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Equitable List Vertex Colourability and Arboricity of Grids

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Abstract. A graph G is equitably k-list arborable if for any k-uniform list assignment L, there is an equitable L-colouring of G whose each colour class induces an acyclic graph. The smallest number k admitting such a coloring is named equitable list vertex arboricity and is denoted by $\rho_l^=(G)$. Zhang in 2016 posed the conjecture that if $k \geq \lceil(\Delta(G)+1)/2\rceil$ then G is equitably k-list arborable. We give some new tools that are helpful in determining values of k for which a general graph is equitably k-list arborable. We use them to prove the Zhang's conjecture for d-dimensional grids where $d \in \{2,3,4\}$ and give new bounds on $\rho_l^=(G)$ for general graphs and for d-dimensional grids with $d \geq 5$.

1. Introduction

All graphs considered in this paper are simple and undirected. For a graph G, we use $V(G)$, $E(G)$, and $\Delta(G)$ to denote vertex set, edge set, and the maximum degree of G, respectively. By $G[V']$ we mean the subgraph of G induced by a vertex subset V'. To simplify the notation we write $G-V'$ instead of $G[V(G)\setminus V']$. Analogously, we write $G-E'$ to denote the graph obtained from G by the deletion of an edge subset E'. By $G_1 \cup G_2$ we mean the union of disjoint graphs G_1, G_2, i.e. the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$.

The symbol $\mathbb{N}$ stands for the set of positive integers, and moreover $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $a, b \in \mathbb{N}_0$. If $a < b$ then $[a, b]$ denotes the set $\{a, a+1, \dots, b-1, b\}$, if $a = b$ then $[a, b] = \{a\}$, and if $a > b$ then $[a, b] = \emptyset$. We adopt the convention $[1, b] = [b]$, moreover $[b]_{ODD}$ and $[b]_{EVEN}$ denote the sets of odd integers and even integers in $[b]$, respectively.

A *colouring* of a graph G is a mapping $c : V(G) \to \mathbb{N}$. A *coloured graph* is then a pair (G, c), where G is a graph and c is its colouring. A colouring of a graph G is *proper* if each colour class induces an edgeless graph. A *k-colouring* of a graph G is a mapping $c : V(G) \to [k]$. A graph G is *properly k-colourable* if there is a proper k-colouring of G. A graph G is *k-arborable* if there is a k-colouring of G such that each colour class induces an acyclic graph.

Let L be a *list assignment* (*for a graph G*), i.e. a mapping that assigns to each vertex $v \in V(G)$ a set $L(v)$ of allowable colours. An *L-colouring* of G is a colouring of G such that for every $v \in V(G)$ the colour on v belongs to $L(v)$. A list assignment L is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$. A graph G is *k-choosable* if for each k-uniform list assignment L, we can find a proper L-colouring of G. A graph G is *k-list arborable* if,

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given a k-uniform list assignment L, we can find an L-colouring of G so that each colour class induces an acyclic subgraph of G. By $\chi(G)$, $\rho(G)$, $ch(G)$, $\rho_l(G)$ we denote the minimum $k \in \mathbb{N}$ such that G is: properly k-colourable, k-arborable, k-choosable, k-list arborable, respectively. We call these numbers the *chromatic number* of G, the *vertex arboricity* of G, the *choice number* of G, the *list vertex arboricity* of G, respectively. The invariant $\rho(G)$ was first introduced by Beineke in 1964 [1] and then it was investigated by many researchers. For example, Chartrand, Kronk, and Wall in 1968 [3] proved that $\rho(G) \leq \lceil(\Delta(G) + 1)/2\rceil$ for every graph G. Next, in 1995, Borowiecki, Drgas-Burchardt, and Mihók [2] introduced the list version of these problem. They showed that $\rho_l(G) \leq \lceil(\Delta(G))/2\rceil$ for every connected graph G excluding cycles and complete graphs of odd order.

In this paper we are mostly interested in a non-classical model of graph colouring, known as equitable. A k-colouring of a graph G is *equitable* when each of its colour classes is of the cardinality either $\lceil|V(G)|/k\rceil$ or $\lfloor|V(G)|/k\rfloor$. A graph G is *equitably properly k-colourable* if there exists an equitable proper k-colouring of G. The definition was firstly introduced by Meyer [9] in 1973. Recently, Wu, Zhang and Li [12] introduced the equitable version of vertex arborocity. A graph G is *equitably k-arborable* if there exists an equitable k-colouring of G whose each colour class induces an acyclic graph. In the list version, given a k-uniform list assignment L for G, we call an L-colouring of G *equitable* when each colour class has the cardinality at most $\lceil|V(G)|/k\rceil$ (see [7]). A graph G is *equitably k-choosable* when for any k-uniform list assignment L, there is an equitable proper L-colouring of G. A graph G is *equitably k-list arborable* when for any k-uniform list assignment L, there is an equitable L-colouring of G whose each colour class induces an acyclic graph. The last definition was given by Zhang [13] in 2016. By $\chi^=(G)$, $\rho^=(G)$, $ch^=(G)$, $\rho_l^=(G)$ we denote the minimum $k \in \mathbb{N}$ such that G is: equitably properly k-colourable, equitably k-arborable, equitably k-choosable, equitably k-list arborable, respectively. The numbers $\chi^=(G)$, $\rho^=(G)$, $ch^=(G)$, $\rho_l^=(G)$ are called the *equitable chromatic number* of G, the *equitable vertex arboricity* of G, the *equitable choice number* of G, the *equitable list vertex arboricity* of G, respectively.

Hajnál and Szemerédi ([5]) proved that a graph G is equitably properly k-colourable whenever $k \geq \Delta(G) + 1$. It caused a question posed by P. Erdös. Kostochka, Pelsmajer, and West [7] conjectured the list version of this theorem.

Conjecture 1.1 ([7]). *If $k \in \mathbb{N}$ and $k \geq \Delta(G) + 1$ then every graph G is equitably k-choosable.*

It has to be mentioned herein that equitable k-colouring is not monotone with respect to k. It means that there are graphs that are equitably k-colourable and not equitably t-colourable for some $t < k$. To the best of our knowledge there are no results of this type on equitable k-choosability nor equitable k-list arborability.

On the other hand, Zhang [13] formulated in 2016 the following conjectures.

Conjecture 1.2 ([13]). *For every graph G it holds $\rho_l^=(G) \leq \lceil(\Delta(G) + 1)/2\rceil$.*

Conjecture 1.3 ([13]). *If $k \in \mathbb{N}$ and $k \geq \lceil(\Delta(G) + 1)/2\rceil$ then every graph G is equitably k-list arborable.*

Zhang [13] confirmed above two conjectures for complete graphs, 2-degenerate graphs, 3-degenerate claw-free graphs with maximum degree at least 4, and planar graphs with maximum degree at least 8. Our results confirm above conjectures for some Cartesian products of paths, i.e. for some grids.

Given two graphs G_1 and G_2, the *Cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$, is defined to be a graph whose vertex set is $V(G_1) \times V(G_2)$ and edge set consists of all the edges joining vertices (x_1, y_1) and (x_2, y_2) when either $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1x_2 \in E(G_1)$. Note that the Cartesian product is commutative and associative. Hence the graph $G_1 \square \cdots \square G_d$ is unambiguously defined for any $d \in \mathbb{N}$. Let P_n denote a path on n vertices. Notice that when $G = G_1 \square \cdots \square G_d$ and each of the factors G_i of G is P_2 then G is a *d-dimensional hypercube*. Similarly, when each of the factors G_i is a path on at least two vertices then G is a *d-dimensional grid* (cf. Fig. 1). By *grids* we mean the class of all d-dimensional grids taken over all $d \in \mathbb{N}$.

Nakprasit and Nakprasit [10] proved that the problem of equitable vertex arboricity is NP-hard. Thus the problem of equitable list vertex arboricity cannot be easier. We are interested in determining polynomially solvable cases. We will use the following known lemmas. By $N_G(x)$ we denote *neighborhood* of a vertex x in G, i.e. the set of adjacent vertices to x.

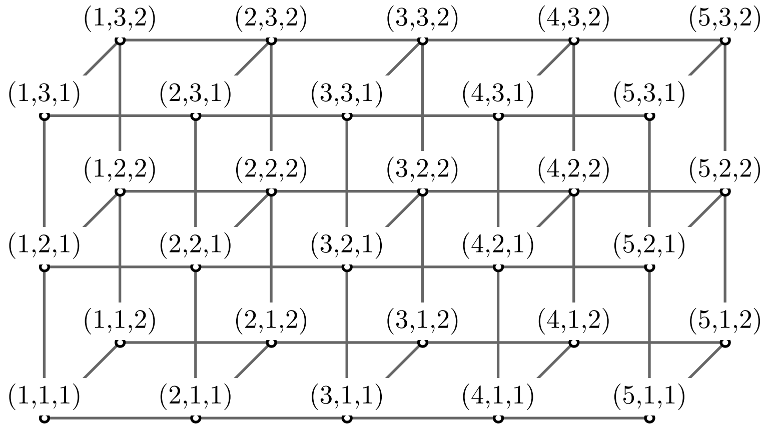

Figure 1: 3-dimensional grid $P_5 \square P_3 \square P_2$.

Lemma 1.4 ([7, 11]). *Let $k \in \mathbb{N}$ and $S = \{x_1, \ldots, x_k\}$, where $x_1, \ldots, x_k$ are distinct vertices of G. If $G - S$ is equitably k-choosable and*

$$|N_G(x_i) \setminus S| \leq i - 1 \tag{1}$$

holds for every $i \in [k]$ then G is equitably k-choosable.

Lemma 1.5 ([13]). *Let $k \in \mathbb{N}$ and $S = \{x_1, \ldots, x_k\}$, where $x_1, \ldots, x_k$ are distinct vertices of G. If $G - S$ is equitably k-list arborable and*

$$|N_G(x_i) \setminus S| \leq 2i - 1 \tag{2}$$

holds for every $i \in [k]$ then G is equitably k-list arborable.

In this paper we investigate the problem of equitable list vertex arboricity of graphs. The remainder of the paper is organized as follows. In Section 2 we generalize Lemmas 1.4 and 1.5 in such a way that their new versions guarantee the continuity of the equitable choosability and equitable list vertex arboricity of graphs. We give also a new tool using the equitable choosability of a subgraph H covering graph G (Lemma 2.7). These tools (Lemmas 2.5, 2.6, and 2.7) lead to new bounds on $\rho_l^=(G)$, for any graph G. Since the new tool uses the notation of equitable choosability we dedicate Section 3 to this notation for some graphs related to grids. Finally, we apply all the lemmas to confirm the correctness of Zhang's conjectures for d-dimensional grids, $d \in \{2, 3, 4\}$, and to give new bounds on $\rho_l^=(G)$ for d-dimensional grids with $d \geq 5$ (Section 4). We conclude the paper with posing some new conjectures concerning equitable list vertex arboricity of graphs.

2. Some auxiliary tools and general bounds on $\rho_l^=(G)$

In the literature a lot of iproofs of results on equitable choosability are done by induction on the number of vertices of a graph and by usage of Lemma 1.4. It means, to show that G is equitably k-choosable, the set $S \subseteq V(G)$ that fulfills the inequality (1) is determined and next the induction hypothesis is applied to the graph $G - S$. Repeated application of this approach defines a partition $S_1 \cup \cdots \cup S_{\eta+1}$ of $V(G)$ such that the following both conditions hold:

- $|S_1| \leq k$ and $|S_j| = k$ for $j \in [2, \eta + 1]$;
- for each $j \in [2, \eta + 1]$ there is an ordering of vertices of S_j, say $x_1^j, \ldots, x_k^j$, that fulfills the inequality $|N_G(x_i^j) \cap (S_1 \cup \cdots \cup S_{j-1})| \leq i - 1$ for every $i \in [k]$.

In this section we prove that if G has such a partition then G is not only equitably k-choosable but also is equitably t-choosable for every $t \in \mathbb{N}$ satisfying $t \geq k$. Next, we observe that the similar result for a graph to be equitably k-list arborable can be formulated.

Let $k \in \mathbb{N}$. A *k-partition* of a graph G is a partition of the vertex set of G into $\lceil |V(G)|/k \rceil$ sets. The k-partition is *special* if all sets of the k-partition, except at most one, have k elements. Let G be a graph and c be its vertex colouring (not necessarily proper). A set $S \subseteq V(G)$ is *rainbow* in the coloured graph (G, c) if all vertices in S are coloured differently. A k-partition of the coloured graph (G, c) is *rainbow* if every set of the k-partition is rainbow. It is easy to see the following fact.

Observation 2.1. *Let $k \in \mathbb{N}$ and (G, c) be a coloured graph. If there is a rainbow k-partition of (G, c) then each colour appears on at most $\lceil |V(G)|/k \rceil$ vertices of G.*

Lemma 2.2. *Let $k \in \mathbb{N}$. A graph G is equitably k-choosable if and only if for every k-uniform list assignment L there is a proper L-colouring c of G such that (G, c) has a rainbow k-partition.*

Proof. Obviously, if for every k-uniform list assignment L there is a proper L-colouring c of G such that (G, c) has a rainbow k-partition then each colour class has the cardinality at most $\lceil |V(G)|/k \rceil$, by Observation 2.1. It means that this L-colouring c is equitable, and hence G is equitably k-choosable.

To prove the opposite implication, suppose that G is equitably k-choosable and L is a k-uniform list assignment for G. It follows that there is a proper L-colouring c of G such that each colour class has at most $\lceil |V(G)|/k \rceil$ elements. Let $|V(G)| = \eta k + r$, where $\eta \in \mathbb{N}_0$, $r \in [k]$. Thus $\eta + 1 = \lceil |V(G)|/k \rceil$, and so each colour class contains at most $\eta + 1$ vertices. Assume, on the contrary, that there is no rainbow k-partition of (G, c). Among all partitions of (G, c) into rainbow sets, let $V_1 \cup \cdots \cup V_t$ be one with the smallest t. Since there is no rainbow k-partition, we have $t > \eta + 1$. Without loss of generality, we may assume that $V_1 \cup \cdots \cup V_t$ is the rainbow partition with $|V_1| \leq \cdots \leq |V_t|$ and with the minimum cardinality of V_1. Let $|V_1| = s$ and $x \in V_1$. Since we have at most $\eta + 1$ vertices coloured with $c(x)$ and $t > \eta + 1$, there is a set V_i such that $V_i \cup \{x\}$ is rainbow. If $s = 1$ then $V_2 \cup V_3 \cup \cdots \cup V_{i-1} \cup (V_i \cup \{x\}) \cup V_{i+1} \cup \cdots \cup V_t$ is the partition with less number of rainbow sets, a contradiction. If $s > 1$ then we get the rainbow partition $V_1 \setminus \{x\} \cup V_2 \cup \cdots \cup V_{i-1} \cup (V_i \cup \{x\}) \cup V_{i+1} \cup \cdots \cup V_t$ that contradicts with the minimum cardinality of V_1. □

Lemma 2.3. *Let $k \in \mathbb{N}$. A graph G is equitably k-list arborable if and only if for every k-uniform list assignment L there is an L-colouring c in which every colour class induces an acyclic graph and such that (G, c) has a rainbow k-partition.*

Proof. We repeat all the steps of the proof of Lemma 2.2, but in each case when we refer to the colouring c of a graph G we assume or state that each colour class in c is acyclic instead of the assumption that c is proper. Additionally, we substitute the notion of equitable k-choosability by the notion of equitable k-list arborability. □

Lemma 2.4. *Let $k \in \mathbb{N}$ and (G, c) be a coloured graph. If there is a rainbow special k-partition of (G, c) then there is also a rainbow special x-partition of (G, c) for every integer x such that $x \leq k$.*

Proof. Let $|V(G)| = \eta k + r_1$, where $\eta \in \mathbb{N}_0$, $r_1 \in [k]$. Let $S_1 \cup S_2 \cup \cdots \cup S_{\eta+1}$ be a rainbow special k-partition of (G, c) such that $|S_1| = r_1$ and $|S_i| = k$ for $i \in [2, \eta + 1]$. We show that there is a rainbow special x-partition, for every $x \leq k$.

Arrange vertices of G in the list in such a way that:

- vertices from S_i are placed before vertices from S_j for $i < j$;
- vertices from S_1 are placed in any order at the top of the list;
- each vertex from S_i, for $i > 1$, is placed in the list in such a way that its colour is different from the colours of $k - 1$ previous vertices in the list or its colour is different from the colours of all previous vertices in the list, if the number of previous vertices is smaller than $k - 1$.

Since sets S_i are rainbow, for every i, then the above described arrangement of vertices is possible. Assume that $(v_1, v_2, \ldots, v_{|V(G)|})$ is the list of vertices created in such a way. Let $|V(G)| = \beta x + r_2$, where $\beta \in \mathbb{N}_0$, $r_2 \in [x]$.

Sets $R_i = \{v_{(i-1)\cdot x+1}, \ldots, v_{i\cdot x}\}$, for $1 \le i \le \beta$ and $R_{\beta+1} = \{v_{\beta\cdot x+1}, \ldots, v_{|V(G)|}\}$ form an x-partition. It is easy to see that this partition is rainbow and special. □

Lemma 2.5. *Let $k \in \mathbb{N}$. If a graph G has a special k-partition $S_1 \cup \cdots \cup S_{\eta+1}$ such that $|S_1| \le k$ and $|S_j| = k$ for $j \in [2, \eta+1]$, moreover, if for every $j \in [2, \eta+1]$ there is an ordering $x^j_1, \ldots, x^j_k$ of vertices of the set S_j that for every $i \in [k]$ the inequality*

$$|N_G(x^j_i) \cap (S_1 \cup \cdots \cup S_{j-1})| \le i - 1 \tag{3}$$

is fullfilled then G is equitably t-choosable for every integer t satisfying $t \ge k$.

Proof. Let k, t be fixed and L be a t-uniform list assignment for G. We show that there is a proper L-colouring c of G such that the coloured graph (G, c) has a rainbow special t-partition. Since L is chosen freely, it will follow that G is equitably t-choosable, by Lemma 2.2. Let

- $|V(G)| = \eta k + r_1$, where η, r_1 are non-negative integers, $r_1 \in [k]$, and
- $|V(G)| = \beta t + r_2$, where β, r_2 are non-negative integers, $r_2 \in [t]$, and
- $t = \gamma k + r$ where γ, r are non-negative integers, $r \in [k]$.

Thus $|V(G)| = \beta(\gamma k+r)+r_2 = \beta\gamma k+\beta r+r_2$. We split $V(G)$ into two subsets V_1 and V_2, where $V_1 = S_1 \cup \cdots \cup S_{\eta+1-\beta\gamma}$ and $V_2 = S_{\eta+1-(\beta\gamma-1)} \cup \cdots \cup S_{\eta+1}$. Observe that $|V_1| = \beta r + r_2$ and $|V_2| = \beta\gamma k$. First, we properly colour the vertices in V_1, next we spread the colouring on V_2. We colour vertices in each set S_i of V_1 in such a way that we obtain a rainbow set. It is easy to see that we can colour vertices from S_1 such that we obtain a rainbow set, since each vertex has assigned a list of length t and $|S_1| = r_1 \le t$. Next, we colour vertices $x^2_1, \ldots, x^2_k$ in S_2. We assign to x^2_k a colour from its list that is not used in S_1. Since $|N_G(x^2_k) \cap S_1| \le k - 1$ and $|L(x_k)| = t \ge k$, this may be done. Next we assign to $x^2_{k-1}, \ldots, x^2_1$ (in the sequence) a colour from its list that is different from the ones assigned to the vertices with higher subscript and not used in S_1. All these steps may be completed since $|N(x^2_i) \cap S_1| \le i - 1$ and $|L(x_i)| = t \ge k$. Similarly, we colour the vertices of each set S_j ($j \in [3, \eta+1-\beta\gamma]$). Consider the coloured subgraph (G_1, c), where $G_1 = G[V_1]$. Since each set S_j ($j \in [\eta+1-\beta\gamma]$) is rainbow, we obtain a rainbow k-partition of (G_1, c). If $r_2 \le r_1$, we take r_2 vertices of S_1 and denote this set by R. Otherwise, we additionally choose $r_2 - r_1$ vertices from S_2 that have colours different than colours of vertices in S_1 and then these vertices together with S_1 form R. Observe that also $(G_1 - R, c)$ has a rainbow k-partition. Furthermore, $|(S_1 \cup \cdots \cup S_{\eta+1-\beta\gamma}) \setminus R| = |V(G_1 - R)| = \beta r$. By Lemma 2.4, $G_1 - R$ has a rainbow r-partition. Let $T_1, \ldots, T_\beta$ be a rainbow r-partition of $(G_1 - R, c)$.

Now we colour the vertices in V_2. Recall that $|V_2| = \beta\gamma k$. Let us divide V_2 into β subsets, each containing γ k-sets S_i, in the following way:

$$H_1 = S_{\eta+1-(\beta\gamma-1)} \cup S_{\eta+1-(\beta\gamma-2)} \cup \cdots \cup S_{\eta+1-(\beta-1)\gamma}$$
$$H_2 = S_{\eta+1-((\beta-1)\gamma-1)} \cup S_{\eta+1-((\beta-1)\gamma-2)} \cup \cdots \cup S_{\eta+1-(\beta-2)\gamma}$$
$$\vdots$$
$$H_i = S_{\eta+1-((\beta-i+1)\gamma-1)} \cup S_{\eta+1-((\beta-i+1)\gamma-2)} \cup \cdots \cup S_{\eta+1-(\beta-i)\gamma}$$
$$\vdots$$
$$H_\beta = S_{\eta+1-(\gamma-1)} \cup S_{\eta+1-(\gamma-2)} \cup \cdots \cup S_{\eta+1}.$$

We will properly colour vertices in $H_1, \ldots, H_\beta$ from their lists, step by step, in such a way that each set $T_i \cup H_i$ for $i \in [\beta]$ is rainbow.

First, consider a colouring of vertices of H_i. To simplify the notation let $A = \alpha + 1 - ((\beta - i + 1)\gamma - 1)$. Thus $H_i = S_A \cup S_{A+1} \cup \cdots \cup S_{A+\gamma-1}$. Recall that vertices $x^A_1, \ldots, x^A_k$ in S_A fulfill the inequality (3). We delete colours that are used on vertices in T_i from lists of vertices in S_A. Now the lists of vertices in S_A are shorter than t, however each vertex still has at least γk colours on the list. Assign to x^A_k a colour from its list that is not used on vertices from $S_1 \cup \cdots \cup S_{A-1}$. Since $|N_G(x_k^A) \cap (S_1 \cup \cdots \cup S_{A-1})| \le k-1$ and $|L(x_k^A)| = \gamma k \ge k$, this may be done. Then assign to $x_{k-1}^A, \ldots, x_1^A$ (in a sequence) a colour from its list that is different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_{A-1}$. All these steps may be done since $|N_G(x_i^A) \cap (S_1 \cup \cdots \cup S_{A-1})| \le i-1$ and $|L(x_i^A)| = \gamma k \ge k$. Now, we colour vertices in S_{A+1}, where $S_{A+1} = \{x_1^{A+1}, \ldots, x_k^{A+1}\}$. We delete colours that are used on vertices in T_i and S_A from lists of vertices in S_{A+1}. Observe that after deleting colours from lists, each vertex in S_{A+1} has at least $(\gamma-1)k$ colours on the list. Similarly as above, first we colour the vertex x_k^{A+1} with a colour from its list that is not used in $S_1 \cup \cdots \cup S_A$ and then we colour, one by one, vertices $x_{k-1}^{A+1}, \ldots, x_1^{A+1}$ with colours from their lists that are different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_A$. We can do this since $|N_G(x_i^{A+1}) \cap (S_1 \cup \cdots \cup S_A)| \le i-1$ and $|L(x_i^{A+1})| = (\gamma-1)k \ge k$. Observe that in the same way we can colour vertices from sets $S_{A+2}, \ldots, S_{A+\gamma-1}$. Indeed, let $S_{A+j} = \{x_1^{A+j}, \ldots, x_k^{A+j}\}$. We delete from lists of vertices in S_{A+j} colours that are used on vertices in $T_i \cup S_A \cup \cdots \cup S_{A+j-1}$ and then we assign the colour different from the ones assigned to the vertices with higher subscript and not used in $S_1 \cup \cdots \cup S_{A+j-1}$.

Thus finally, we have obtained a proper colouring c that admits a rainbow t-partition of (G, c) which completes the proof. □

To stress the difference between Lemma 1.4 and Lemma 2.5 let us consider the equitable choice number of a special family of graphs, which we denote by $\mathcal{K}$. Assume $K_3 \in \mathcal{K}$ and if $G \in \mathcal{K}$, then the graph obtained from G by adding three new vertices, say x, y, z, and six edges such that x, y, z are pairwise adjacent and additionally the vertex x is adjacent to any two vertices of G and the vertex y is adjacent to any one vertex of G is also in $\mathcal{K}$. The definition of $\mathcal{K}$ very naturally indicates the special 3-partition satisfying the assumptions of Lemma 2.5 of every graph $G \in \mathcal{K}$. Thus Lemma 2.5 implies that if $G \in \mathcal{K}$, then G is equitably t-choosable for every integer t satisfying $t \ge 3$. However, such a result cannot be easily proved by Lemma 1.1. Obviously, Lemma 1.1 implies that G is equitably 3-choosable, but to prove that G is equitably t-choosable for every integer $t \ge 4$ we have to involve more arguments.

The next result generalizes Lemma 1.5. We give only a sketch of its proof because it imitates the proof of Lemma 2.5.

Lemma 2.6. *Let $k \in \mathbb{N}$. If a graph G has a special k-partition $S_1 \cup \cdots \cup S_{\eta+1}$ such that $|S_1| \le k$ and $|S_j| = k$ for $j \in [2, \eta+1]$, moreover, if for every $j \in [2, \eta+1]$ there is an ordering $x_1^j, \ldots, x_k^j$ of vertices of the set S_j that for every $i \in [k]$ the inequality*

$$|N_G(x_i^j) \cap (S_1 \cup \cdots \cup S_{j-1})| \le 2i - 1, \tag{4}$$

is fulfilled then G is equitably t-list arborable for any integer t satisfying $t \ge k$.

Proof. For fixed k, t and a t-uniform list assignment L for G, we construct an L-colouring c of G such that the coloured graph (G, c) has a rainbow special t-partition and each colour class in c induces an acyclic graph. We do it in the same manner as in the proof of Lemma 2.5, but if we put a colour on the vertex x_i^j, $i \in [k]$, $j \in [2, \eta+1]$ then we use Lemma 1.5 (instead of Lemma 1.4) to guarantee that each colour class in c induces an acyclic graph (instead of to guarantee that the constructed colouring is proper). □

Next, we give new tool that help us in proving further results concerning exact values as well as bounds on equitable list vertex arboricity of graphs.

A *spanning* graph H of a graph G is any subgraph of G such that $V(H) = V(G)$. We say that a graph H *covers all cycles* of G if it is spanning and for any cycle C contained in G there are $x, y \in V(C)$ such that $xy \in E(H)$.

Lemma 2.7. *Let $k \in \mathbb{N}$. If H is a graph that covers all cycles of G and H is equitably k-choosable then G is equitably k-list arborable.*

Proof. Let L be any k-list assignment for G. Let c be an equitable proper L-colouring of H. We show that each colour class induces an acyclic subgraph of G. Let C be a cycle of G. By our assumption on H there are $x, y \in V(C)$ such that $xy \in E(H)$. Thus C contains two vertices which have different colours in c. Since G has no monochromatic cycle in c, each colour class induces an acyclic graph. □

Lemma 2.7 states that we can use known results related to equitable choosability for determining results on equitable list vertex arboricity. Let us recall results proven in [6].

Theorem 2.8 ([6]). *Let $r \in \mathbb{N}$ and G be a graph such that $\Delta(G) \leq r$.*

(i) *If $r \leq 7$ and $k \geq r+1$ then G is equitably k-choosable.*

(ii) *If $k \geq r + \begin{cases} 1 + \frac{r-1}{7} & if \quad r \leq 30 \\ \frac{r}{6} & if \quad r \geq 31 \end{cases}$ then G is equitably k-choosable.*

(iii) *If $|V(G)| \geq r^3$ and $k \geq r+2$ then G is equitably k-choosable.*

(iv) *If $\omega(G) \leq r$ and $|V(G)| \geq 3(r+1)r^8$ then G is equitably $(r+1)$-choosable ($\omega(G)$ is the clique number of G).*

Theorem 2.8 and Lemma 2.7 imply the general upper bound on equitable list vertex arboricity.

Theorem 2.9. *Let $r \in \mathbb{N}$ and G be a graph with at least one edge and $\Delta(G) - 1 \leq r$.*

(i) *If $r \leq 7$ and $k \geq r+1$ then G is equitably k-list arborable.*

(ii) *If $k \geq r + \begin{cases} 1 + \frac{r-1}{7} & if \quad r \leq 30 \\ \frac{r}{6} & if \quad r \geq 31 \end{cases}$ then G is equitably k-list arborable.*

(iii) *If $|V(G)| \geq r^3$ and $k \geq r+2$ then G is equitably k-list arborable.*

(iv) *If $\omega(G) \leq r$ and $|V(G)| \geq 3(r+1)r^8$ then G is equitably $(r+1)$-list arborable.*

Proof. Let F be a spanning forest of G such that the numbers of connected components of F and G are the same. Thus $G - F$ covers all cycles of G. By Lemma 2.7, if $G - F$ is equitably k-choosable then G is equitably k-list arborable. Since $\Delta(G - F) \leq \Delta(G) - 1$, the theorem follows directly from Theorem 2.8. □

If we restrict our consideration to particular graph classes or to graphs with particular properties, we get even better bounds on equitable list arboricity that, in addition, confirm Zhang's conjecture.

Theorem 2.10 ([7]). *Let $k \in \mathbb{N}$ and let F be a forest. If $k \geq \Delta(F)/2 + 1$ then F is equitably k-choosable.*

We can apply Theorem 2.10 to show an upper bound on equitable list vertex arboricity of graphs with (edge) arboricity equal to 2. The *(edge) arboricity* of a graph G is the minimum number of forests into which its edges can be partitioned.

Theorem 2.11. *Let $k \in \mathbb{N}$ and let G be a graph with arboricity 2. If $k \geq \lceil(\Delta(G)+1)/2\rceil$ then G is equitably k-list arborable.*

Proof. Let $F_1 = (V(G), E_1)$ and $F_2 = (V(G), E_2)$ be two forests into which $E(G)$ was partitioned. Of course, $E(G) = E_1 \cup E_2$. It is clear that F_1 covers all cycles of G. If $\Delta(F_1) < \Delta(G)$ then by Theorem 2.10 and Lemma 2.7 G is equitably k-list arborable for $k \geq \Delta(F_1)/2 + 1$. It means that G is equitably k-list arborable for $k \geq \lceil(\Delta(G)+1)/2\rceil$. Suppose that $\Delta(F_1) = \Delta(G)$. Let D be the set of vertices of maximum degree in F_1. Observe that every vertex in D is adjacent only with edges from E_1. Let $E_1' \subseteq E_1$ be the minimal set of edges such that $D \subseteq \bigcup_{e \in E_1'} e$. Since E_1' is minimal, the subgraph induced by E_1' is a star-forest. Furthermore, in the subgraph induced by $E_2 \cup E_1'$ every edge in E_1' is a pendant edge. Thus the subgraph induced by $E_2 \cup E_1'$ is acyclic and so $F_1 - E_1'$ covers all cycles of G. Since $\Delta(F_1 - E_1') < \Delta(G)$, by Theorem 2.10 and Lemma 2.7, G is equitably k-list arborable for $k \geq \Delta(F_1 - E_1')/2 + 1$. It means that G is equitably k-list arborable for $k \geq \lceil(\Delta(G)+1)/2\rceil$. □

A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. Since every 2-degenerate graph has arboricity 2, Theorem 2.11 confirms the result for 2-degenerate graphs obtained by Zhang [13].

Corollary 2.12 ([13]). *Let $k \in \mathbb{N}$ and let G be a 2-degenerate graph. If $k \geq \lceil(\Delta(G)+1)/2\rceil$ then G is equitably k-list arborable.*

3. Equitable choosability of grids

Since our new tool (Lemma 2.7) uses the notion of equitable choosability we dedicate this section to this notion for some graphs related to grids. Nethertheless, before we consider it, we give some sufficient conditions for graphs to be equitably 2-choosable.

Lemma 3.1. *If G has a matching of size $\lfloor |V(G)|/2 \rfloor$ and G is 2-choosable then G is equitably 2-choosable.*

Proof. Observe that the assumption that G has a matching of size $\lfloor |V(G)|/2 \rfloor$ implies that $\alpha(G) \leq \lceil |V(G)|/2 \rceil$ ($\alpha(G)$ denotes the cardinality of the largest independent vertex set of G). Thus each colour class has at most $\lceil |V(G)|/2 \rceil$ vertices in any proper colouring of G. Let L be a 2-uniform list assignment for G. Since G is 2-choosable, there is a proper L-colouring c of G. Furthermore, every colour class in c has at most $\lceil |V(G)|/2 \rceil$ vertices, and so c is equitable proper L-colouring of G. □

The graphs that are 2-choosable were characterized by Erdös, Rubin and Taylor in [4]. The *core* of G is a graph obtained from G by recursive removing all vertices of degree one. Thus the core of G has no vertices of degree one. A graph is called a $\Theta_{2,2,p}$-*graph* if it consists of two vertices x and y and three internally disjoint paths of lengths 2, 2 and p, joining x and y.

Theorem 3.2 ([4]). *A connected graph G is 2-choosable if and only if the core of G is either K_1, or an even cycle, or a $\Theta_{2,2,2r}$-graph, where $r \in \mathbb{N}$.*

Lemma 3.3. *Let $k \in \mathbb{N}$ with $k \geq 2$. If G is a bipartite graph with $\Delta(G) \leq 2$ then G is equitably k-choosable.*

Proof. Observe first that each component of G is either an even cycle or a path. If G has more than one component that is a path, let G' be a graph obtained from G by adding edges so that G' has one component that is a path and all other components are even cycles. In the case when G has at most one component that is a path, we assume $G' = G$. We will show that G' is equitably k-choosable for any $k \geq 2$. By Theorem 3.2, being applied to each connected component of G', G' is 2-choosable (it is clear that if each component is 2-choosable then the whole graph is also 2-choosable). Since G' has a matching of size $\lfloor |V(G')|/2 \rfloor$ then G' is equitably 2-choosable by Lemma 3.1. Furthermore, Theorem 2.8(i) follows that G' is equitably k-choosable for every $k \geq 3$ (since $\Delta(G') \leq 2$). Hence the arguments that G' is equitably k-choosable for any $k \geq 2$ and that G is a spanning subgraph of G' imply that G is equitably k-choosable for any $k \geq 2$. □

Now, we define $\mathcal{G}_1$ to be a family of all grids $P_{n_1} \square P_2$ and all graphs resulting from grids $P_{n_1} \square P_2$ by removing one vertex of minimum degree, taken over all $n_1 \in \mathbb{N}$. The following results will be used in the next section to determine equitable list vertex arboricity of grids.

Lemma 3.4. *Let $k \in \mathbb{N}$ with $k \geq 3$. If every component of a graph G is in $\mathcal{G}_1$ then G is equitably k-choosable.*

Proof. We show that there is a special 3-partition of G that fulfills the assumptions of Lemma 2.5, i.e. there are disjoint sets $S_1, \ldots, S_{\eta+1}$ such that the following conditions hold:

- $V(G) = S_1 \cup \cdots \cup S_{\eta+1}$;
- $|S_1| \leq 3$ and $|S_j| = 3$ for $j \in [2, \eta + 1]$;
- there is an ordering of vertices of each set S_j, say x_1^j, x_2^j, x_3^j, fulfilling the inequality $|N_G(x_i^j) \cap (S_1 \cup \ldots \cup S_{j-1})| \leq i - 1$ for $i \in [3]$;

and hence, by Lemma 2.5, G is equitably k-choosable for any $k \geq 3$. We prove the existence of the partition by induction on the number of vertices of G. It is easy to see that it is true for a graph with at most 3 vertices. Thus suppose that if every component of a graph is in $\mathcal{G}_1$ and the graph has less than n vertices, $n \geq 4$, then it has a special 3-partition that fulfills the assumptions of Lemma 2.5. Let G be an n-vertex graph having

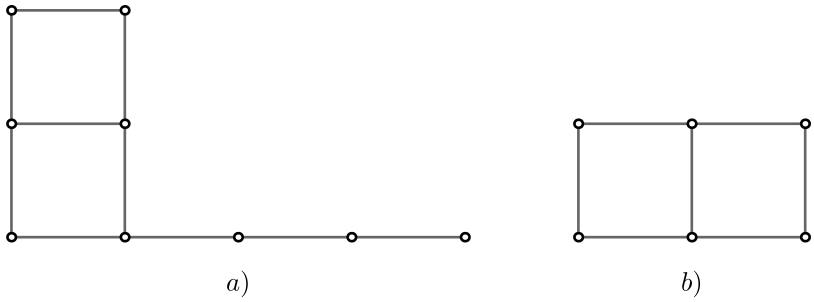

Figure 2: A graph a) $(P_5\square P_3, 3)$ and b) $(P_5\square P_2, 2)$ being isomorphic to $P_3\square P_2$.

every component in $\mathcal{G}_1$. We show that there is a set S in G, say $\{x_1, x_2, x_3\}$, such that $|N_G(x_i) \setminus S| \leq i-1$ for $i \in [3]$ and every component of $G-S$ is in $\mathcal{G}_1$. Thus, by induction, the lemma follows.

Let x_1 be a vertex of the minimum degree in G, thus $\deg_G(x_1) \leq 2$. Suppose first that $\deg_G(x_1) = 2$. In this case each component has at least four vertices. Let x_2, x_3 be the neighbors of x_1 such that $\deg_G(x_2) = 2$ and $\deg_G(x_3) \leq 3$. Let $S = \{x_1, x_2, x_3\}$, then $|N_G(x_1) \setminus S| = 0, |N_G(x_2) \setminus S| \leq 1$ and $|N_G(x_3) \setminus S| \leq 2$. Observe that every component of $G-S$ is in $\mathcal{G}_1$, so by our induction hypothesis $G-S$ has a special 3-partition that fulfills the assumptions of Lemma 2.5, and so we are done.

Suppose now that $\deg_G(x_1) = 1$. Let x_2 be the neighbor of x_1. If $\deg_G(x_2) = 3$ then let x_3 be the neighbor of x_2 of degree 2. Let $S = \{x_1, x_2, x_3\}$. Hence every component of $G-S$ is in $\mathcal{G}_1$ and we see that the vertices of S satisfy $|N_G(x_i) \setminus S| \leq i-1$ for $i \in [3]$. If $\deg_G(x_2) = 2$ then let x_3 be the neighbor of x_2, other than x_1. Observe that in this case the vertices x_1, x_2, x_3 form a component of G. Again $S = \{x_1, x_2, x_3\}$ satisfies $|N_G(x_i) \setminus S| = 0 \leq i-1$ for $i \in [3]$, and so, by induction hypothesis, G has a special 3-partition that fulfills the assumptions of Lemma 2.5. If $\deg_G(x_2) = 1$ then as x_3 in S we put a vertex of the minimum degree in $G - \{x_1, x_2\}$.

Finally suppose that $\deg_G(x_1) = 0$. In this case let x_2, x_3 be two adjacent vertices of degree at most two. If there are no such vertices then G is an edgeless graph and we can choose x_2, x_3 arbitrarily. Similarly as above we can see that every component of $G-S$ is in $\mathcal{G}_1$ and S satisfies $|N_G(x_i) \setminus S| \leq i-1$ for $i \in [3]$. It implies that G has a special 3-partition that fulfills the assumptions of Lemma 2.5, and so G is equitably k-choosable for $k \geq 3$. □

It should be mentioned here that for each component of graph G in $\mathcal{G}_1$, we have $\Delta(G) \leq 3$. Thus, by Theorem 2.8, such a graph is equitably k-choosable for $k \geq 4$. Hence Lemma 3.4 extends this result to $k \geq 3$.

Let $n_1, n_2 \in \mathbb{N}$, $n_2 \geq 2$, and $\ell \in [0, n_1 - 1]$. The symbol $(P_{n_1}\square P_{n_2}, \ell)$ denotes a graph obtained from $P_{n_1}\square P_{n_2}$ by the deletion of a set V' (cf. Fig. 2), where

$$V' = \{(n_1 - p, n_2) :\ p \in [0, \ell - 1]\} \cup \{(n_1 - p, n_2 - 1) :\ p \in [0, \ell - 1]\}.$$

Observe that $(P_{n_1}\square P_{n_2}, 0)$ is a grid $P_{n_1}\square P_{n_2}$.

Let $\mathcal{G}_2 = \{(P_{n_1}\square P_{n_2}, \ell) : n_1 \geq 1, n_2 \geq 2, \ell \in [0, n_1 - 1]\}$.

Lemma 3.5. *Let $k \in \mathbb{N}$ with $k \geq 4$. If each component of a graph G is in $\mathcal{G}_2$ then G is equitably k-choosable.*

Proof. We show that there is a special 4-partition of G that fulfills the assumptions of Lemma 2.5. We prove it by induction on the number of vertices. Observe that every graph in $\mathcal{G}_2$ has at least two vertices and it is easy to see that if G has at most 4 vertices then G has a special 4-partition that fulfills the assumptions of Lemma 2.5. Suppose that the assertion is true for graphs with less than n vertices, $n \geq 5$. Let G be a graph with n vertices that satisfies assumptions of the lemma. We show that there is a set S, say $\{x_1, x_2, x_3, x_4\}$, such that $|N_G(x_i) \setminus S| \leq i-1$ for $i \in [4]$ and each component of $G-S$ is in $\mathcal{G}_2$.

We choose the set S as follows. First suppose that there is a component $(P_{n_1}\square P_{n_2}, \ell)$ of G such that $n_1 - \ell \geq 2$ and $n_2 \geq 2$. Let us consider the set $S = \{x_1, x_2, x_3, x_4\}$ with $x_1 = (n_1 - \ell, n_2)$, $x_2 = (n_1 - \ell - 1, n_2)$, $x_3 = (n_1 - \ell, n_2 - 1)$, and $x_4 = (n_1 - \ell - 1, n_2 - 1)$.

Thus $|N_G((n_1-\ell, n_2))\setminus S| = 0$, $|N_G((n_1-\ell-1, n_2))\setminus S| \le 1$, $|N_G((n_1-\ell, n_2-1))\setminus S| \le 1$ and $|N_G((n_1-\ell-1, n_2-1))\setminus S| \le 2$. Furthermore, every component of $G-S$ is in $\mathcal{G}_2$ and hence, by the induction hypothesis, G has 4-partition of G that fulfills the assumptions of Lemma 2.5. If there is a component $(P_{n_1}\square P_{n_2}, \ell)$ of G such that $n_1 - \ell = 1$ and $n_2 \ge 4$ then we put $x_1 = (1, n_2)$, $x_2 = (1, n_2 - 1)$, $x_3 = (1, n_2 - 2)$, and $x_4 = (1, n_2 - 3)$. Every component of $G-S$ is in $\mathcal{G}_2$ and $|N_G((1, n_2)) \setminus S| = 0$, $|N_G((1, n_2-1)) \setminus S| = 0$, $|N_G((1, n_2-2)) \setminus S| \le 1$ and $|N_G((1, n_2-3)) \setminus S| \le 2$, so by the induction hypothesis, the assumptions of Lemma 2.5 are satisfied. Otherwise, every component of G is a path. If there is a component with at least four vertices then four consecutive vertices of the path form the set S that satisfies $|N_G(x_i) \setminus S| \le i - 1$ for $i \in [4]$. If each component of G has less than four vertices then, to obtain S, we take all vertices of one component and we next complete the set S by vertices of some other component or even components, if the number of vertices chosen to set S is still to small. It is easy to see that also in such a case the assumptions of Lemma 2.5 are fulfilled, which finishes the proof. □

Lemma 3.6. *Let $n_1, n_2, t \in \mathbb{N}$. If G is a graph with t components such that each one is isomorphic to $P_{n_1}\square P_{n_2}$ then G is equitably 3-choosable.*

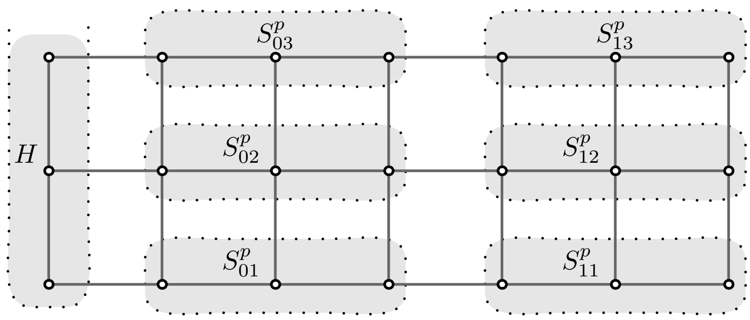

Figure 3: Illustration for the proof of Lemma 3.6; $G^p = P_7\square P_3$.

Proof. If $n_1 \le 2$ or $n_2 \le 2$ then the proof follows from Lemma 3.4. Thus we may assume that $n_1 \ge 3$ and $n_2 \ge 3$. Let $G^p = P^p_{n_1}\square P^p_{n_2}$ for $p \in [t]$ be components of G and $\{(i, j)^p : i \in [n_1], j \in [n_2]\}$ be the vertex set of the component G^p. Let $n_1 = 3q + r$ where $r \in [0, 2]$ and let L be a 3-uniform list assignment for the graph G. We show that there is a proper L-colouring c such that (G, c) has a rainbow 3-partition. Let H be a subgraph of G induced by the set $\{(1, j)^p : j \in [n_2], p \in [t]\}$ if $r = 1$, and induced by the set $\{(1, j)^p, (2, j)^p : j \in [n_2], p \in [t]\}$ if $r = 2$. Moreover, let $S^p_{ij} = \{(3i + 1 + r, j)^p, (3i + 2 + r, j)^p, (3i + 3 + r, j)^p\}$ where $i \in [0, q - 1]$, $j \in [n_2]$, $p \in [t]$ (cf. Fig. 3).

First, we colour the vertices of H. Let c' be an equitable proper L-colouring of H guaranteed by Lemma 3.4. Thus, by Lemma 2.2, there is a rainbow 3-partition of (H, c'). After this step all vertices of the first and the second column are coloured if $r = 2$, all vertices of the first column are coloured if $r = 1$, and graph is uncoloured if $r = 0$. Next, in each component, we colour uncoloured vertices of the first row, i.e., $(r + 1, 1)^p, (r + 2, 1)^p, \ldots, (n_1, 1)^p$ for $p \in [t]$. We properly colour these vertices in such a way that the sets S^p_{i1}, $i \in [0, q-1]$ are rainbow. Now we divide the uncoloured vertices of each component into 3-element subsets S^p_{ij} where $i \in [0, q - 1]$, $j \in [2, n_2]$, and $p \in [t]$. In each component we define linear ordering $\prec^p$ on these sets in the following way: $S^p_{ij} \prec S^p_{rs}$ if $(j < s)$ or $(j = s$ and $i < r)$. According to this ordering, we properly colour vertices of each set S^p_{ij} with the following rules:

- if it is only possible, we colour vertices in S^p_{ij} in such a way that vertices of this set obtain different colours;
- if we cannot colour vertices in S^p_{ij} in such a way that S^p_{ij} is rainbow then we color vertices in S^p_{ij} in such a way that two vertices have the same colour, let us say c_1, and there is no vertex coloured with c_1 in

S^p_{ij-1}; moreover, if also the set S^p_{ij-1} is not rainbow, i.e. two vertices in S^p_{ij-1} are coloured with the same colour, let us say c_2, then there is no vertex coloured with c_2 in S^p_{ij}.

We will show that such a colouring exists. Let c'' be a proper L-colouring of $G-H$ such that these rules are maintained. Suppose that we are at the step when we have just coloured vertices in S^p_{ij}, so vertices in every set that precedes S^p_{ij}, with respect to $\prec^p$, and the vertices in S^p_{ij} are coloured, the vertices in S^p_{i+1j} (or S^p_{0j+1} when $i = q-1$) are uncoloured. To simplify notation let $S^p_{ij} = \{(x, j), (x+1, j), (x+2, j)\}$ and $S^p_{ij-1} = \{(x, j-1), (x+1, j-1), (x+2, j-1)\}$. Let $c''((x, j)) = c''((x+2, j)) = c_1$ and $c''((x+1, j)) = b_1$. First we show that there is no vertex coloured with c_1 in S^p_{ij-1}. Since vertices (x, j) and $(x, j-1)$ are adjacent, it follows that $c_1 \neq c''((x, j-1))$. Similarly, $c_1 \neq c''((x+2, j-1))$. Now we need to show that $c_1 \neq c''((x+1, j-1))$. Since we use c_1 to colour $(x+2, j)$ then we necessarily have $L((x+2, j)) = \{c_1, b_1, c''((x+2, j-1))\}$. If $c_1 = c''((x+1, j-1))$ then we could colour $(x+1, j)$ with colour different from c_1 and b_1 and next colour $(x+2, j)$ with b_1 and so we would colour the vertices in S^p_{ij} with different colours, a contradiction. To finish the reasoning we show that if $c''((x, j-1)) = c''((x+2, j-1)) = c_2$ then there is no vertex coloured with c_2, in S^p_{ij}. It is easy to see that $c_2 \neq c''((x, j))$ and $c_2 \neq c''((x+2, j))$. As we observed above $L((x+2, j)) = \{c_1, b_1, c''((x+2, j-1))\}$. Since each vertex has the list consisting of three different colours, we have $b_1 \neq c''((x+2, j-1))$ and so $c''((x+1, j)) \neq c_2$.

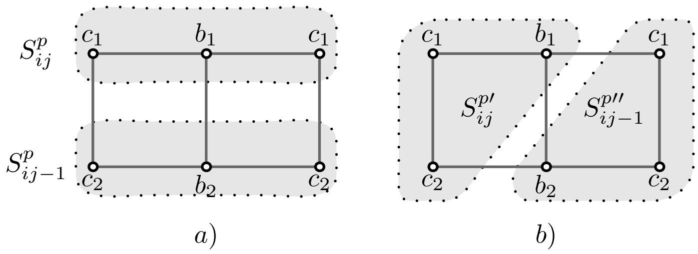

Figure 4: a) A part of G^p with depicted non-rainbow sets S^p_{ij} and S^p_{ij-1}, $b_2 \neq c_1$, $b_1 \neq c_2$. b) Repartition of $S^p_{ij} \cup S^p_{ij-1}$ into two rainbow sets $S^{p'}_{ij}$ and $S^{p''}_{ij-1}$.

Above described rules imply that either S^p_{ij} is rainbow or $S^p_{ij} \cup S^p_{ij-1}$ can be divided into two 3-element rainbow sets in $(G-H, c'')$: $S^{p'}_{ij} \cup S^{p''}_{ij-1}$ (cf. Fig. 4). We use this property to show that there is a rainbow special 3-partition of $(G-H, c'')$. We divide $V(G-H)$ in the following way:

- the set of vertices of each component is divided step by step;
- in each component G^p, we start with the last set, with respect to $\prec^p$, and go down due to this ordering;
- if S^p_{ij} is rainbow then it forms a set of the rainbow special 3-partition of $(G-H, c'')$; otherwise, we partite $S^p_{ij} \cup S^p_{ij-1}$ into two 3-element rainbow sets $S^{p'}_{ij} \cup S^{p''}_{ij-1}$ (cf. Fig. 4); we modify $\prec^p$ by removing sets that are already included in the rainbow 3-partition.

Recall that the sets S^p_{i1} for $i \in [0, q-1]$ (sets of the first row) are rainbow, so the above partition results in a rainbow special 3-partition of $(G-H, c'')$. Thus together with the rainbow 3-partition of (H, c') we obtain the rainbow 3-partition of $(G, c' \cup c'')$. Hence for every 3-uniform list assignment L there is a proper L-colouring c such that (G, c) has a rainbow 3-partition and next, by Lemma 2.2, G is equitably 3-choosable. □

Lemma 3.5 and Lemma 3.6 immediately imply the following result.

Lemma 3.7. *Let $n_1, n_2, k \in \mathbb{N}$ with $k \geq 3$. If each component of a graph G is isomorphic to $P_{n_1} \Box P_{n_2}$ then G is equitably k-choosable.*

If each component of graph G is in $P_{n_1}\square P_{n_2}$ then $\Delta(G) \le 4$. Thus, by Theorem 2.8, such a graph is equitably k-choosable for $k \ge 5$. Hence Lemma 3.7 extends this result to $k \ge 3$.

Remark 3.8. *Observe that* Lemma 3.6 *and* Lemma 3.7 *are still true if each component of G is an arbitrary* 2-*dimensional grid* (*components are not necessarily of the same sizes*). *Furthermore, the bound in* Lemma 3.7 *is tight, since $P_2\square P_3$ is not* 2-*choosable.*

Lemma 3.9. *Let $n_1, n_2 \in \mathbb{N}$ and $t, s \in \mathbb{N}_0$. If G is a graph with t components such that each one is isomorphic to $P_{n_1}\square P_{n_2}\square P_2$ and with s components being isomorphic to $P_{n_1}\square P_{n_2}$ then G is equitably* 4-*choosable.*

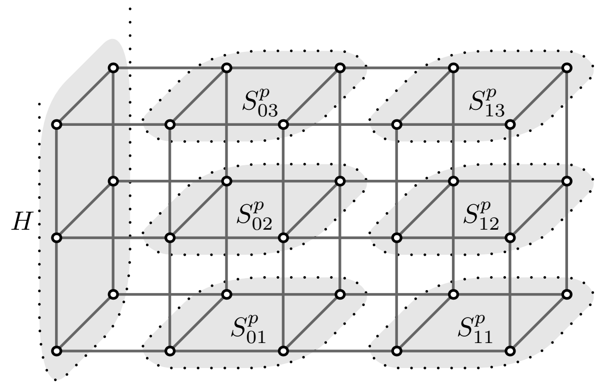

Figure 5: Illustration for the proof of Lemma 3.9; $G^p = P_5\square P_3\square P_2$.

Proof. If $n_1 = 1$ or $n_2 = 1$ then the proof follows from Lemma 3.7. Thus, without loss of generality, we may assume that $n_1, n_2 \ge 2$. Let $G^p = P_{n_1}\square P_{n_2}\square P_2$, $F^u = P_{n_1}\square P_{n_2}$ for $p \in [t], u \in [s]$ be components of G and $V(G^p) = \{(i, j, \ell)^p : i \in [n_1], j \in [n_2], \ell \in [2]\}$, $V(F^u) = \{(i, j)^u : i \in [n_1], j \in [n_2]\}$.
Let $n_1 = 2q + r$ where $r \in [0, 1]$. Let L be a 4-uniform list assignment for a graph G. We show that there is a proper L-colouring c such that (G, c) has a rainbow 4-partition. If $r = 1$ then let H be a subgraph induced in G by the set $\{(1, j, \ell)^p : j \in [n_2], p \in [t], \ell \in [2]\} \cup \{(i, j)^u : i \in [n_1], j \in [n_2], u \in [s]\}$. If $r = 0$ then let H be a subgraph induced in G by the set $\{(i, j)^u : i \in [n_1], j \in [n_2], u \in [s]\}$. By Lemma 3.7 there is an equitable proper L-colouring c' of H, and so by Lemma 2.2 there is a rainbow 4-partition of (H, c'). Now we start with colouring vertices of $G - H$ (vertices of G, if $r = 0$ and G has no component isomorphic to $P_{n_1}\square P_{n_2}$).

We divide the set of uncoloured vertices of each component into 4-element subsets.
$S^p_{ij} = \{(2i+1+r, j, 1)^p, (2i+1+r, j, 2)^p, (2i+2+r, j, 1)^p, (2i+2+r, j, 2)^p\}$, where $i \in [0, q-1], j \in [n_2], p \in [t]$ (cf. Fig 5). In each component we define a linear ordering $\prec^p$ on the family of these sets in the following way: $S^p_{ij} \prec S^p_{rs}$ if $(j < s)$ or $(j = s$ and $i < r)$. According to this ordering we properly colour vertices of each set with the following rules:

- if it is only possible, we colour vertices in S^p_{ij} in such a way that the vertices from this set get different colours;
- if we cannot colour vertices in S^p_{ij} in such a way that S^p_{ij} is rainbow then we colour vertices in this set in such a way that two vertices have the same colour, let us say colour c, other vertices are coloured differently and there is no vertex coloured with c in S^p_{ij-1}.

We show that there exists a proper L-colouring of $G - H$ such that these rules are maintained. It is easy to see that we can colour vertices in sets $\{S^p_{i1} : i \in [0, q-1]\}$ such that these sets are rainbow. Suppose that we are at the step when we colour vertices in S^p_{ij}, $j \ge 2$, so vertices of every set that precedes S^p_{ij} are coloured, the vertices in S^p_{ij} are uncoloured. Let c'' be a proper L-colouring of the coloured part of $G - H$ constructed up to now. To simplify the notation let $S^p_{ij} = \{(x, j, 1), (x, j, 2), (x + 1, j, 1), (x + 1, j, 2)\}$. Thus each vertex in $\{(x, j, 1), (x, j, 2)\}$ has at most two coloured neighbours that are not in S^p_{ij} and each vertex in $\{(x+1, j, 1), (x+1, j, 2)\}$ has one coloured neighbour that is not in S^p_{ij}. Suppose that we cannot colour vertices in S^p_{ij} such that S^p_{ij} is rainbow. Since every vertex has four colours on its list, we can always colour three vertices in S^p_{ij} with different colours, only the last vertex being coloured in S^p_{ij} obtains the colour just used on S^p_{ij}. Let $c''((x, j, 1)) = c_1, c''((x, j, 2)) = c_2, c''((x + 1, j, 1)) = c_3, c''((x + 1, j, 2)) = c_1$. If there is no vertex coloured with c_1 in S^p_{ij-1} then we are done. Suppose that there is a vertex coloured with c_1 in S^p_{ij-1}. Since we are forced to use the colour c_1 on $(x+1, j, 2)$, we necessarily have $L((x+1, j, 2)) = \{c_1, c_2, c_3, c''(x+1, j-1, 2)\}$. If in $L((x + 1, j, 1))$ there is a colour b such that $b \notin \{c_1, c_2, c_3, c''((x + 1, j - 1, 1))$ then we can colour $(x + 1, j, 1)$ with b and next we colour $(x + 1, j, 2)$ with c_3, to obtain a rainbow set S^p_{ij}, a contradiction. Thus $L((x + 1, j, 1)) = \{c_1, c_2, c_3, c''(x+1, j-1, 1)\}$. Since each vertex has four different colours on the list, we have $c_1 \neq c''(x+1, j-1, 2)$ and $c_1 \neq c''(x+1, j-1, 1)$. Furthermore, $(x, j-1, 1)$ has a neighbour coloured with c_1, thus $c''((x, j-1, 1)) \neq c_1$. However, by our assumption in S^p_{ij-1} there is a vertex coloured with c_1, so $c''((x, j - 1, 2)) = c_1$. Observe that also $c_2 \neq c''((x + 1, j - 1, 2))$ and $c_2 \neq c''((x + 1, j - 1, 1))$. Thus if $c_2 \neq c''((x, j - 1, 1))$ then we can colour $(x + 1, j, 1)$ with c_2 and $(x + 1, j, 2)$ with c_3 to obtain desired colouring. Assume that $c_2 = c''((x, j - 1, 1))$. Observe that there is no vertex coloured with c_3 in S^p_{ij-1}. If $c_3 \in L((x, j, 1))$ then we colour $(x, j, 1)$ with c_3 and next $(x+1, j, 1)$ with c_2 to obtain a desired colouring. Otherwise, $(x, j, 1)$ has a colour b different from c_1, c_2, c_3 and $c''(x - 1, j, 1)$ on its list. If we colour $(x, j, 1)$ with b, the S^p_{ij} is rainbow, a contradiction.

Claim 3.10. *If the set S^p_{ij} is not rainbow and S^p_{ij-1} is not rainbow, i.e., in S^p_{ij-1} there are two vertices coloured with b_1, then in S^p_{ij} there is no vertex coloured with b_1.*

Proof. Without loss of generality we may assume $c''((x, j, 1)) = c_1, c''((x, j, 2)) = c_2, c''((x+1, j, 1)) = c_3, c''((x + 1, j, 2)) = c_1$. Similarly as above we observe that $L((x+1, j, 1)) = \{c_1, c_2, c_3, c''((x+1, j-1, 1))\}$ and $L((x+1, j, 2)) = \{c_1, c_2, c_3, c''((x + 1, j - 1, 2))\}$. Since the colours on lists are different, $c''((x + 1, j - 1, 1)) \notin \{c_1, c_2, c_3\}$ and $c''(x + 1, j - 1, 2) \notin \{c_1, c_2, c_3\}$ and hence neither $c''((x + 1, j - 1, 1))$ nor $c''((x + 1, j - 1, 2))$ is used on S^p_{ij}. The argument that $c''((x, j - 1, 1)) \neq c''((x, j - 1, 2))$ completes the proof. □

Previous arguments imply that either S^p_{ij} is rainbow or $S^p_{ij} \cup S^p_{ij-1}$ can be divided into two 4-elements rainbow sets in $(G - H, c'')$, as it has been shown that each colour is used in $S^p_{ij} \cup S^p_{ij-1}$ at most twice.

We use the similar method as in the proof of Lemma 3.6 to show that there is a rainbow 4-partition of $(G - H, c'')$. We divide $V(G - H)$ in the following way (cf. Fig. 5):

- the set of vertices of each component is divided step by step;
- in each component G^p, we start with the last set due to $\prec^p$ and go down according this ordering;
- if S^p_{ij} is rainbow then it forms a set of the rainbow special 4-partition of $(G-H, c'')$, otherwise, we partite $S^p_{ij} \cup S^p_{ij-1}$ into two rainbow 4-element sets that form two sets of the rainbow 4-partition of $(G - H, c'')$, we modify $\prec^p$ by removing sets that have been already included into the rainbow 4-partition.

Recall that for $i \in [0, q - 1]$ the sets S^p_{i1} are rainbow, so the above partition results in a rainbow special 4-partition of $(G - H, c'')$. Thus, together with the rainbow 4-partition of (H, c'), we obtain the rainbow 4-partition of $(G, c' \cup c'')$. Hence for every 4-uniform list assignment L there is a proper L-colouring c such that (G, c) has a rainbow 4-partition, and so G is equitably 4-choosable, by Lemma 2.2. □

Remark 3.11. *Lemma* 3.9 *is still true when components of G are of different size.*

Observe that the 4-partition given in the proof of Lemma 3.9 does not meet the assumptions of Lemma 2.5, thus from that proof we cannot conclude that such a graph is equitably k-choosable for $k > 4$. However, if each component of G is isomorphic to $P_{n_1}\Box P_{n_2}\Box P_2$ or $P_{n_1}\Box P_{n_2}$ then $\Delta(G) \leq 5$ and by Theorem 2.8 we have that G is equitably k-choosable for $k \geq 6$.

4. Equitable list vertex arboricity of grids

In this section we apply tools described in the previous sections what causes in giving new results concerning equitable list arboricity of d-dimensional grids $P_{n_1}\Box\cdots\Box P_{n_d}$.

First, observe that every 2-dimensional grid has a spanning linear forest, i.e. a union of disjoint paths), that covers all cycles. Since every linear forest is equitably k-choosable for any $k \geq 2$ (cf. Lemma 3.3) then, using Lemma 2.7, we have the following

Theorem 4.1. *Let $k \in \mathbb{N}$. If $k \geq 2$ then every 2-dimensional grid is equitably k-list arborable.*

4.1. 3-dimensional grids

Theorem 4.2. *Let $k, n_2, n_3 \in \mathbb{N}$ with $n_2 \geq 2$, $n_3 \geq 2$. If $k \geq 2$ then $P_2\Box P_{n_2}\Box P_{n_3}$ is equitably k-list arborable.*

Proof. We will prove that $P_2\Box P_{n_2}\Box P_{n_3}$ contains a subgraph H with maximum degree at most two that covers all cycles. Since $P_2\Box P_{n_2}\Box P_{n_3}$ is bipartite then H is also bipartite so, by Lemma 3.3, H is certainly equitably k-choosable for any $k \geq 2$. Hence, by Lemma 2.7, the proof will follow.

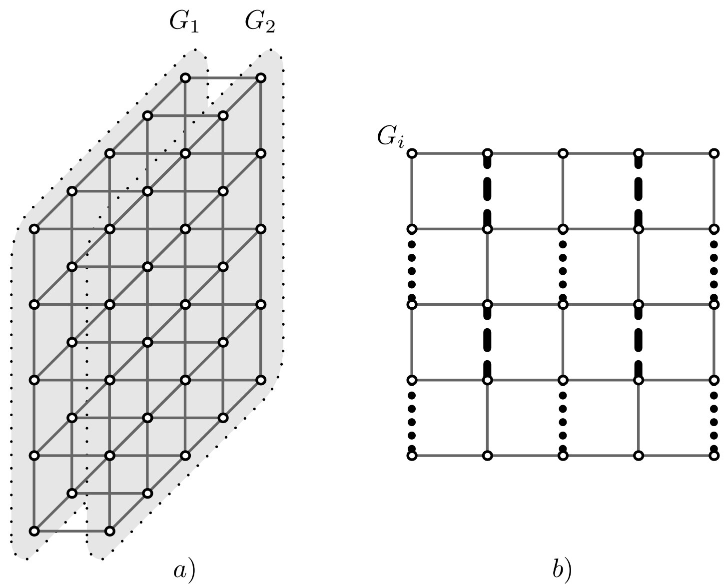

Figure 6: Illustration for the proof of Theorem 4.2; a) $P_2\Box P_5\Box P_5$ with depicted layers G_1 and G_2; b) layer G_i with depicted set M_i' (dotted line) and set M_i'' (dashed line).

We can see $P_2\Box P_{n_2}\Box P_{n_3}$ as two copies of $P_{n_2}\Box P_{n_3}$ (we call them layers G_1 and G_2) joined by some edges. Let $V(G_1) = \{(1, y, z) : y \in [n_2], z \in [n_3]\}$ be the vertex set of the layer G_1 and let $V(G_2) = \{(2, y, z) : y \in [n_2], z \in [n_3]\}$ be the vertex set of the layer G_2 (cf. Fig. 6a)). In each layer we choose a maximal matching in the following way. In each column we choose a maximal matching. We start with the first edge if the column is odd and with the second edge if the column is even. More formally, for $i \in [2]$, $M_i' = \{(i, 2p+1, r)(i, 2p+2, r) : p \in [0, \lfloor (n_2-2)/2 \rfloor], r \in [n_3], r \text{ is odd}\}$ and $M_i'' = \{(i, 2p, r)(i, 2p+1, r) : p \in [\lfloor (n_2-1)/2 \rfloor], r \in [n_3], r \text{ is even}\}$ (cf. Fig. 6b)). Let M_i be a spanning subgraph of G_i such that $V(M_i) = V(G_i)$ and $E(M_i) = M_i' \cup M_i''$. We show that M_i covers all cycles in G_i. Since both G_1, G_2 are isomorphic to $P_{n_2} \square P_{n_3}$ we simplify notation and show that $M = M' \cup M''$ covers all cycles in $P_{n_2} \square P_{n_3}$, where $M' = \{(2p+1, r)(2p+2, r) : p \in [0, \lfloor (n_2 - 2)/2 \rfloor], r \in [n_3], r \text{ is odd}\}$ and $M'' = \{(2p, r)(2p+1, r) : p \in [\lfloor (n_2 - 1)/2 \rfloor], r \in [n_3], r \text{ is even}\}$. We prove it by induction on n_3. It is obviously true for $n_3 = 2$. Thus by induction hypothesis we may assume that such a spanning subgraph covers all cycles of $P_{n_2} \square P_{n_3 - 1}$. Suppose that $P_{n_2} \square P_{n_3}$ contains a cycle C not covered by M. Thus C contains an edge whose vertices have second coordinates n_3, say $(x, n_3)(x+1, n_3)$. So $(x, n_3)(x+1, n_3) \notin M$, however by our choice of M we have $(x-1, n_3)(x, n_3) \in M$ and $(x+1, n_3)(x+2, n_3) \in M$ (whenever such edges exist in $P_{n_2} \square P_{n_3}$). Thus C must contain vertices $(x, n_3 - 1), (x+1, n_3 - 1)$ but $(x, n_3 - 1)(x+1, n_3 - 1) \in M$, which contradicts that M does not cover C. Now we construct a spanning subgraph H of $P_2 \square P_{n_2} \square P_{n_3}$ in the following way. Let us denote the set of edges in $P_2 \square P_{n_2} \square P_{n_3}$ joining vertices between G_1 and G_2 by $E(G_1, G_2)$. We set $E(H) = M_1 \cup M_2 \cup E(G_1, G_2)$. Thus H covers all cycles of $P_2 \square P_{n_2} \square P_{n_3}$ and $\Delta(H) = 2$, and so $P_2 \square P_{n_2} \square P_{n_3}$ is equitably k-list arborable for every $k \geq 2$. □

Theorem 4.3. *Let $n_3, k \in \mathbb{N}$. If $k \geq 2$ then $P_3 \square P_3 \square P_{n_3}$ is equitably k-list arborable.*

Proof. Similarly as in the proof of Theorem 4.2, we prove that $P_3 \square P_3 \square P_{n_3}$ contains a spanning subgraph $HP_{3 \times 3 \times n_3}$ with maximum degree at most two that covers all cycles. Since $P_3 \square P_3 \square P_{n_3}$ is bipartite, $HP_{3 \times 3 \times n_3}$ is also bipartite so, by Lemma 3.3, $HP_{3 \times 3 \times n_3}$ is equitably k-choosable for any $k \geq 2$. Thus, by Lemma 2.7, the proof will follow.

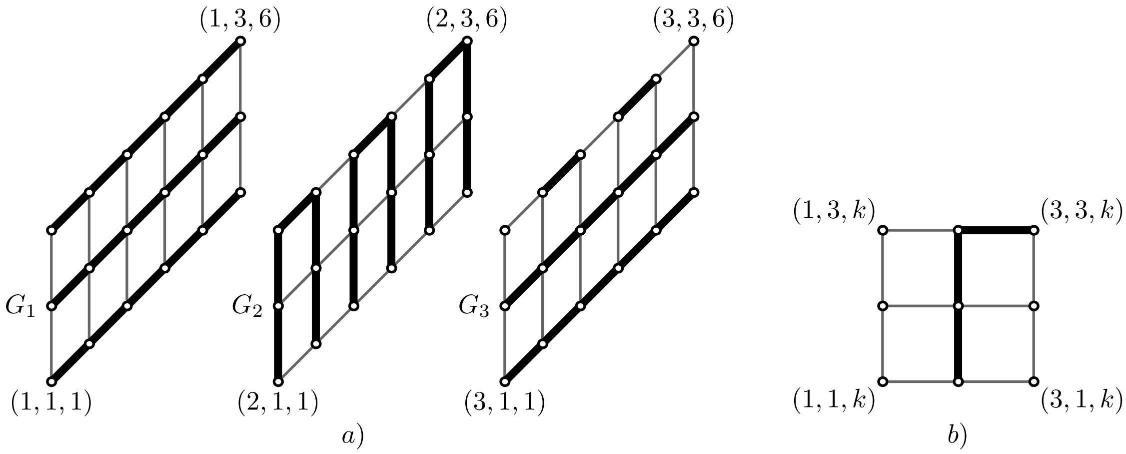

Figure 7: Illustration for the proof of Theorem 4.3

Let G_1, G_2 and G_3 be layers of $P_3 \square P_3 \square P_{n_3}$ such that $V(G_i) = \{(i, y, z) : y \in [3], z \in [n_3]\}$ for $i \in [3]$. In each layer G_i we choose the spanning subgraph M_i in the following way (cf. Fig. 7a)):

- $E(M_1) = \{(1, i, j)(1, i, j+1) : i \in [3], j \in [n_3 - 1]\}$;
- $E(M_2) = \{(2, 1, i)(2, 1, i+1) : i \in [n_3 - 1]_{ODD}\} \cup \{(2, 1, i)(2, 2, i), (2, 2, i)(2, 3, i) : i \in [n_3]\}$;
- $E(M_3) = \{(3, 1, i)(3, 1, i+1) : i \in [n_3 - 1]\} \cup \{(3, 2, i)(3, 2, i+1) : i \in [n_3 - 1]\} \cup \{(3, 3, i)(3, 3, i+1) : i \in [n_3 - 1]_{EVEN}\}$.

Moreover,

- $E_{2,3} = \{(2, 3, i)(3, 3, i) : i \in [n_3]\}$.

The subgraph $HP_{3 \times 3 \times n_3}$ is defined in the following way: $V(HP_{3 \times 3 \times n_3}) = V(P_3 \square P_3 \square P_{n_3})$ and $E(HP_{3 \times 3 \times n_3}) = E(M_1) \cup E(M_2) \cup E(M_3) \cup E_{2,3}$.

We show that $HP_{3\times3\times n_3}$ covers all cycles of $P_3\Box P_3\Box P_{n_3}$. Let L_i for $i \in [n_3]$ be layers that are isomorphic to $P_3\Box P_3$, so $V(L_i) = \{(j, \ell, i) : j \in [3], \ell \in [3]\}$. Observe that the subgraphs induced by $V(HP_{3\times3\times n_3}) \cap V(L_i)$ are isomorphic (cf. Fig. 7b)).

If a cycle in $P_3\Box P_3\Box P_{n_3}$ contains an edge from $HP_{3\times3\times n_3}$ then obviously it is covered by $HP_{3\times3\times n_3}$. Thus we focus only on cycles in $P_3\Box P_3\Box P_{n_3} - E(HP_{3\times3\times n_3})$. We use the induction method to proof that every cycle in $P_3\Box P_3\Box P_{n_3} - E(HP_{3\times3\times n_3})$ contains two vertices u and v such that $uv \in E(HP_{3\times3\times n_3})$.

It is easy to see that $HP_{3\times3\times1}$ covers all cycles in $P_3\Box P_3\Box P_1$. Let $n_3 \geq 2$, assume that $HP_{3\times3\times(n_3-1)}$ covers all cycles in $P_3\Box P_3\Box P_{n_3-1}$ and consider $HP_{3\times3\times n_3}$ in $P_3\Box P_3\Box P_{n_3}$. Thus if there is an uncovered cycle in $P_3\Box P_3\Box P_{n_3} - E(HP_{3\times3\times n_3})$ then it must contain vertices from layer L_{n_3}. First observe that the only cycle of L_{n_3} that contains no edge from $HP_{3\times3\times n_3}$ contains vertices $(2, 1, n_3)$ and $(2, 2, n_3)$. Since $(2, 1, n_3)(2, 2, n_3) \in E(HP_{3\times3\times n_3})$, all cycles of L_{n_3} are covered by $HP_{3\times3\times n_3}$. Thus if there is an uncovered cycle C in $P_3\Box P_3\Box P_{n_3} - E(HP_{3\times3\times n_3})$ then it must contain vertices from layers L_{n_3} and L_{n_3-1}. We consider two cases.

Case 1. n_3 is even. C must go through two out of three following edges: $a = (2, 3, n_3 - 1)(2, 3, n_3)$, $b = (2, 2, n_3 - 1)(2, 2, n_3)$, $c = (3, 3, n_3 - 1)(3, 3, n_3)$. If C contains edges a and b (edges a and c, resp.) then it is covered by the edge $(2, 2, n_3)(2, 3, n_3)$ ($(2, 3, n_3)(3, 3, n_3)$, resp.). If C goes through the edges b and c then it must contain the vertex $(3, 2, n_3)$. On the other hand, edges $(3, 3, n_3 - 2)(3, 3, n_3 - 1)$ and $(2, 3, n_3 - 1)(3, 3, n_3 - 1)$ belong to $HP_{3\times3\times n_3}$. Hence C must go through $(3, 2, n_3 - 1)(3, 3, n_3 - 1)$. This implies that the cycle is covered by the edge $(3, 2, n_3 - 1)(3, 2, n_3)$.

Case 2. n_3 is odd. C must go through two out of three following edges: $a = (2, 3, n_3 - 1)(2, 3, n_3)$, $b = (2, 2, n_3 - 1)(2, 2, n_3)$, $c = (2, 1, n_3 - 1)(2, 1, n_3)$. If C contains edges a and b (b and c, resp.) then it is covered by the edge $(2, 2, n_3)(2, 3, n_3)$ ($(2, 1, n_3)(2, 2, n_3)$, resp.). If the cycle contains the edges a and c then, to avoid vertex $(2, 2, n_3)$, it consecutively goes through the edge a, vertices $(1, 3, n_3)$, $(1, 2, n_3)$, $(1, 1, n_3)$, $(2, 1, n_3)$ and edge c. Observe that $(2, 3, n_3 - 1)$ is incident with exactly two edges $(1, 3, n_3 - 1)(2, 3, n_3 - 1)$ and $(2, 3, n_3 - 2)(1, 3, n_3 - 1)$ that are not in $E(HP_{3\times3\times n_3})$. Due to '$n_3$ even' case the cycle C cannot go through the second one. If it goes through the first one then $(1, 3, n_3 - 1) \in V(C)$ and C is covered by $(1, 3, n_3 - 1)(1, 3, n_3)$.

Thus $HP_{3\times3\times n_3}$ covers all cycles of $P_3\Box P_3\Box P_{n_3}$. $\Delta(HP_{3\times3\times n_3}) = 2$, and so $P_3\Box P_3\Box P_{n_3}$ is equitably k-list arborable for every $k \geq 2$. □

Theorem 4.4. *Let $n_1, n_2, n_3, k \in \mathbb{N}$. If $k \geq 3$ then $P_{n_1}\Box P_{n_2}\Box P_{n_3}$ is equitably k-list arborable.*

Proof. Let $G = P_{n_1}\Box P_{n_2}\Box P_{n_3}$ be a 3-dimensional grid. Let us define a set of edges $X_{ij} = \{(\ell, i, j)(\ell + 1, i, j) : \ell \in [n_1 - 1]\}$ for $i \in [n_2]$ and $j \in [n_3]$. First, observe that the graph $(V(G), X)$, where $X = \bigcup_{i\in[n_2], j\in[n_3]} X_{ij}$, is a linear forest. Thus $G - X$ covers all cycles of G. Furthermore, every component of $G - X$ is isomorphic to $P_{n_2}\Box P_{n_3}$. Thus, by Lemma 3.7, $G - X$ is equitably k-choosable for every $k \geq 3$. Finally, Lemma 2.7 implies that G is equitably k-list arborable for every $k \geq 3$. □

4.2. 4-dimensional grids

Theorem 4.5. *Let $n_4, k \in \mathbb{N}$. If $k \geq 2$ then $P_2\Box P_2\Box P_2\Box P_{n_4}$ is equitably k-list arborable.*

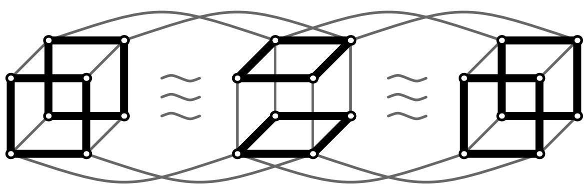

Figure 8: Illustration for the proof of Theorem 4.5

Proof. Let $G = P_2\square P_2\square P_2\square P_{n_4}$. We can see G as n_4 3-dimensional cubes $Q^1, \ldots, Q^{n_4}$ joined by some edges. Let H be a spanning subgraph of G that contains two cycles of length 4 of each cube Q^i: 'front' and 'back' cycles of Q^i with i odd, 'top' and 'bottom' cycles of Q^i with i even (cf. Fig. 8). More formally, let us define a spanning subgraph H of G in the following way $E(H) = E_1 \cup E_2$, where

$$E_1 = \{(1,1,1,i)(1,2,1,i), (1,1,1,i)(2,1,1,i), (1,2,1,i)(2,2,1,i), (2,1,1,i)(2,2,1,i), \\ (1,1,2,i)(1,2,2,i), (1,1,2,i)(2,1,2,i), (1,2,2,i)(2,2,2,i), (2,1,2,i)(2,2,2,i) : i \in [n_4]_{ODD}\}$$

$$E_2 = \{(1,1,1,j)(1,1,2,j), (1,1,1,j)(2,1,1,j), (2,1,1,j)(2,1,2,j), (1,1,2,j)(2,1,2,j), \\ (1,2,1,j)(1,2,2,j), (1,2,1,j)(2,2,1,j), (2,2,1,j)(2,2,2,j), (1,2,2,j)(2,2,2,j) : j \in [n_4]_{EVEN}\}.$$

We prove by induction on n_4 that H covers all cycles of $P_2\square P_2\square P_2\square P_{n_4}$. It is obviously true for $n_4 = 1$. Assume that it is true for $P_2\square P_2\square P_2\square P_{n_4-1}$. Without loss of generality we may assume that n_4 is even. Suppose that there is a cycle C in G that has no two vertices adjacent by an edge in H. Since there is no such a cycle in $P_2\square P_2\square P_2\square P_{n_4-1}$, it follows that C contains an edge of the cube Q^{n_4} induced by the vertices of the form (i, j, ℓ, n_4), $i \in [2]$, $j \in [2]$, $\ell \in [2]$ that is not in H. By symmetry we may assume that C contains $(1,1,1,n_4)(1,2,1,n_4)$. Thus C must also contain vertices $(1,1,1,n_4-1)$ and $(1,2,1,n_4-1)$, however $(1,1,1,n_4-1)(1,2,1,n_4-1) \in E(H)$, a contradiction. Since H is equitably k-choosable for $k \geq 2$ by Lemma 3.3, G is equitably k-list arborable for $k \geq 2$ by Lemma 2.7. □

Theorem 4.6. *Let $n_3, n_4, k \in \mathbb{N}$. If $k \geq 3$ then $P_2\square P_2\square P_{n_3}\square P_{n_4}$ is equitably k-list arborable.*

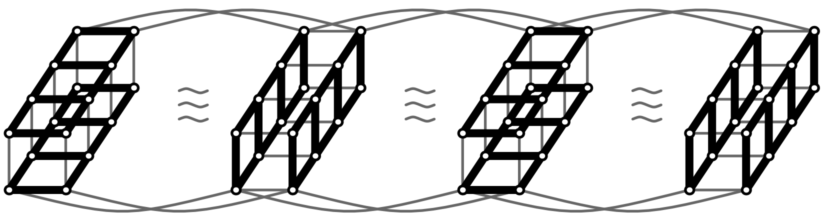

Figure 9: Illustration for the proof of Theorem 4.6

Proof. Let $G = P_2\square P_2\square P_{n_3}\square P_{n_4}$. We show that there is a spanning subgraph H of G that covers all cycles of G such that each component of H is isomorphic to $P_2\square P_{n_3}$. Since H is equitably k-choosable for $k \geq 3$, by Lemma 3.4, we apply Lemma 2.7 to show that G is equitably k-list arborable for every $k \geq 3$. We can cf. G as n_4 layers $G_1, \ldots, G_{n_4}$, each of which is isomorphic to a 3-dimensional grid $P_2\square P_2\square P_{n_3}$, joined by some edges. To obtain H from every grid G_i we take two disjoint $P_2\square P_{n_3}$, if i is odd we take 'top' and 'bottom' $P_2\square P_{n_3}$, if i is even we take 'left' and 'right' $P_2\square P_{n_3}$ (cf. Fig. 9). Let $H = \bigcup_{i\in[n_4]_{ODD}}(H_{1i} \cup H_{2i}) \cup \bigcup_{j\in[n_4]_{EVEN}}(H'_{1j} \cup H'_{2j})$ be a spanning subgraph of G, where

- $H_{1i} = G[\{(1,1,p,i), (2,1,p,i) : p \in [n_3]]$ ('bottom');
- $H_{2i} = G[\{(1,2,p,i), (2,2,p,i) : p \in n_3]]$ ('top');
- $H'_{1j} = G[\{(1,1,p,j), (1,2,p,j) : p \in [n_3]]$ ('left');
- $H'_{2j} = G[\{(2,1,p,j), (2,2,p,j) : p \in [n_3]]$ ('right').

We prove by induction on n_4 that H covers all cycles of G. It is easy to see that if $n_4 = 1$, the subgraph H covers all cycles of G. Now, suppose that H covers all cycles of $P_2\square P_2\square P_{n_3}\square P_{n_4-1}$. Without loss of generality we may assume that n_4 is odd. If G contains a cycle C not covered by H then there is an edge in C whose end vertices have the last coordinate n_4 and that are not in H. Let $(1,1,p,n_4)(1,2,p,n_4)$ be such an edge. Since all edges adjacent to the edge $(1,1,p,n_4)(1,2,p,n_4)$ except $(1,1,p,n_4)(1,1,p,n_4-1)$ and $(1,2,p,n_4)(1,2,p,n_4-1)$ are in H then the vertices $(1,1,p,n_4-1)$ and $(1,2,p,n_4-1)$ must be in C. However, $(1,1,p,n_4-1)(1,2,p,n_4-1) \in E(H)$, which contradicts the assumption that H does not cover C. Thus, by Lemma 3.4 and Lemma 2.7, the theorem holds. □

Theorem 4.7. *Every* 4*-dimensional grid is equitably* 4*-list arborable.*

Proof. Let $G = P_{n_1} \Box P_{n_2} \Box P_{n_3} \Box P_{n_4}$. Again, we determine a graph H, whose every component is isomorphic to $P_2 \Box P_{n_2} \Box P_{n_3}$ or $P_{n_2} \Box P_{n_3}$, that covers all cycles of G. Next we apply Lemmas 3.9 and 2.7, so G is equitably 4-list arborable.

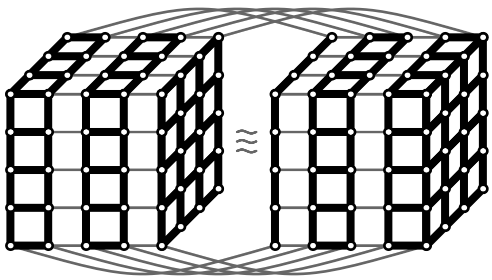

Figure 10: Illustration for the proof of Theorem 4.7.

We can see G as 3-dimensional grids $G_i = P_{n_1} \Box P_{n_2} \Box P_{n_3}$, $i \in [n_4]$ joined by some edges, i.e. $G_i = G[\{(r,s,t,i) : r \in [n_1], s \in [n_2], t \in [n_3]\}], ; i \in [n_4]$. To obtain H we take all copies of G_i after removing the matching E_i defined as follows (cf. Fig. 10).

$$E_i = \begin{cases} \{(r,s,t,i)(r+1,s,t,i) : r \in [n_1-1]_{ODD}, s \in [n_2], t \in [n_3]\} & \text{if } i \text{ is odd,} \\ \{(r,s,t,i)(r+1,s,t,i) : r \in [n_1-1]_{EVEN}, s \in [n_2], t \in [n_3]\} & \text{if } i \text{ is even.} \end{cases}$$

Now, $H = \bigcup_{i \in [n_4]} (G_i - E_i)$. We prove by induction on n_4 that H covers all cycles of G. Since E_1 is a matching, $G_1 - E_1$ obviously covers all cycles of G_1. Let $G' = P_{n_1} \Box P_{n_2} \Box P_{n_3} \Box P_{n_4-1}$ and $H' = \bigcup_{i \in [n_4-1]} (G_i - E_i)$. Assume that H' covers all cycles of G'. Without of loss generality we may assume that n_4 is odd. On the contrary, suppose that G contains a cycle C not covered by H. Thus C contains an edge e of E_{n_4}, say $e = (2r+1,s,t,n_4)(2r+2,s,t,n_4)$. So vertices $(2r+1,s,t,n_4), (2r+2,s,t,n_4)$ are in $V(C)$. Since all edges of G_{n_4} incident with $(2r+1,s,t,n_4)$ and $(2r+2,s,t,n_4)$, except e, are in H, we must have that $(2r+1,s,t,n_4-1)$ is a neighbour of $(2r+1,s,t,n_4)$ in C and $(2r+2,s,t,n_4-1)$ is a neighbour of $(2r+2,s,t,n_4)$ in C. Thus $(2r+1,s,t,n_4-1), (2r+2,s,t,n_4-1) \in V(C)$, however $(2r+1,s,t,n_4-1)(2r+2,s,t,n_4-1) \in E(H)$, which contradicts that C is not covered by H. □

In the proof of the next theorem we use Lemma 2.6. We determine a special 5-partition of a graph to show that the graph is equitably k-list arborable for every $k \geq 5$.

Theorem 4.8. *Let* $k \in \mathbb{N}$*. If* $k \geq 5$ *then every* 4*-dimensional grid is equitably* k*-list arborable.*

Proof. Let $G = P_{n_1} \Box P_{n_2} \Box P_{n_3} \Box P_{n_4}$ and $V(G) = \{(i,j,k,l) : i \in [n_1], j \in [n_2], k \in [n_3], l \in [n_4]\}$, We determine a special 5-partition $S_1 \cup \cdots \cup S_{\eta+1}$ of G, with $|V(G)| = 5\eta + r$ and $r \in [5]$, that fulfills the assumptions of Lemma 2.6. So, by Lemma 2.6, the theorem will follow. We depict sets S_j of size 5 step by step in decreasing order, starting with determining a set $S_{\eta+1}$ and next, in the same manner, sets $S_\eta, \ldots, S_2$. The last set S_1 is formed by vertices in $V(G) \backslash (S_2 \cup \cdots \cup S_{\eta+1})$, so its size is less than or equal to 5. Since the assumtions of Lemma 2.6 are obviously fulfilled for each 4-dimentional grid G satisfying $|V(G)| \leq 5$ we may assume that $|V(G)| \geq 6$.

Let $j \in [2, \eta+1]$. To determine a set S_j consisting of elements $x_1^j, \ldots, x_5^j$, we use the sets $S_{j+1}, \ldots, S_{\eta+1}$ constituted in the previous steps. Let $G_j = G - (S_{j+1} \cup \cdots \cup S_{\eta+1})$. Thus G_j is the graph induced in G by the union of sets $S_1, \ldots, S_j$, whose forms are unknown at this moment. Observe that $V(G_{j-1})$ is equal to

$S_1 \cup \cdots \cup S_{j-1}$. Hence $V(G_{j-1})$ is the set involved in the condition (4) of Lemma 2.6. Precisely, this condition can be rewritten here in the form

$$|N_{G_{j-1}}(x_i^j)| \leq 2i - 1.$$

To find $x_1^j, \ldots, x_5^j$ that satisfy the condition (4) of Lemma 2.6, let us do as follows.

Let L_{lex} be the list of all vertices of $V(G_{j-1})$ ordered lexicographically. Note that if vertex (a, b, c, d) is the first in the list then it has at most four neighbours in the list: $(a + 1, b, c, d)$, $(a, b + 1, c, d)$, $(a, b, c + 1, d)$, $(a, b, c, d + 1)$, moreover if it has exactly four neighbours then $(a, b, c, d + 1)$ is the second in the list.

Let x_1^j be the first, x_2^j the second and x_3^j the third vertex in the list L_{lex}. Remove those vertices from the list. If there is still any neighbour of x_1^j in the list then let x_4^j be this neighbour, otherwise let x_4^j be the first element in the list. Remove x_4^j from the list and similarly choose x_5^j. If there is any neighbour of x_1^j in the list then let x_5^j be this neighbour, otherwise let x_5^j be the first element in the list.

We will prove that the set S_j, determined in the way described above, fulfill the assumption of Lemma 2.6. We know that $|N_{G_j}(x_1^j)| \leq 4$. If $|N_{G_j}(x_i^j)| = 4$ then we have chosen to S_j at least three of the neighbours of x_1^j: x_2^j, x_4^j, x_5^j. On the other hand, if $2 \leq |N_{G_j}(x_1^j)| \leq 3$ then at least two neighbours of x_1^j are chosen to S_j. In every case we have $|N_{G_{j-1}}(x_1^j)| \leq 1$. If $|N_{G_j}(x_2^j)| = 4$ then x_2^j and x_3^j are adjacent, so $|N_{G_{j-1}}(x_2^j)| \leq 3$. After removing x_1^j and x_2^j, the vertex x_3^j was the first in the list so $|N_{G_{j-1}}(x_3^j)| \leq 4 \leq 5$. If x_4^j was chosen as the first in the list then $|N_{G_{j-1}}(x_4^j)| \leq 4$, otherwise at least one of its neighbours, i.e. x_1^j, is in S_j, so $|N_{G_{j-1}}(x_4^j)| \leq 7$. Obviously $|N_{G_{j-1}}(x_5^j)| \leq 9$. □

4.3. *d-dimensional grids, the general upper bound*

In Section 2 we give a general upper bound on the equitable list vertex arboricity of all graphs. Now we improve this bound for d-dimensional grids.

Assume that $d \geq 3$ and $n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\}$. Let us define the following family of graphs.

$$\mathcal{H}(n_1, \ldots, n_{d-2}) = \{G : \text{each component of } G \text{ is isomorphic to } P_{n_1} \square \cdots \square P_{n_{d-2}} \square P_2 \text{ or } P_{n_1} \square \cdots \square P_{n_{d-2}}\}.$$

Lemma 4.9. *Let $d \in \mathbb{N}$ with $d \geq 3$, $n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\}$ and $G = P_{n_1} \square \ldots \square P_{n_d}$. There is a graph $H \in \mathcal{H}(n_1, \ldots, n_{d-2})$ that covers all cycles of G.*

Proof. The idea of determining a graph H is the same as in the proof of Theorem 4.7. We can see G as n_d copies of a $(d-1)$-dimensional grid $P_{n_1} \square \cdots \square P_{n_{d-1}}$ joined by some edges. Let $G_i = G[\{(y_1, \ldots, y_{d-1}, i) : y_j \in [n_j], j \in [d-1]\}]$, $i \in [n_d]$. To obtain H, we delete from every G_i the matching E_i defined as follows.

Case 1 i is odd

$$E_i = \{(y_1, y_2, \ldots, y_{d-1}, i)(y_1 + 1, y_2, \ldots, y_{d-1}, i) : y_1 \in [n_1 - 1]_{ODD}\}.$$

Case 2 i is even

$$E_i = \{(y_1, y_2, \ldots, y_{d-1}, i)(y_1 + 1, y_2, \ldots, y_{d-1}, i) : y_1 \in [n_1 - 1]_{EVEN}\}.$$

In both cases we take $y_j \in [n_j]$ for $j \in [2, d-1]$. Put $H = \bigcup_{i \in [n_d]} (G_i - E_i)$. Note that $H \in \mathcal{H}(n_1, \ldots, n_{d-2})$. We prove by induction on n_d that H covers all cycles of G. Since E_1 is a matching of G_1, obviously $G_1 - E_1$ covers all cycles of G_1. Let $G' = P_{n_1} \square \cdots \square P_{n_d - 1}$ and $H' = \bigcup_{i \in [n_d - 1]} (G_i - E_i)$. By the induction hypothesis, H' covers all cycles of G'. Without loss of generality we may assume that n_d is odd. On the contrary, suppose that G contains a cycle C not covered by H. Thus C contains an edge e of E_{n_d}, say $e = (2r + 1, y_2, \ldots, y_{d-1}, n_d)(2r + 2, y_2, \ldots, y_{d-1}, n_d)$. So vertices $(2r + 1, y_2, \ldots, y_{d-1}, n_d), (2r + 2, y_2, \ldots, y_{d-1}, n_d)$ are in $V(C)$. Since all edges of G_{n_d} incident with $(2r + 1, y_2, \ldots, y_{d-1}, n_d)$ and $(2r + 2, y_2, \ldots, y_{d-1}, n_d)$, except e, are in H, we must have that $(2r + 1, y_2, \ldots, y_{d-1}, n_d - 1)$ is a neighbour of $(2r + 1, y_2, \ldots, y_{d-1}, n_d)$ in C and $(2r + 2, y_2, \ldots, y_{d-1}, n_d - 1)$ is a neighbour of $(2r + 2, y_2, \ldots, y_{d-1}, n_d)$ in C. Thus $(2r + 1, y_2, \ldots, y_{d-1}, n_d - 1), (2r + 2, y_2, \ldots, y_{d-1}, n_d - 1) \in V(C)$, however $(2r + 1, y_2, \ldots, y_{d-1}, n_d - 1)(2r + 2, y_2, \ldots, y_{d-1}, n_d - 1) \in E(H)$, which contradicts that C is not covered by H. □

Observation 4.10. *Let $d \in \mathbb{N}$, $n_1, \ldots, n_{d-2} \in \mathbb{N} \setminus \{1\}$ and $H \in \mathcal{H}(n_1, \ldots, n_{d-2})$. If $d \geq 3$ then $\Delta(H) \leq 2d - 3$.*

Observation 4.10 together with Theorem 2.8(i)-(ii) and Lemma 2.7 imply the following result.

Theorem 4.11. *Let $d, k \in \mathbb{N}$.*

(i) *If $k \geq 8$ then every 5-dimensional grid is equitably k-list arborable.*
(ii) *If $d \in [6, 16]$ and $k \geq 2d - 2 + \frac{2d-4}{7}$ then every d-dimensional grid is equitably k-list arborable.*
(iii) *If $d \geq 17$ and $k \geq 2d - 3 + \frac{2d-3}{6}$ then every d-dimensional grid is equitably k-list arborable.*

5. Concluding remarks

Note that our results confirm Zhang's conjectures for d-dimensional grids, when $d \in [2, 4]$. For many cases they are even stronger than the conjectures. More precisely, we have obtained the following facts.

Corollary 5.1. *Let $k \in \mathbb{N}$ and $d \in \{2, 3, 4\}$. If G is a d-dimensional grid and $k \geq \lceil (\Delta(G) + 1)/2 \rceil$ then G is equitably k-list arborable.*

Corollary 5.2. *Let $d, k \in \mathbb{N}$ with $d \geq 2$ and $k \geq 2$. If G is a d-dimensional grid with $\Delta(G) \leq 5$ then G is equitably k-list arborable.*

Corollary 5.3. *Let $k \in \mathbb{N}$, $d \in \{2, 3, 4\}$, and let G be a d-dimensional grid with $\Delta(G) \geq 6$ that is different from $P_{n_1} \square P_{n_2} \square P_{n_3} \square P_2$, $n_1, n_2, n_3 \in \mathbb{N} \setminus \{1, 2\}$. If $k \geq \lfloor (\Delta(G))/2 \rfloor$ then G is equitably k-list arborable.*

Since d-dimensional grids have many special properties, we expect that the results that are better than Zhang's conjectures hold for almost all of them. Among others, d-dimensional grids are bipartite and d-degenerate. The equitable colouring of such classes of graphs is analyzed in many papers. For instance, it was proven in [8] that the inequality $\chi^=(G) \leq \Delta(G)$ holds for every connected bipartite graph G. We improve this result for all d-dimensional grids. The following two theorems will help us to post some conjectures.

Theorem 5.4. *Let $d, k \in \mathbb{N}$ with $d \geq 2$, and let G be a d-dimensional grid. If $k \geq 2$ then there exists an equitable proper k-colouring of G.*

The concept of layers in d-dimensional grids, used until now, must be extended on the purpose of the proof of Theorem 5.4. Let $G = P_{n_1} \square \cdots \square P_{n_d}$ and $\{i_1, \ldots, i_s\}$ be any s-subset of indexes from $[d]$. Moreover, let $(a_{i_1}, \ldots, a_{i_s})$ be a fixed s-tuple from $[n_{i_1}] \times \cdots \times [n_{i_s}]$. Then each graph induced in G by the set

$$\{(y_1, \ldots, y_d) : \ y_{i_1} = a_{i_1}, \ldots, y_{i_s} = a_{i_s}\}$$

is called an *s-layer* of G. Note that the layers used until now are 1-layers.

Proof. Let k be fixed and $G = P_{n_1} \square \cdots \square P_{n_d}$ with $n_1, \ldots, n_d \in \mathbb{N} \setminus \{1\}$. We construct a proper k-colouring of G in which every colour class has the cardinality either $\lceil |V(G)|/k \rceil$ or $\lfloor |V(G)|/k \rfloor$. The construction is given in d stages. For $i \in [d]$, in the $i-th$ stage we describe a proper k-colouring c_i of an i-dimensional grid $P_{n_1} \square \cdots \square P_{n_i}$ which is a $(d - i)$-layer G_i of G induced in G by the set of vertices V_i, where

$$V_i = \{(y_1, \ldots, y_i, \underbrace{1, \ldots, 1}_{d-i}) : \ y_1 \in [n_1], \ldots, y_i \in [n_i]\}.$$

We construct a proper k-coloring c_{i+1} on V_{i+1} as an extention of a proper k-colouring c_i on V_i. Finally, we obtain a proper k-colouring c_d of G. For each $i \in [d]$ we care for c_i to be equitable, which means that each colour class of c_i is of the cardinality either $\lceil (n_1 \cdots n_i)/k \rceil$ or $\lfloor (n_1 \cdots n_i)/k \rfloor$.

Let us start with the construction of c_1. In this case $G_1 = P_{n_1}$ and we put $c_1((y_1, \underbrace{1, \dots, 1}_{d-1})) \equiv y_1 (\text{mod } k)$.

Thus, depending on n_1, each of k colours arises either $\lceil n_1/k \rceil$ or $\lfloor n_1/k \rfloor$ times and moreover, c_1 is a proper k-colouring of G_1. Note that this time we use colors from $[0, k-1]$.

Suppose that, for some $i \in [d-1]$, the colouring c_i is constructed. Of course c_i satisfies all requirements mentioned before. Now we permute coloures used in c_i on vertices in V_i (recall that $|V_i| = n_1 \cdots n_i$) in such a way that each of the coloures $1, \dots, p$ is used $\lceil (n_1 \cdots n_i)/k \rceil$ times and each of the remaining $k - p$ coloures $p+1, \dots, k$ is used $\lfloor (n_1 \cdots n_i)/k \rfloor$ times. Of course it could be $p = k$. Now let us define c_{i+1} for each tuple $(y_1, \dots, y_{i+1}) \in [n_1] \times \cdots \times [n_{i+1}]$. We put

$$c_{i+1}((y_1, \dots y_i, y_{i+1}, \underbrace{1, \dots, 1}_{d-i-1})) = \begin{cases} (c_i((y_1, \dots y_i, \underbrace{1, \dots, 1}_{d-i})) + p(y_{i+1} - 1))(\text{mod } k), & \text{if } p \neq k, \\ (c_i((y_1, \dots y_i \underbrace{1, \dots, 1}_{d-i})) + y_{i+1} - 1)(\text{mod } k), & \text{if } p = k. \end{cases}$$

Note that c_{i+1} is proper. Indeed, the graph induced in G_{i+1} by vertices with fixed coordinate y_{i+1} is isomorphic to G_i and is coloured according to c_i (with permuted coloures). Moreover, each edge e of G_{i+1} that is not an edge of any copy of G_i (any of the n_{i+1} layers of G_{i+1} that are isomorphic to G_i), joins vertices from the consecutive copies of G_i that are consecutive layers of G_{i+1}. Hence e has end vertices coloured with j and $(j+p)(\text{mod } k)$, when $p \neq k$ and j and $(j+1)(\text{mod } k)$, when $k = p$ (for some $j \in [k]$). In both cases these two coloures are different. Thus c_{i+1} is proper.

Next we have to observe that c_{i+1} is equitable. Suppose that $p = k$. In this case each of k coloures arises in c_i on the same number of vertices in V_i. Since in G_{i+1} each of n_{i+1} copies of G_i is coloured in the same manner (with permuted coloures) we can see that in the whole graph G_{i+1} each colour arises the same number $(n_1 \cdots n_{i+1})/k$ of times. Consequently c_{i+1} is equitable in this case. Now, suppose that $p \neq k$. Recall that the vertices of the first layer of G_{i+1} are coloured in such a way that coloures $1, \dots, p$ arise one more than coloures $p+1, \dots k$. In the second layer the coloures $(p+1)(\text{mod } k), \dots, (p+p)(\text{mod } k)$ arise one more than the remaining $k - p$ coloures $(p+p+1)(\text{mod } k), \dots (p+p+k-p)(\text{mod } k)$ and so on. Thus we use coloures cyclically, which guarantees that c_{i+1} is equitable also in this case. □

It is very easy to observe the following fact valid for all d-degenerate graphs.

Theorem 5.5. *Let $d, k \in \mathbb{N}$. If $k \geq \lceil (d+1)/2 \rceil$ then every d-degenerate graph is k-list arborable.*

Proof. Let k be fixed. We order vertices $v_i, \dots, v_n$ of G such that $\deg_{G[\{v_1, \dots, v_i\}]}(v_i) \leq d$. Such an ordering always exists since G is d-degenerate. Let L be an arbitrary k-uniform list assignment for G. We construct an L-colouring of G whose each colour class induces an acyclic subgraph of G. We do it, step by step, putting on a vertex v_i a colour from its list that is not present more than once on previously coloured vertices $v_1, \dots, v_{i-1}$. Since the size of each list is at least $\lceil (d+1)/2 \rceil$, such a colour exists. Obviously, we obtained an L-colouring for G. Moreover, putting the colour on v_i we do not produce any monochromatic cycle since v_i has at most one neighbour in the colour of v_i. □

As we mentioned previously, a d-dimensional grid is d-degenerate graph and hence it is k-list arborable for every $k \geq \lceil (d+1)/2 \rceil$, by Theorem 5.5. Furthermore, when $k \neq 1$, by Theorem 5.4, for a d-dimensional grid there is a k-colouring, in which, each colour class is of the cardinality at most $\lceil |V(G)|/k \rceil$ and induces an acyclic graph (each edgless graph is acyclic). These two facts and some other investigation yield the proposition of a general conjecture. If the conjecture is true then it improves our results for 3-dimensional and 4-dimensional grids.

Conjecture 5.6. *Let $k, d \in \mathbb{N}$. If $k \geq \lceil (d+1)/2 \rceil$ then every d-dimensional grid is equitably k-list arborable.*

In general, we think that if a graph has a k-colouring in which each colour class is of the cardinality at most $\lceil |V(G)|/k \rceil$ and induces an acyclic graph, then it may not be equitably k-list arborable, even if it is k-list arborable. Thus we propose the following conjecture.

Conjecture 5.7. *There is a graph G and $k \in \mathbb{N}$ such that G is k-list arborable and G has a k-colouring in which each colour class is of the cardinality at most $\lceil |V(G)|/k \rceil$ and induces an acyclic graph, however G is not equitably k-list arborable.*

Note that the motivation of the paper came from Zhang's conjectures, but along the way, we have obtained some new results on equitable k-choosability of grids.

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