\bigcap_{i \in I} F_i$.

(c) So, if we choose $\sqcap_{i \in I} F_i = \bigcup \mathfrak{X}$ and $\sqcup_{i \in I} F_i = \bigcup_{i \in I} F_i$, then $(\mathfrak{F}, \sqcap, \sqcup)$ is a completely lattice. □

5. Some Types of BZ-Filters

In the theory of algebras, important role play ideals. Filters in these algebras can also be important since they are highly related to the ideals. Our intent in this section is to determine the various types of filters in the BZ-algebras and establish the relationships between them.

In what follows, we analyze the conditions that a BZ-filter could satisfy. Marks for these conditions we borrowed from the corresponding ideals in BZ-algebras since we have been given them by modification of the logical atoms in the formulas that are determined these ideals.

(T) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in F) \wedge x \cdot z \in F) \implies y \in F)$;
(*) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \implies x \in F$;
(R) $(\forall x, y \in A)((\neg(x \in F) \wedge y \in F) \implies xy \in F)$;
(S) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge y \in F) \implies x \in F)$;
(A) $(\forall x, y \in A)(x \cdot y \in F \implies y \cdot \varphi(x) \in F)$;
(C) $(\forall x \in A)(\varphi(x) \in F \implies x \in F)$;
(AG) $(\forall x \in A)(x \in F \implies \varphi^2(x) \in F)$.

Our first proposition refers to BZ-filter which satisfies condition (T).

Proposition 5.1. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (T). *Then:*
(T1) $(\forall x, y \in A)(x \cdot \varphi(y) \in F \vee \neg(x \cdot y \in F))$.

Proof. If we put $y = 0$ and $z = y$ in formula (T), we get the following

$$(\forall x, y \in A)((\neg(x \cdot \varphi(y) \in F) \wedge x \cdot y \in F) \implies 0 \in F)$$

which is a contradiction. So, we have

$$(\forall x, y \in A)\neg(\neg(x \cdot \varphi(y) \in F) \wedge x \cdot y \in F).$$

Therefore, for any BZ-filter F satisfying condition (T) have to be

$$(\forall x, y \in A)(x \cdot \varphi(y) \in F \vee \neg(x \cdot y \in F))$$

□

Corollary 5.2. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (T). *Then:*
(T2) $(\forall y \in A)(\varphi^2(y) \in F \vee \neg(\varphi(y) \in F))$.

Proof. Putting $x = 0$ in (T1) we get (T2). □

Our second proposition relates to a BZ-filter that satisfies the condition (*)

Proposition 5.3. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (*). *Then*
(*1) $(\forall y \in A)(\neg(\varphi(y) \in F) \vee \neg(y \in F))$.

Proof. If we put $x = 0$ in formula (*) we get the following

$$(\forall y \in A)((0 \cdot y \in F \wedge y \in F) \implies 0 \in F)$$

which is a contradiction. So, we have

$$(\forall y \in A)\neg(\varphi(y) \in F \wedge y \in F).$$

Therefore, for any BZ-filter F satisfying the condition (*), the condition (*1) is valid also. □

Proposition 5.4. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (R). *Then:*
(R1) $(\forall y \in A)(y \in F \implies \varphi(y) \in F)$.

Proof. If we put $x = 0$ in the formula (R) we have $(\neg(0 \in F) \wedge y \in F) \implies \varphi(y) \in F$. Since the condition $\neg(0 \in F)$ is valid by (6), for any $y \in A$ we have $y \in F \implies \varphi(y) \in F$. So, for any BZ-filter in BZ-algebra A satisfying the condition (R), the formula (R1) is valid. □

Our third proposition refers to the BZ-filter that satisfies the condition (S).

Proposition 5.5. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (S). *Then*
(S1) $(\forall y \in A)(\varphi(y) \in F \vee \neg(y \in F))$.

Proof. Putting $x = 0$ in (S) we get the following

$$(\forall y \in A)((\neg(\varphi(y) \in F) \wedge y \in F) \implies 0 \in F).$$

We got a contradiction. This contradiction negates the hypothesis in the previous implication. So, have to be $\neg(\neg(\varphi(y) \in F) \wedge y \in F)$. Therefore, the following $\varphi(y) \in F \vee \neg(y \in F)$ is proven for any $y \in A$. □

Our next assertion refers to the BZ filter in the BZ algebra that satisfies condition (A).

Proposition 5.6. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (A). *Then:*
(A1) $(\forall x \in A)(x \in F \implies \varphi^2(x) \in F)$, *and*
(A2) $(\forall y \in A)(\varphi(y) \in F \implies y \in F)$.

Proof. Putting $y = 0$ in (A) we get (A1). If put $x = 0$ in (A) we will get (A2). □

With (mark) we mark any of the conditions mentioned above. We will write $F \in$ (*mark*) if we want to say that the BZ-filter F satisfies the condition (*mark*). The obtained results in the preceding propositions allow us to make a summary of the interdependence of the conditions that we intend BZ-filters satisfies.

Corollary 5.7. *For any BZ-filter F of BZ-algebra A the following holds*

$$F \in (A) \implies F \in (C) \cap (AG).$$

Before we expose the first theorem about the BZ-filters in the BZ-algebra, we recall the readers to the term 'consistent subset': For a subset X of a algebra A we say that it is a *consistent subset* in A if and only if the following is valid

$$(\forall x, y \in A)(x \cdot y \in X \implies (x \in X \vee y \in X)).$$

Theorem 5.8. *A BZ-filter F of BZ-algebra A satisfies the condition (C) if and only if F is a consistent subset of A.*

Proof. Assume that a BZ-filter F is a consistent subset in A. Let $x \in A$ be an arbitrary element such that $\varphi(x) \in F$. Thus from $0 \cdot x \in F$ follows $0 \in F$ or $x \in F$. Since the first option is impossible by (6), we have $x \in F$. So, $F \in (C)$.

Conversely, Let $F \in (C)$ be holds for a BZ-filter F of a BZ-algebra A and let $x, y \in A$ be arbitrary elements such that $x \cdot y \in F$. We have two options:

(i) Suppose $x \cdot y \in F$ and $\neg(y \in F)$. Thus $\neg(\varphi(y) \in F)$. Then $x \in F$ by (9).

(ii) Let $x \cdot y \in F$ and $\neg(x \in F)$ is valid. If we suppose $\neg(y \in F)$, thus $\neg(\varphi(y) \in F)$ is valid. Then from $x \cdot y \in F$ and $\neg(\varphi(y) \in F)$ would get $x \in F$ according to (9). We got a contradiction. Therefore, it must be $y \in F$.

Finally, the filter F is a consistent subset of A. □

Theorem 5.9. *For a BZ-filter F of BZ-algebra A the following holds*

$$F \in (C) \cap (*) \text{ if and only if } (\forall x \in A)\neg(\varphi(x) \in F).$$

Proof. Suppose $(\forall x \in A)\neg(\varphi(x) \in F)$ holds. Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in F$ and $y \in F$. Then $\neg(\varphi(y) \in F)$ by hypothesis. Thus from $x \cdot y \in F$ and $\neg(\varphi(y) \in F)$follows $x \in F$ by (9). So, $F \in (*)$.

Further, suppose $\varphi(x) \in F$ and $\neg(x \in F)$. Thus $x \in A\backslash F$ and $\neg(\varphi(x) \in F)$. We got a contradiction. So, have to be $x \in F$. So, $F \in (C)$.

Opposite, suppose (C) and $(*)$ hold. Let $x \in A$ be an arbitrary element. If we suppose $\varphi(x) \in F$, thus $x \in F$ by the condition (C). On the other side, from $0 \cdot x \in F$ and $x \in F$ follows $0 \in F$. We got a contradiction. So, have to be $\neg(\varphi(x) \in F)$. Therefore, for a BZ-filter F satisfies conditions (C) and $(*)$ the following $(\forall x \in A)\neg(\varphi(x) \in F)$ holds. □

Theorem 5.10. *For a BZ-filter F of a BZ-algebra A the following are equivalent:*

(i) $F \in (AG)$;

(ii) $(\forall x, y \in A)((y \in F \wedge x \leqslant y) \implies x \in F)$;

(iii) $(\forall x, y, z \in A)((x \in F \wedge \neg(y \in F)) \implies (x \cdot z) \cdot (y \cdot z) \in F)$;

(iv) $(\forall x, z \in A)(x \in F \implies (x \cdot z) \cdot \varphi(z) \in F)$.

Proof. (i) $\Longrightarrow$ (ii). L$y \in F$ and $x \leqslant y$. Thus $x \cdot y = 0$. Suppose $y \cdot x \in F$. Then $\varphi^2(y \cdot x) \in F$ because $F \in (AG)$. From this follows $0 = \varphi(0) = \varphi(x \cdot y) = \varphi^2(y \cdot x) \in F$ by (h). We got a contradiction. So, $\neg(y \cdot x \in F)$. Now, from $\neg(y \cdot x \in F)$ and $y \in F$ follows $x \in F$ by (8).

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds. Let $x, y \in A$ be arbitrary elements such that $x \in F$ and $\neg(y \in F)$. Thus $x \cdot y \in F$. Indeed. If there were $\neg(x \cdot y \in F)$ then from $\neg(x \cdot y \in F)$ and $x \in F$ would follow $y \in F$ by (8). It would have been contradictory. Therefore, it must be $x \cdot y \in F$. Since $(x \cdot z) \cdot (y \cdot z) \leqslant x \cdot y$ by (BZ-1) and $x \cdot y \in F$ we have $(x \cdot z) \cdot (y \cdot z) \in F$, by (ii).

(iii) $\Longrightarrow$ (iv). Putting $y = 0$ we get (iv).

(iv) $\Longrightarrow$ (i). Putting $z = x$ we get (i). □

Theorem 5.11. *A BZ-filter F of a BZ-algebra A satisfies condition (C) and (AG) if and only if for every $x \in A$ both x and $\varphi(x)$ belong or not belong to F.*

Proof. Suppose $F \in (C) \cap (AG)$. Let $x \in A$ be an arbitrary element such that $x \in F$. Thus $\varphi^2(x) \in F$ by (AG). Then $\varphi(x) \in F$ by (C). On the other hand, if $\varphi(x) \in F$ then $x \in F$ according to (c). Therefore, both x $\varphi(x)$ belong or not belong to F.

Conversely, let a BZ-filter F has the property that both x and $\varphi(x)$ belong or not belong to F for any $x \in A$. Suppose $x \in F$ and $\varphi(x) \in F$. Thus $\varphi^2(x) \in F$ again. So, F obviously satisfy the condition (C) and the condition (AG). Analogously, the second option can be demonstrated. □

Theorem 5.12. *For any BZ-filter F of a BZ-algebra A the following holds*

$$F \in (S) \iff F \in (C) \cap (AG).$$

Proof. Suppose $F \in (S)$.

Let $x \in A$ be arbitrary element such that $x \in F$. Since $\varphi^2(x) \leqslant x$ by (b), from $\neg(\varphi^2(x) \cdot x = 0 \in F)$ and $x \in F$ follows $\varphi^2(x) \in F$ by (S). So, $F \in (AC)$.

Let $x \in A$ be arbitrary element such that $\varphi(x) \in F$. Suppose $\neg(\varphi^2(x) \in F)$. Thus $0 \in F$ by the condition (S). So, it have to be $\varphi^2(x) \in F$. Since $\varphi^2(x) \leqslant x$, it follows $x \in F$ by Corollary 4.1. Therefore, $F \in (C)$.

Suppose $F \in (C)$ and $F \in (AG)$. Let $x, y \in A$ be arbitrary elements such that $\neg(x \cdot y \in F)$ and $y \in F$. Suppose that $\neg(x \in F)$. Thus $\varphi^2(y) \in F$ by (AG). Further on, then by (d) $\varphi^2(y) = \varphi(x \cdot y) \cdot \varphi(x) \in F$ and $\varphi(x \cdot y) \in F$ or $\varphi(x) \in F$ by Theorem 5.1 and $x \cdot y \in F$ or $x \in F$. Since both cases are in contradictions with hypothesis, then it must be $x \in F$. Finally, $F \in (S)$. □

Theorem 5.13. *For any BZ-filter F of a BZ-algebra A the following holds*

$$F \in (R) \iff F \in (C) \cap (AG).$$

Proof. Suppose $F \in (R)$.

Let $x \in A$ is an arbitrary element such that $x \in F$ and $\neg(\varphi^2(x) \in F)$. Thus $\varphi^2(x) \cdot x \in F$ by (R). On the other hand, according to (b), we have $\varphi^2(x) \cdot x = 0$ and $\neg(\varphi^2(x) \cdot x \in F)$ We got a contradiction. So, it must be $\varphi^2(x) \in F$. This means $F \in (AG)$.

Suppose $\varphi(x) \in F$. But, from $\neg(0 \in F)$ and $\varphi(x) \in F$ follows $\varphi^2(x) = 0 \cdot \varphi(x) \in F$ by (R) Further, from $\varphi^2(x) \leqslant x$ and $\varphi^2(x) \in F$ follows $x \in F$ by Corollary 4.1. So, $F \in (C)$.

Opposite, suppose $F \in (C) \cap (AG)$ holds. Let $x, y \in A$ be arbitrary elements such that $\neg(x \in F)$ and $y \in F$. Thus $\neg(\varphi(c) \in F)$ by (C) and $\neg(\varphi^2(x) \in F)$ by (C) again. Besides we have $\varphi^2(y) \in F$ from $y \in F$ by (AG). On the other hand we have $\varphi^2(y) = \varphi(x \cdot y) \cdot \varphi(x) \in F$ by (d). Now, from $\varphi(x \cdot y) \cdot \varphi(x) \in F$ and $\neg(\varphi^2(x) \in F)$ follows $\varphi(x \cdot y) \in F$ by (9). Finally, thus $x \cdot y \in F$ by (C). Therefore, $F \in (R)$. $\square$

Corollary 5.14. *For any BZ-filter F of BZ-algebra A the following holds*

$$F \in (S) \iff F \in (R).$$

Theorem 5.15. *For any BZ-filter of a BZ-algebra the following conditions are equivalent:*

(T) $(\forall x, y, z \in F)(\neg(x \cdot (y \cdot z) \in F) \wedge x \cdot z \in F) \implies y \in F)$,

(Ta) $(\forall x, z \in A)(x \cdot z \in F \implies x \cdot \varphi(z) \in F)$,

(Tb) $(\forall x, z \in A)(x \cdot \varphi^2(z) \in F \implies x \cdot \varphi(z) \in F)$,

(Tc) $(\forall x \in A)\neg(\varphi(x) \cdot x \in F)$.

Proof. (T) $\Longrightarrow$ (Ta). Putting $y = 0$ in (T) we obtain (T1). Thus (Ta).

(Ta) $\Longrightarrow$ (T). Suppose (Ta). Let $x, z \in A$ be arbitrary elements such that $\neg(x \cdot (y \cdot z) \in F$ for some $y \in A$ and $x \cdot z \in F$. Thus $x \cdot \varphi(z) \in F$. On the other hand, if we put $x = y$ and $y = 0$ in (BZ-1) we get $(y \cdot z) \cdot (0 \cdot z) \leqslant y \cdot 0$. Again, if we put $y = y \cdot z$ and $z = \varphi(z)$ in (BZ-1) we get $(x \cdot \varphi(z)) \cdot ((y \cdot z)\varphi(z)) \leqslant x \cdot (y \cdot z)$. Since $\neg(x \cdot (y \cdot z) \in F)$ we have $\neg((x \cdot \varphi(z)) \cdot ((y \cdot z)\varphi(z)) \in F)$. Now, from the last formula and $x \cdot \varphi(z) \in F$ we get $(y \cdot z) \cdot \varphi(z) \in F$ by (8). Again, from $(y \cdot z) \cdot \varphi(z) \leqslant y$ we get $y \in F$ by Corollary 4.1.

(Ta) $\Longrightarrow$ (Tb). Putting $z = \varphi^2(z)$ in (Ta) we obtain $x \cdot \varphi^3(z) \in F$ and $x \cdot \varphi(z) \in F$ by (e).

(Tb) $\Longrightarrow$ (Ta). Let $x, z \in A$ be arbitrary elements such that $x \cdot z \in F$. Since $\varphi^2(z) \leqslant z$ by (b), we have $x \cdot z \leqslant x \cdot \varphi^2(z)$. Thus $x \cdot \varphi^2(z) \in F$ by Corollary 4.1 and $x \cdot \varphi(z) \in F$ by (Tb).

(T) $\Longrightarrow$ (Tc). By Proposition 5.1 we have (T) $\Longrightarrow$ (T1). Putting $x = \varphi(y)$ in (T1) we obtain $0 = \varphi(y) \cdot \varphi(y) \in F$ or $\neg(\varphi(y) \cdot y \in F)$. Since, the first option is impossible, we have $\neg(\varphi(y) \cdot y \in F)$.

(Tc) $\Longrightarrow$ (Ta). Let (Tc) be holds. Let $x, y, z \in A$ be arbitrary elements such that $x \cdot z \in F$. Suppose $\neg(x \cdot \varphi(z) \in F)$. On the other hand, from (BZ-a) with $y = \varphi 9z)$ we have $(x \cdot z) \cdot (\varphi(z) \cdot z) \leqslant x \cdot \varphi(z)$. Since $\neg(x \cdot \varphi(z) \in F)$, then $\neg((x \cdot z) \cdot (\varphi(z) \cdot z) \in F$ by Corollary 4.1. Now, from $\neg((x \cdot z) \cdot (\varphi(z) \cdot z) \in F$ and $x \cdot z \in F$ follows $\varphi(z) \cdot z \in F$ by (8). We got a contradiction. Therefore, it have to be $x \cdot \varphi(z) \in F$. So, we are proven (Ta). $\square$

6. Final Observation

In this paper, we try to develop the theory of filters in the BZ algebras. First, we introduce the concept of the BZ filters in the BZ algebra (Definition 4.1). In addition, we analyze some of the additional conditions that we assume that BZ-filters can satisfy them. In addition, we analyze some interrelations between these additional conditions (Theorems 5.1 - 5.7). The author is convinced that this analysis enriches our knowledge of BZ-algebras.

By the fluctuation of logic atoms in the formulas that determine the particular types of ideals in these algebra it can be obtained some other types of filters in the weak BCC-algebras. Some of these additional conditions that can be imposed on BZ-filters in BZ-algebra are shown below:

(O) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge y \in F) \implies \neg(x \in F))$,

(I) $(\forall x, y, z \in A)(\neg((x \cdot y) \cdot z \in F) \wedge x \cdot z \in F) \implies y \cdot z \in F)$,

(As) $(\forall x, y, z \in A)((\neg((x \cdot y) \cdot z \in F) \wedge x \in F) \implies y \cdot z \in F)$. It is immediately apparent that if we put $z = 0$ in (I) and (As) we get (8) in both cases.

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