Filomat 32:18 (2018), 6167–6176
https://doi.org/10.2298/FIL1818167K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Leaping Cauchy numbers

Takao Komatsu[a]

[a] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072 China*

Abstract. We introduce leaping Cauchy numbers, that are generalizations of the analogous numbers to Euler numbers, as Cauchy numbers corresponds to Bernoulli numbers, in particular, in terms of determinant expressions. We also give several properties including sums of products.

The theory of determinants has been established and studied by many Mathematicians through the history. For example, T. Muir, who was remembered as an authority on determinants, studied and collected nearly all of the known facts about determinants up to the early 1930s. The basic facts, including permutations and combinations, general principles of simple determinants, compound determinants, co-factors, adjugates, rectangular arrays and matrices, and linear dependence can be found in [12]. His book the history of determinants [11], whose Volume 1 covered the origins to Leibniz in 1693 until 1840, was published in 1890. The remaining volumes were Volume 2 1840-1860 (1911), Volume 3 1860-1880 (1920), Volume 4 1880-1900 (1923), Volume 5 1900-1920 (1929). Muir was working on Volume 6 1920-1940 at the time of his death. Although many theories about determinants had been established, they have been unknown, ignored, or forgotten. Unfortunately, many Mathematicians has declared new results, though they had already been established. One can safely says that we have rediscovered the original results. However, it would be important to remember the first or the original results. Nowadays, it is really difficult to get the original papers. Nevertheless, we should or cite them through the definite sources including [2], [4] and [15] by way of [11].

In this paper by remembering the classical results, that have been often ignored, we shall introduce one kind of generalized numbers of the original Cauchy numbers, which are related to the Bernoulli numbers of the second kind, in particular, in terms of determinants.

The classical Cauchy numbers c_n ([3, p.293]) may be defined by the generating function:

$$\frac{x}{\log(1+x)}=\sum_{n=0}^\infty c_n\frac{x^n}{n!}\,. \tag{1}$$

The numbers $b_n=c_n/n!$ are sometimes called Bernoulli numbers of the second kind (e.g. [5, 13]).

2010 *Mathematics Subject Classification*. Primary 11B75; Secondary 11B37, 11B50, 11C20; 05A15; 05A19
Keywords. Cauchy numbers, leaping Cauchy numbers, recurrence, determinant, sums of products
Received: 11 October 2017; Accepted: 07 September 2018
Communicated by Dragan Djordjević
Email address: komatsu@whu.edu.cn (Takao Komatsu)

It is known that Cauchy numbers have a determinant expression ([4, p.50]):

$$c_n = n!\left|\begin{array}{ccccc} \frac{1}{2} & 1 & 0 & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{array}\right| . \tag{2}$$

On the other hand, Bernoulli numbers B_n, determined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

also have a determinant expression ([4, p.53]):

$$B_n = (-1)^n n!\left|\begin{array}{ccccc} \frac{1}{2!} & 1 & 0 & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{array}\right| . \tag{3}$$

If we choose only the factorials of even numbers, this is reduced to a determinant expression of Euler numbers (*cf.* [4, p.52]):

$$E_{2n} = (-1)^n (2n)!\left|\begin{array}{ccccc} \frac{1}{2!} & 1 & 0 & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{array}\right| . \tag{4}$$

Here, classical Euler numbers E_n are defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

One of the natural questions is what the number U_n is from Cauchy number c_n, as Euler number E_n from Bernoulli number B_n. Namely, what kind of number U_n is, expressing the following determinant expression?

$$U_{2n} = (2n)!\left|\begin{array}{ccccc} \frac{1}{2} & 1 & 0 & & \\ \frac{1}{4} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{2n-2} & \frac{1}{2n-4} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{2n} & \frac{1}{2n-2} & \cdots & \frac{1}{4} & \frac{1}{2} \end{array}\right| . \tag{5}$$

In this paper, we shall answer this natural question by introducing the leaping Cauchy numbers as one natural extension of the classical Cauchy numbers, in particular, in terms of determinanal expressions.

1. Definitions and preliminary properties

For a nonnegative integer m, let $F_m(x)$ be the partial summation of the logarithm function $\log(1+x)$, defined by

$$F_m(x)=\sum_{n=1}^{m}(-1)^{n-1}\frac{x^n}{n}$$

and $F_{\mathrm{odd},m}(x)$ be its odd version, defined by

$$F_{\mathrm{odd},m}(x)=\sum_{n=1}^{m}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\,.$$

Define the *leaping Cauchy numbers* $U_n^{(m)}$ by

$$\sum_{n=0}^{\infty}U_n^{(m)}\frac{x^n}{n!}=\left(1-\frac{(-1)^{\frac{m}{2}}\left(\log(1+x^2)-F_{\frac{m}{2}}(x^2)\right)}{2x^m}\right)^{-1}$$

if m is even, and

$$\sum_{n=0}^{\infty}U_n^{(m)}\frac{x^n}{n!}=\left(1+\frac{(-1)^{\frac{m-1}{2}}\left(\arctan x-F_{\mathrm{odd},\frac{m+1}{2}}(x)\right)}{x^m}\right)^{-1}$$

if m is odd.

In particular, we have

$$\sum_{n=0}^{\infty}U_n^{(0)}\frac{x^n}{n!}=\left(1-\frac{\log(1+x^2)}{2}\right)^{-1}$$

and

$$\sum_{n=0}^{\infty}U_n^{(1)}\frac{x^n}{n!}=\frac{x}{\arctan x}\,.$$

In [8, Theorem 1], different types of Cauchy numbers $c_{n,\le m}$ and $c_{n,\ge m}$ are defined in terms of the partial function $F_m(x)$:

$$\sum_{n=0}^{\infty}c_{n,\le m}\frac{x^n}{n!}=\frac{e^{F_m(x)-1}}{F_m(x)}$$

and

$$\sum_{n=0}^{\infty}c_{n,\ge m}\frac{x^n}{n!}=\frac{e^{\log(1+x)-F_{m-1}(x)}}{\log(1+x)-F_{m-1}(x)}\,,$$

so that $c_n=c_{n,\le\infty}=c_{n,\ge 1}$.

There is a recurrence relation among the leaping Cauchy numbers.

Lemma 1.1. *For integers $n\ge 1$ and $m\ge 0$,*

$$U_{2n}^{(m)}=\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}(2n)!}{(2n-2k+m)(2k)!}U_{2k}^{(m)}\,.$$

with $U_0^{(m)}=1$.

Proof. By the definition, we see that for any nonnegative integer m,

$$\sum_{n=0}^{\infty} U_n^{(m)} \frac{x^n}{n!} = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n+m}\right)^{-1} . \tag{6}$$

By this definition, $U_n^{(m)} = 0$ if n is odd. Therefore,

$$\begin{aligned} 1 &= \left(\sum_{n=0}^{\infty} U_{2n}^{(m)} \frac{x^{2n}}{(2n)!}\right)\left(1 + \sum_{l=1}^{\infty} \frac{(-1)^l x^{2l}}{2l+m}\right) \\ &= \sum_{n=0}^{\infty} U_{2n}^{(m)} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^{n-k} U_{2k}^{(m)}}{(2n-2k+m)(2k)!} x^{2n} . \end{aligned}$$

Comparing the coefficients on both sides, we have $U_0^{(m)} = 1$ and for $n \ge 1$

$$\frac{U_{2n}^{(m)}}{(2n)!} + \sum_{k=0}^{n-1} \frac{(-1)^{n-k} U_{2k}^{(m)}}{(2n-2k+m)(2k)!} = 0 .$$

□

The leaping Cauchy numbers have an explicit expression.

Theorem 1.2. *For integers $n \ge 1$ and $m \ge 0$,*

$$U_{2n}^{(m)} = (2n)! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{1}{(2i_1+m)\cdots(2i_k+m)} .$$

Proof. We shall prove that

$$U_{2n}^{(m)} = (2n)! \sum_{k=1}^{n} (-1)^{k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{(-1)^{i_1+\cdots+i_k}}{(2i_1+m)\cdots(2i_k+m)} . \tag{7}$$

When $n = 1$, by Lemma 1.1, we have

$$U_2^{(m)} = \frac{2}{m+2} U_0^{(m)} = \frac{2}{m+2} .$$

This matches the result of (7) for $n = 1$. Assume that (7) is valid up to $n-1$. Then by Lemma 1.1, we have

$$\begin{aligned} \frac{U_{2n}^{(m)}}{(2n)!} &= \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{(2n-2l+m)(2l)!} U_{2l}^{(m)} \\ &= \frac{(-1)^{n-1}}{2n+m} U_0^{(m)} + \sum_{l=1}^{n-1} \frac{(-1)^{n-l-1}}{2n-2l+m} \sum_{k=1}^{l} (-1)^k \sum_{\substack{i_1+\cdots+i_k=l \\ i_1,\dots,i_k \ge 1}} \frac{(-1)^{i_1+\cdots+i_k}}{(2i_1+m)\cdots(2i_k+m)} \\ &= \frac{(-1)^{n-1}}{2n+m} + \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{l=k}^{n-1} \frac{(-1)^{n-l}}{2n-2l+m} \sum_{\substack{i_1+\cdots+i_k=l \\ i_1,\dots,i_k \ge 1}} \frac{(-1)^{i_1+\cdots+i_k}}{(2i_1+m)\cdots(2i_k+m)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{n-1}}{2n+m} + \sum_{k=2}^{n}(-1)^k \sum_{l=k-1}^{n-1} \frac{(-1)^{n-l}}{2n-2l+m} \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_{k-1}\ge 1}} \frac{(-1)^{i_1+\cdots+i_{k-1}}}{(i_1+m)\cdots(i_{k-1}+m)}\\
&= \frac{(-1)^{n-1}}{2n+m} + \sum_{k=2}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{1}{(2i_1+m)\cdots(2i_k+m)} \quad (i_k = n-l \ge 1)\\
&= \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{(-1)^{i_1+\cdots+i_k}}{(2i_1+m)\cdots(2i_k+m)}\,.
\end{aligned}
$$

□

2. Deteminantal expressions

Leaping Cauchy numbers are naturally extended from the original Cauchy numbers in terms of determinant.

Theorem 2.1. *For integers $n \ge 1$ and $m \ge 0$,*

$$
U_{2n}^{(m)} = (2n)! \begin{vmatrix}
\frac{1}{m+2} & 1 & 0 & & \\
\frac{1}{m+4} & \frac{1}{m+2} & & & \\
\vdots & \vdots & \ddots & 1 & 0\\
\frac{1}{m+2n-2} & \frac{1}{m+2n-4} & \cdots & \frac{1}{m+2} & 1\\
\frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+4} & \frac{1}{m+2}
\end{vmatrix}. \tag{8}
$$

Remark 2.2. *When $m = 0$ in Theorem 2, the result is reduced to (5), where $U_{2n} = U_{2n}^{(0)}$. It is known ([7, 10]) that Euler numbers of the second kind $\widehat{E}_n$, defined by*

$$
\frac{x}{\sinh x} = \sum_{n=0}^{\infty} \widehat{E}_n \frac{x^n}{n!}\,,
$$

have the determinant expression

$$
\widehat{E}_{2n} = (-1)^n (2n)! \begin{vmatrix}
\frac{1}{3!} & 1 & 0 & & \\
\frac{1}{5!} & \frac{1}{3!} & 1 & & \\
\vdots & & \ddots & \ddots & 0\\
\frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & & \frac{1}{3!} & 1\\
\frac{1}{(2n+1)!} & \frac{1}{(2n-1)!} & \cdots & \frac{1}{5!} & \frac{1}{3!}
\end{vmatrix}.
$$

When $m = 1$ in Theorem 2, we have its analogous expression:

$$
U_{2n}^{(1)} = (2n)! \begin{vmatrix}
\frac{1}{3} & 1 & 0 & & \\
\frac{1}{5} & \frac{1}{3} & 1 & & \\
\vdots & & \ddots & \ddots & 0\\
\frac{1}{2n-1} & \frac{1}{2n-3} & & \frac{1}{3} & 1\\
\frac{1}{2n+1} & \frac{1}{2n-1} & \cdots & \frac{1}{5} & \frac{1}{3}
\end{vmatrix}.
$$

Proof.] For simplicity, put $\tilde{U}_n^{(m)} = U_n^{(m)}/n!$ and prove that

$$\tilde{U}_{2n}^{(m)} = \begin{vmatrix} \frac{1}{m+2} & 1 & 0 & & \\ \frac{1}{m+4} & \frac{1}{m+2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{m+2n-2} & \frac{1}{m+2n-4} & \cdots & \frac{1}{m+2} & 1 \\ \frac{1}{m+2n} & \frac{1}{m+2n-2} & \cdots & \frac{1}{m+4} & \frac{1}{m+2} \end{vmatrix} . \tag{9}$$

By Theorem 1.2, we see that

$$\tilde{U}_2^{(m)} = \frac{1}{m+2} .$$

Assume that (9) is valid up to $n-1$. By Lemma 1.1,

$$\tilde{U}_{2n}^{(m)} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{2n-2k+m} \tilde{U}_{2k}^{(m)}$$

with $\tilde{U}_0^{(m)} = 1$. Expanding at the first row of the right-hand side of (9), we have

$$\frac{\tilde{U}_{2n-2}^{(m)}}{m+2} - \begin{vmatrix} \frac{1}{m+4} & 1 & & & \\ \frac{1}{m+6} & \frac{1}{m+2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{m+2n-2} & \frac{1}{m+2n-6} & \cdots & \frac{1}{m+2} & 1 \\ \frac{1}{m+2n} & \frac{1}{m+2n-4} & \cdots & \frac{1}{m+4} & \frac{1}{m+2} \end{vmatrix} = \frac{\tilde{U}_{2n-2}^{(m)}}{m+2} - \frac{\tilde{U}_{2n-4}^{(m)}}{m+4} + \cdots + (-1)^{n-2} \begin{vmatrix} \frac{1}{m+2n-2} & 1 \\ \frac{1}{m+2n} & \frac{1}{m+2} \end{vmatrix}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{2n-2k+m} \tilde{U}_{2k}^{(m)} = \tilde{U}_{2n}^{(m)} .$$

□

3. Table of $U_n^{(m)}$

n	0	2	4	6	8	10	12
$U_n^{(0)}$	1	1	0	30	-840	60480	6153840
$U_n^{(1)}$	1	$\frac{2}{3}$	$-\frac{32}{15}$	$\frac{704}{21}$	$-\frac{54784}{45}$	$\frac{2610176}{33}$	$-\frac{10976325632}{1365}$
$U_n^{(2)}$	1	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{165}{4}$	$-\frac{2989}{2}$	$\frac{384615}{4}$	$-\frac{38732265}{4}$
$U_n^{(3)}$	1	$\frac{2}{5}$	$-\frac{432}{175}$	$\frac{7808}{175}$	$-\frac{16143872}{9625}$	$\frac{13685271552}{125125}$	$-\frac{4400505311232}{398125}$
$U_n^{(4)}$	1	$\frac{1}{3}$	$-\frac{7}{3}$	$\frac{136}{3}$	$-\frac{15974}{9}$	$\frac{355010}{3}$	$-\frac{36298042}{3}$
$U_n^{(5)}$	1	$\frac{2}{7}$	$-\frac{320}{147}$	$\frac{168640}{3773}$	$-\frac{800765440}{441441}$	$\frac{42494991360}{343343}$	$-\frac{1573960917340160}{122573451}$
$U_n^{(6)}$	1	$\frac{1}{4}$	$-\frac{81}{40}$	$\frac{1389}{32}$	$-\frac{290553}{160}$	$\frac{16202079}{128}$	$-\frac{17062159653}{1280}$

The following properties are easily seen.

Theorem 3.1. *For $m \ge 0$*

$$U_2^{(m)} = \frac{2}{m+2} .$$

The sequence of numerators of $U_{2n}^{(1)}/(2n)!$ ($n \ge 1$) is seen in [16, A216272], and the sequence of denominators of $U_{2n}^{(1)}/(2n)!$ ($n \ge 2$) is seen in [16, A195466]. We have an explicit expression:

$$\frac{U_{2n}^{(1)}}{(2n)!}=\frac{(-1)^{n+1}}{2n-1}\sum_{l=0}^{2n-1}\frac{2^{l+1}}{l+1}\binom{2n-1}{l}\sum_{k=0}^{l+1}k!\left\{ {l+1 \atop k} \right\}\left[{l+k \atop l} \right]\frac{1}{(l+k)!}\quad (n\ge 1)\,,$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind given by

$$\left\{ {n \atop k} \right\}=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

and $\left[{n \atop k} \right]$ denotes the (unsigned) Stirling numbers of the first kind arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n}\left[{n \atop k} \right]x^k\,.$$

4. Applications by the Trudi's formula

We shall use the Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $B_n^{(m)}$.

Lemma 4.1. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-a_0)^{n-t_1-\cdots-t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [11, Vol.3, p.214],[15] and the case $a_0=1$ of this formula is known as Brioschi's formula [2],[11, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g. [9]), which is based upon the relation:

$$\sum_{k=0}^{n}(-1)^{n-k}\alpha_k D(n-k)=0\quad (n\ge 1)\,.$$

Lemma 4.2. *If $\{\alpha_n\}_{n\ge 0}$ is a sequence defined by $\alpha_0=1$ and*

$$\alpha_n=\begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix},\ \textit{then } D(n)=\begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}\,.$$

By applying these lemmata to Theorem 8, we obtain an explicit expression for leaping Cauchy numbers.

Theorem 4.3. *For $n \ge m \ge 1$, we have*

$$U_{2n}^{(m)} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} \times (-1)^{n-t_1-\cdots-t_n} \left(\frac{1}{m+2}\right)^{t_1} \cdots \left(\frac{1}{m+2n}\right)^{t_n}\,.$$

By applying the inversion relation in Lemma 4.2 to Theorem 8, we have the following.

Theorem 4.4. *For $n \ge 1$, we have*

$$\frac{1}{m+2n} = \begin{vmatrix} \frac{U_2^{(m)}}{2!} & 1 & 0 & & \\ \frac{U_4^{(m)}}{4!} & \frac{U_2^{(m)}}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{U_{2n-2}^{(m)}}{(2n-2)!} & \frac{U_{2n-4}^{(m)}}{(2n-4)!} & \cdots & \frac{U_2^{(m)}}{2!} & 1 \\ \frac{U_{2n}^{(m)}}{(2n)!} & \frac{U_{2n-2}^{(m)}}{(2n-2)!} & \cdots & \frac{U_4^{(m)}}{4!} & \frac{U_2^{(m)}}{2!} \end{vmatrix}\,.$$

5. Convolution identities

It is known ([17, Theorem 2.1]) that Cauchy numbers c_n satisfy the identity

$$\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = -(n-1)c_n - n(n-2)c_{n-1} \quad (n \ge 0)\,.$$

Leaping Cauchy numbers satisfy the following relations.

Theorem 5.1. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k} U_k^{(0)} U_{n-k}^{(0)} = \frac{U_{n+2}^{(0)}}{n+1} + nU_n^{(0)}\,, \tag{10}$$

$$\sum_{k=0}^{n} \binom{n}{k} U_k^{(1)} U_{n-k}^{(1)} = -(n-1)U_n^{(1)} - n(n-1)(n-3)U_{n-2}^{(1)} \tag{11}$$

and for $m \ge 2$

$$\sum_{k=0}^{n} \binom{n}{k} U_k^{(m)} U_{n-k}^{(m)} = \frac{m-n}{m-1} U_n^{(m)} + \frac{1}{m(m-1)} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^l \left(\frac{m-1}{m}\right)^l \frac{n!(n-2l-m)}{(n-2l)!} U_{n-2l}^{(m)}\,. \tag{12}$$

Remark 5.2. *When n is odd, both sides of any of (10), (11) and (12) are equal to 0.*

Proof. Put

$$u(x)=\left(1+\sum_{n=1}^\infty\frac{(-1)^n x^{2n}}{2n+m}\right)^{-1}.$$

We have

$$\begin{aligned}u'(x)&=-u^2(x)\sum_{n=1}^\infty\frac{(-1)^n 2n x^{2n-1}}{2n+m}\\&=-u^2(x)\left(\sum_{n=1}^\infty(-1)^n x^{2n-1}-\frac{m}{x}\sum_{n=1}^\infty\frac{(-1)^n x^{2n}}{2n+m}\right)\\&=-u^2(x)\left(\frac{-x}{1+x^2}-\frac{m}{x}\left(\frac{1}{u(x)}-1\right)\right)\\&=u^2(x)\left(\frac{x}{1+x^2}-\frac{m}{x}\right)+\frac{m}{x}u(x).\end{aligned}$$

If $m=0$, we have

$$u'(x)=u^2(x)\frac{x}{1+x^2}$$

or

$$u^2(x)=\frac{1}{x}u'(x)+xu'(x).$$

Hence, we get

$$\begin{aligned}\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}U_k^{(0)}U_{n-k}^{(0)}\frac{x^n}{n!}&=\frac{1}{x}\sum_{n=0}^\infty U_{n+1}^{(0)}\frac{x^n}{n!}+x\sum_{n=0}^\infty U_{n+1}^{(0)}\frac{x^n}{n!}\\&=\sum_{n=0}^\infty\frac{U_{n+2}^{(0)}}{n+1}\frac{x^n}{n!}+\sum_{n=0}^\infty nU_n^{(0)}\frac{x^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we obtain the identity (10).

If $m=1$, we have

$$u'(x)=-\frac{u^2(x)}{x(1+x^2)}+\frac{u(x)}{x}$$

or

$$u^2(x)=(1+x^2)u(x)-x(1+x^2)u'(x).$$

Hence, we get

$$\begin{aligned}\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}U_k^{(1)}U_{n-k}^{(1)}\frac{x^n}{n!}&=\sum_{n=0}^\infty U_n^{(1)}\frac{x^n}{n!}+x^2\sum_{n=0}^\infty U_n^{(1)}\frac{x^n}{n!}-x\sum_{n=0}^\infty U_{n+1}^{(1)}\frac{x^n}{n!}-x^3\sum_{n=1}^\infty U_{n+1}^{(1)}\frac{x^n}{n!}\\&=\sum_{n=0}^\infty U_n^{(1)}\frac{x^n}{n!}+\sum_{n=0}^\infty\frac{n!}{(n-2)!}U_{n-2}^{(1)}\frac{x^n}{n!}-\sum_{n=0}^\infty nU_n^{(1)}\frac{x^n}{n!}-\sum_{n=1}^\infty\frac{n!}{(n-3)!}U_{n-2}^{(1)}\frac{x^n}{n!}\\&=-\sum_{n=0}^\infty(n-1)U_n^{(1)}\frac{x^n}{n!}-\sum_{n=0}^\infty n(n-1)(n-3)U_{n-2}^{(1)}\frac{x^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we obtain the identity (11).

If $m\ge 2$, we have

$$u^2(x)=\frac{m(1+x^2)}{m+(m-1)x^2}u(x)-\frac{x(1+x^2)}{m+(m-1)x^2}u'(x).$$

The first term on the right-hand side is equal to

$$
\begin{aligned}
\frac{m}{m-1}\left(1-\frac{1}{m}\frac{1}{1+\frac{m-1}{m}x^2}\right)u(x)&=\frac{m}{m-1}u(x)-\frac{1}{m-1}\sum_{l=0}^\infty(-1)^l\left(\frac{m-1}{m}\right)^l x^{2l}u(x)\\
&=\frac{m}{m-1}\sum_{n=0}^\infty U_n^{(m)}\frac{x^n}{n!}-\frac{1}{m-1}\sum_{l=0}^\infty(-1)^l\left(\frac{m-1}{m}\right)^l\sum_{n=0}^\infty\frac{(n+2l)!}{n!}U_n^{(m)}\frac{x^{n+2l}}{(n+2l)!}\\
&=\frac{m}{m-1}\sum_{n=0}^\infty U_n^{(m)}\frac{x^n}{n!}-\frac{1}{m-1}\sum_{n=0}^\infty\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^l\left(\frac{m-1}{m}\right)^l\frac{n!}{(n-2l)!}U_{n-2l}^{(m)}\frac{x^n}{n!}\,.
\end{aligned}
$$

The second term on the right-hand side is equal to

$$
\begin{aligned}
&-\frac{x}{m-1}\left(1-\frac{1}{m}\frac{1}{1+\frac{m-1}{m}x^2}\right)u'(x)\\
&=-\frac{x}{m-1}\sum_{n=0}^\infty U_{n+1}^{(m)}\frac{x^n}{n!}+\frac{1}{m(m-1)}\sum_{l=0}^\infty(-1)^l\left(\frac{m-1}{m}\right)^l\sum_{n=0}^\infty\frac{(n+2l+1)!}{n!}U_{n+1}^{(m)}\frac{x^{n+2l+1}}{(n+2l+1)!}\\
&=-\frac{1}{m-1}\sum_{n=0}^\infty nU_n^{(m)}\frac{x^n}{n!}+\frac{1}{m(m-1)}\sum_{n=0}^\infty\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^l\left(\frac{m-1}{m}\right)^l\frac{n!}{(n-2l-1)!}U_{n-2l}^{(m)}\frac{x^n}{n!}\,.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}U_k^{(m)}U_{n-k}^{(m)}\frac{x^n}{n!}\\
&=\frac{1}{m-1}\sum_{n=0}^\infty(m-n)U_n\frac{x^n}{n!}+\frac{1}{m(m-1)}\sum_{n=0}^\infty\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^l\left(\frac{m-1}{m}\right)^l\frac{n!}{(n-2l)!}(n-2l-m)U_{n-2l}\frac{x^n}{n!}\,.
\end{aligned}
$$

Comparing the coefficients on both sides, we obtain the identity (12). □

References

[1] M. Aoki, T. Komatsu, Remarks on hypergeometric Cauchy numbers, Math. Rep. (Bucur.) (to appear). arXiv:1802.05455
[2] F. Brioschi, Sulle funzioni Bernoulliane ed Euleriane, Annali de Mat., i. (1858), . 260–263; Opere Mat., i. pp. 343–347.
[3] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.
[4] J. W. L. Glaisher, Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants, Messenger (2) 6 (1875), 49–63.
[5] Ch. Jordan, Sur des polynomes analogues aux polynomes de Bernoulli et sur des formules de sommation analogues a celle de MacLaurin-Euler, Acta Sci. Math. (Szeged), 4 (1928–29), 130–150.
[6] T. Komatsu, Incomplete poly-Cauchy numbers, Monatsh. Math. 180 (2016), 271–288.
[7] T. Komatsu, Complementary Euler numbers, Period. Math. Hungar. 75 (2017), 302–314.
[8] T. Komatsu, I. Mező, L. Szalay, Incomplete Cauchy numbers, Acta Math. Hungar. 149 (2016), 306–323.
[9] T. Komatsu, J. L. Ramirez, Some determinants involving incomplete Fubini numbers, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 26 (2018) no.3 (to appear). arXiv:1802.06188
[10] T. Komatsu, H. Zhu, Hypergeometric Euler numbers, Chinese J. Contemp. Math. (to appear). arXiv:1612.06210.
[11] T. Muir, The theory of determinants in the historical order of development, Four volumes, Dover Publications, New York, 1960.
[12] T. Muir, A treatise on the theory of determinants, Revised and enlarged by William H. Metzler, Dover Publications, Inc., New York 1960.
[13] N. E. Nörlund, Vorlesungen über Differenzenrechnung, Berlin, Springer, 1924.
[14] G. Polya, Induction and analogy in mathematics, Mathematics and plausible reasoning, vol. I. Princeton University Press, Princeton, N. J., 1954.
[15] N. Trudi, Intorno ad alcune formole di sviluppo, Rendic. dell' Accad. Napoli, (1862), 135–143.
[16] N. J. A. Sloane, The on-line encyclopedia of integer sequences, available at oeis.org. (2018).
[17] F.-Z. Zhao, Sums of products of Cauchy numbers, Discrete Math. 309 (2009) 3830–3842.