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Expressions for the Drazin Inverse of a Modified Matrix

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Abstract. We present some new expressions for the Drazin inverse of a modified matrix $A-CD^DB$ in terms of the Drazin inverse of A and its generalized Shur's complement $D-BA^DC$ under weaker conditions. Some results in recent literature are unified and generalized, and most importantly, we obtain new expressions of generalized Sherman-Morrison-Woodbury formula under weaker restrictions.

1. Introduction

Throughout this paper, let $\mathbb{C}^{n\times m}$ denote the set of all $n\times m$ complex matrices. For $A\in\mathbb{C}^{n\times n}$, the Drazin inverse of A is the unique matrix $A^D\in\mathbb{C}^{n\times n}$ such that

$$A^{k+1}A^D=A^k,\quad A^DAA^D=A^D,\quad AA^D=A^DA,$$

where k is the least non-negative integer such that $rank(A^k)=rank(A^{k+1})$, called index of A and denoted by $ind(A)$. Denote by $A^e=AA^D$ and $A^\pi=I-A^e$, where I denotes the identity matrix of proper size. When $ind(A)=1$, $A^D=A^\#$ is called the group inverse of A. If $ind(A)=0$, then $A^D=A^{-1}$.

If A and D are invertible matrices and B and C are matrices of appropriate size such that $D-BA^{-1}C$ and $A-CD^{-1}B$ are invertible, then original Sherman-Morrison-Woodbury (from now on SMW) formula gives explicit expression for the inverse of a modified matrix $A-CD^{-1}B$ of A in terms of its Schur's complement $D-BA^{-1}C$ [11, 13]. Precisely, SMW formula is expressed as

$$(A-CD^{-1}B)^{-1}=A^{-1}+A^{-1}C(D-BA^{-1}C)^{-1}BA^{-1}.$$

This formula has a lot of applications in statistics, networks, optimization and partial differential equations (see [5, 6, 8]).

Main objective of this article is to study the Drazin inverse of a modified matrix $A-CD^DB$ in terms of the Drazin inverse of A and the Drazin inverse of its generalized Schur complement, since the Drazin inverse has many applications in numerical analysis, singular differential or difference equations, Markov chains, cryptography, etc. Some of mentioned applications can be found in [1, 2].

Under some assumptions, Wei [12] gave representations of the Drazin inverse of a modified matrix $A-CB$ (in this case $D=I$). His results were generalized in [3, 9, 10].

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Zhang and Du [4] relaxed and removed some assumptions of theorems proved in [3, 9, 10, 12] and presented formulae for $(A - CD^DB)^D$ under weaker conditions.

In [4, Lemma 2.2], which is extensively used through [4], Zhang and Du made an assumption $A^\pi J = 0$. We relax this condition and our conditions yield four different versions of [4, Lemma 2.2], four different versions of [4, Theorem 2.5] and its corollaries. Thus, new expressions for $(A - CD^DB)^D$ and generalized SMW formula are obtained under weaker conditions.

2. Main results

In this section, let $A \in \mathbb{C}^{n\times n}$, $D \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times m}$. For simplicity, we introduce following denotations:

$$H = BA^D, \quad K = A^DC, \quad J = CD^DB, \quad L = BA^DC,$$
$$S = A - J, \quad Z = D - L, \quad S_A = A^eSA^e \quad \text{and} \quad Z_D = D^eZD^e.$$

We will investigate the Drazin inverse of a modified matrix $S = A - CD^DB$ in terms of the Drazin inverse of A and the Drazin inverse of its generalized Schur's complement Z. For every square matrix A, we define $A^0 = I$. If the lower limit of a sum is greater than its upper limit, we define the sum to be 0, i.e. $\sum_{i=0}^{k} * = 0$ when $k < 0$.

Let us first introduce an auxiliary result.

Lemma 2.1. [7, Theorem 2.1] *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ = 0$ then*

$$(P + Q)^D = Q^\pi \sum_{i=0}^{t-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{s-1} (Q^D)^{i+1} P^i P^\pi,$$

where $s = ind(P)$ and $t = ind(Q)$.

Now, we generalize [4, Lemma 2.2].

Lemma 2.2. *If $A^\pi JA^\pi = 0$ and if*

(a) $SA^\pi J = 0$, *then*

$$S^D = (I - A^\pi JS_A^D)S_A^D + \sum_{i=0}^{k-1} ((I - A^\pi JS_A^D)S_A^D)^{i+2} SA^iA^\pi;$$

(b) $SJA^\pi = 0$, *then*

$$S^D = (I - A^\pi JS_A^D)S_A^D + \sum_{i=0}^{k-1} SA^iA^\pi((I - A^\pi JS_A^D)S_A^D)^{i+2};$$

(c) $JA^\pi S = 0$, *then*

$$S^D = S_A^D(I - S_A^D JA^\pi) + \sum_{i=0}^{k-1} A^iA^\pi S(S_A^D(I - S_A^D JA^\pi))^{i+2};$$

(d) $A^\pi JS = 0$, *then*

$$S^D = S_A^D(I - S_A^D JA^\pi) + \sum_{i=0}^{k-1} (S_A^D(I - S_A^D JA^\pi))^{i+2} A^iA^\pi S;$$

where $k = ind(A)$.

Proof. (a) The hypothesis $A^\pi JA^\pi = 0$ implies

$$A^\pi SA^\pi = AA^\pi \quad \text{and} \quad A^\pi J = A^\pi JA^e. \tag{1}$$

Now we compute

$$SA^e = (A^e + A^\pi)SA^e = S_A + A^\pi SA^e = S_A - A^\pi J \quad \text{and} \quad - S_A A^\pi J = 0.$$

Because $A^\pi JA^\pi = 0$, we know that $-A^\pi J$ is a nilpotent matrix and thus $(-A^\pi J)^D = 0$ and $(-A^\pi J)^\pi = I$. Since $ind(-A^\pi J) \leq 2$, by Lemma 2.1,

$$(SA^e)^D = (I - A^\pi JS_A^D)S_A^D. \tag{2}$$

Obviously, $S = SA^\pi + SA^e$ and $SA^\pi SA^e = 0$. Using (1), we have

$$(SA^\pi)^i = S(A^\pi SA^\pi)^{i-1} = S(AA^\pi)^{i-1} = SA^{i-1}A^\pi, \quad i \in \mathbb{N}. \tag{3}$$

Since k is the least non-negative integer such that $A^k A^\pi = 0$ holds, SA^π is a nilpotent matrix and therefore $(SA^\pi)^D = 0$ and $(SA^\pi)^\pi = I$. For all $i > k$, $(SA^\pi)^i = 0$ and, from Lemma 2.1, (2) and (3), we obtain desired result

$$\begin{aligned} S^D &= \sum_{i=0}^{k}((SA^e)^D)^{i+1}(SA^\pi)^i = (SA^e)^D + \sum_{i=0}^{k-1}((SA^e)^D)^{i+2}(SA^\pi)^{i+1} \\ &= (I - A^\pi JS_A^D)S_A^D + \sum_{i=0}^{k-1}((I - A^\pi JS_A^D)S_A^D)^{i+2}SA^iA^\pi. \end{aligned}$$

(b) Similarly like in (a), by

$$JA^\pi = A^eJA^\pi, \quad S = SA^e + SA^\pi, \quad SA^eSA^\pi = 0.$$

(c) This part follows using

$$S = A^eS + A^\pi S, \quad A^eSA^\pi S = 0.$$

(d) We notice

$$S = A^\pi S + A^eS, \quad A^\pi SA^eS = 0.$$

□

Parts (a) and (d) of Lemma 2.2 generalize [4, Lemma 2.2], while Lemma 2.2(a) yields the same result when $A^\pi J = 0$.

In the following example, we present complex matrices A, B, C and D such that for them [4, Lemma 2.2] is not applicable, yet Lemma 2.2(a) is.

Example 2.3. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A^\# = A$, $D^\# = D$,

$$A^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = A - J = \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

give $A^\pi J = J \neq 0$, $A^\pi J A^\pi = 0$ and $SA^\pi J = 0$. Hence, we can not apply [4, Lemma 2.2], but using Lemma 2.2(a), we get

$$S^D = \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

△

We state now a result from [4].

Lemma 2.4. [4, Lemma 2.4] *Let $M = A^D + KZ^DH$. Then following statements are equivalent:*

(a) $KD^\pi Z^D H = KD^D Z^\pi H$;

(b) $S_A M = A^e$;

(c) $MS_A = A^e$;

(d) $KZ^\pi D^D H = KZ^D D^\pi H$.

Furthermore, if one of (a)–(d) *holds, then S_A has the group inverse*

$$S_A^\# = A^D + KZ^DH.$$

Using Lemma 2.2 and Lemma 2.4, we obtain the following result.

Theorem 2.5. *If $A^\pi J A^\pi = 0$, $KD^\pi Z^D H = KD^D Z^\pi H$ and if*

(a) $SA^\pi J = 0$, *then*

$$S^D = (I - A^\pi J(A^D + KZ^DH))(A^D + KZ^DH) + \sum_{i=0}^{k-1} \Big((I - A^\pi J(A^D + KZ^DH))(A^D + KZ^DH)\Big)^{i+2} SA^iA^\pi,$$

or alternatively

$$\begin{aligned} S^D = {} & (I - A^\pi J(A^D + A^DCZ^DBA^D))(A^D + A^DCZ^DBA^D) \\ & - \sum_{i=0}^{k-1} \Big[\Big((I - A^\pi J(A^D + A^DCZ^DBA^D))(A^D + A^DCZ^DBA^D)\Big)^{i+1} \\ & \qquad \times (I - A^\pi J(A^D + A^DCZ^DBA^D))A^DCZ^DBA^iA^\pi\Big] \\ & + \sum_{i=0}^{k-1} \Big[\Big((I - A^\pi J(A^D + A^DCZ^DBA^D))(A^D + A^DCZ^DBA^D)\Big)^{i+1} \\ & \qquad \times (I - A^\pi J(A^D + A^DCZ^DBA^D))A^DC(Z^DD^\pi - Z^\pi D^D)BA^i\Big]; \end{aligned} \tag{4}$$

(b) $SJA^\pi = 0$, *then*

$$S^D = (I - A^\pi J(A^D + KZ^DH))(A^D + KZ^DH) + \sum_{i=0}^{k-1} SA^iA^\pi\Big((I - A^\pi J(A^D + KZ^DH))(A^D + KZ^DH)\Big)^{i+2};$$

(c) $JA^{\pi}S = 0$, *then*

$$S^D = (A^D + KZ^D H)(I - (A^D + KZ^D H)JA^{\pi}) + \sum_{i=0}^{k-1} A^i A^{\pi} S\Big((A^D + KZ^D H)(I - (A^D + KZ^D H)JA^{\pi})\Big)^{i+2},$$

or alternatively

$$\begin{aligned} S^D &= (A^D + A^D CZ^D BA^D)(I - (A^D + A^D CZ^D BA^D)JA^{\pi}) \\ &- \sum_{i=0}^{k-1} \Big[A^{\pi} A^i CZ^D BA^D (I - (A^D + A^D CZ^D BA^D)JA^{\pi}) \\ &\quad \times \Big((A^D + A^D CZ^D BA^D)(I - (A^D + A^D CZ^D BA^D)JA^{\pi})\Big)^{i+1}\Big] \\ &+ \sum_{i=0}^{k-1} \Big[A^i C(D^{\pi} Z^D - D^D Z^{\pi})BA^D (I - (A^D + A^D CZ^D BA^D)JA^{\pi}) \\ &\quad \times \Big((A^D + A^D CZ^D BA^D)(I - (A^D + A^D CZ^D BA^D)JA^{\pi})\Big)^{i+1}\Big]; \end{aligned} \tag{5}$$

(d) $A^{\pi}JS = 0$, *then*

$$S^D = (A^D + KZ^D H)(I - (A^D + KZ^D H)JA^{\pi}) + \sum_{i=0}^{k-1} \Big((A^D + KZ^D H)(I - (A^D + KZ^D H)JA^{\pi})\Big)^{i+2} A^i A^{\pi} S;$$

where $k = ind(A)$.

Proof. (a) This proof is nearly identical to the one in [4, Theorem 2.5]. We provide it here because of completeness. Combining Lemma 2.2 and Lemma 2.4, we obtain the first equality. The second equality follows from the first one by noticing:

$$\begin{aligned} (A^D + KZ^D H)SA^{\pi} &= (A^e - KD^D B + KZ^D BA^e - KZ^D (D - Z)D^D B)A^{\pi} \\ &= -KZ^{\pi} D^D BA^{\pi} - KZ^D (I - D^{\pi})BA^{\pi} \\ &= K(Z^D D^{\pi} - Z^{\pi} D^D)BA^{\pi} - KZ^D BA^{\pi}; \end{aligned} \tag{6}$$

By $KD^{\pi}Z^D H = KD^D Z^{\pi} H$ and Lemma 2.4, we get

$$\begin{aligned} K(Z^D D^{\pi} - Z^{\pi} D^D)BA^{\pi} &= K(Z^D D^{\pi} - Z^{\pi} D^D)B - K(Z^D D^{\pi} - Z^{\pi} D^D)HA \\ &= K(Z^D D^{\pi} - Z^{\pi} D^D)B. \end{aligned}$$

(c) We have in mind parts (a) and (d) of Lemma 2.4 and derive an equality symmetric to (6).
We can similarly verify parts (b) and (d). □

Equality (4) of Theorem 2.5(a) can be also modified in the following way.

Remark 2.6. Using

$$\begin{aligned} (A^D + KZ^D H)SA^{\pi} &= K(Z^D D^{\pi} - Z^{\pi} D^D)BA^{\pi} - KZ^D BA^{\pi} \\ &= -K(Z^D D + Z^{\pi})D^D BA^{\pi}, \end{aligned}$$

we get one more expression for S^D:

$$S^D = (I - A^\pi J(A^D + A^D C Z^D B A^D))(A^D + A^D C Z^D B A^D)$$
$$- \sum_{i=0}^{k-1} \Big[\big((I - A^\pi J(A^D + A^D C Z^D B A^D))(A^D + A^D C Z^D B A^D) \big)^{i+1}$$
$$\times (I - A^\pi J(A^D + A^D C Z^D B A^D)) A^D C (Z^D D + Z^\pi) D^D B A^i A^\pi \Big].$$

In [4, Theorem 2.5 and Theorem 2.9], to give representations of the Drazin inverse of S, Zhang and Du assumed

(I) $A^\pi J = 0$ and $K D^\pi Z^D H = K D^D Z^\pi H$;
(II) $J A^\pi = 0$ and $K Z^\pi D^D H = K Z^D D^\pi H$.

Notice that Theorem 2.5(a,d) relaxes the first condition in (I). When $A^\pi J = 0$, Theorem 2.5(a) yields the same result as conditions in (I), while Theorem 2.5(d) provides entirely new expression for S^D regardless. Similarly, Theorem 2.5(b,c) relaxes the first condition in (II).

We now state another auxiliary result.

Lemma 2.7. [4, Lemma 3.2] *If $D^D B A A^D = D^D D B A^D$, then*

$$S_A^D = A^D + A^D C Z_D^D D^D B A A^D - \sum_{i=0}^{s-1} (A^D)^{i+2} C D D^D Z_D^i Z_D^\pi D^D B A A^D, \tag{7}$$

where $s = ind(Z_D)$.

Using Lemma 2.2 and Lemma 2.7, we obtain the next consequence.

Corollary 2.8. *If $D^D B A A^D = D^D D B A^D$ and $A^\pi J A^\pi = 0$, then statements* (a)–(d) *of Lemma 2.2 hold, where S_A^D is represented by* (7).

Analogously to Lemma 2.2, we have following results.

Lemma 2.9. *If $D^\pi L D^\pi = 0$ and if*

(a) $Z D^\pi L = 0$, *then*

$$Z^D = (I - D^\pi L Z_D^D) Z_D^D + \sum_{i=0}^{l-1} ((I - D^\pi L Z_D^D) Z_D^D)^{i+2} Z D^i D^\pi; \tag{8}$$

(b) $Z L D^\pi = 0$, *then*

$$Z^D = (I - D^\pi L Z_D^D) Z_D^D + \sum_{i=0}^{l-1} Z D^i D^\pi ((I - D^\pi L Z_D^D) Z_D^D)^{i+2};$$

(c) $L D^\pi Z = 0$, *then*

$$Z^D = Z_D^D (I - Z_D^D L D^\pi) + \sum_{i=0}^{l-1} D^i D^\pi Z (Z_D^D (I - Z_D^D L D^\pi))^{i+2};$$

(d) $D^\pi L Z = 0$, *then*

$$Z^D = Z_D^D (I - Z_D^D L D^\pi) + \sum_{i=0}^{l-1} (Z_D^D (I - Z_D^D L D^\pi))^{i+2} D^i D^\pi Z;$$

where $l = ind(D)$.

We now express S_A^D in terms of Z^D.

Lemma 2.10. *If* $D^DBAA^D = D^DDBA^D$, $D^\pi LD^\pi = 0$ *and* ($ZD^\pi L = 0$ *or* $LD^\pi Z = 0$), *then*

$$S_A^D = A^D + A^DCD^eZ^DD^DBAA^D - \sum_{i=0}^{s-1}(A^D)^{i+2}CDD^D(Z + D^\pi L)^i(I - D^eZD^eZ^D)D^DBAA^D, \tag{9}$$

where $s = ind(D^eZD^e)$.

Proof. Assume that $ZD^\pi L = 0$. Obviously, equalities (7) and (8) hold. Since $Z_D = (Z + D^\pi L)D^e$, we easily deduce that $Z_D^i = (Z + D^\pi L)^iD^e$ for all $i \in \mathbb{N}$. Notice that $D^eZ_D^D = Z_D^D$ and $Z_D^DD^e = Z_D^D$. Using (8), we have $Z^DD^e = (I - D^\pi LZ^D)Z_D^D$, and then, $D^eZ^DD^e = Z_D^D$. Also, note that $D^eZ_D^\pi D^D = (I - Z_DZ^D)D^D$. We now obtain desired result by substituting Z_D^D, Z_D^i and Z_D^π in (7).

When $LD^\pi Z = 0$, we apply Lemma 2.9(c) instead of Lemma 2.9(a). □

If we combine Lemma 2.2 and Lemma 2.10, we can get eight different expressions for S^D in terms of Z^D. We give for example the following result.

Corollary 2.11. *If* $D^DBAA^D = D^DDBA^D$, $A^\pi JA^\pi = 0$, $D^\pi LD^\pi = 0$ *and* ($ZD^\pi L = 0$ *or* $LD^\pi Z = 0$), *then statements* (a)–(d) *of Lemma 2.2 hold, where* S_A^D *is represented by* (9).

3. Special cases

Several special cases of Theorem 2.5, which generalize some recent results, will be presented in this section. The first consequence of Theorem 2.5 recovers [4, Corollary 2.7].

Corollary 3.1. *If* $A^\pi JA^\pi = 0$ *and* $CD^\pi Z^DB = 0 = CD^DZ^\pi B$, *then statements* (a)–(d) *of Theorem 2.5 hold.*

Since $D^\pi Z^D = 0$ and $D^DZ^\pi = 0$ iff $D^\pi = Z^\pi$, we obtain the following consequence of Theorem 2.5.

Corollary 3.2. *If* $A^\pi JA^\pi = 0$, $D^\pi = Z^\pi$ *and if*

(a) $SA^\pi J = 0$, *then*

$$\begin{aligned} S^D &= (I - A^\pi J(A^D + A^DCZ^DBA^D))(A^D + A^DCZ^DBA^D) \\ &\quad - \sum_{i=0}^{k-1}\Big[\Big((I - A^\pi J(A^D + A^DCZ^DBA^D))(A^D + A^DCZ^DBA^D)\Big)^{i+1} \\ &\qquad \times (I - A^\pi J(A^D + A^DCZ^DBA^D))A^DCZ^DBA^iA^\pi\Big]. \end{aligned}$$

(c) $JA^\pi S = 0$, *then*

$$\begin{aligned} S^D &= (A^D + A^DCZ^DBA^D)(I - (A^D + A^DCZ^DBA^D)JA^\pi) \\ &\quad - \sum_{i=0}^{k-1}\Big[A^\pi A^iCZ^DBA^D(I - (A^D + A^DCZ^DBA^D)JA^\pi) \\ &\qquad \times \Big((A^D + A^DCZ^DBA^D)(I - (A^D + A^DCZ^DBA^D)JA^\pi)\Big)^{i+1}\Big]. \end{aligned}$$

The Drazin inverse of $A - CD^DB$ is expressed under some conditions, which are listed partially as follows:

(a) $A^\pi C = 0$, $CD^\pi = 0$, $Z^\pi B = 0$, $D^\pi B = 0$ and $CZ^\pi = 0$ (see [3, Theorem 2.1]);
(b) $A^\pi C = 0$, $CD^\pi Z^DB = 0$, $CD^DZ^\pi B = 0$, $CZ^\pi D^DB = 0$ and $CZ^DD^\pi B = 0$ (see [10, Theorem 2.1]);

(c) $A^\pi C = CD^\pi$, $D^\pi B = 0$ and $DZ^\pi = 0$ (see [9, Theorem 2.1]);

(d) $A^\pi C = CD^\pi$, $D^\pi B = 0$, $Z^\pi B = 0$ and $CZ^\pi = 0$ (see [9, Theorem 2.2]).

Hence, Corollary 3.1(a,d) is an unified generalization of [3, Theorem 2.1] and [10, Theorem 2.1], because Corollary 3.1(a,d) relaxes the first condition and drops the last two in each item (a)–(c). Also, Corollary 3.1(a,d) is an extension of [9, Theorem 2.1 and Theorem 2.2], for details see [4]. In the case that $D = I$, Corollary 3.1(a,d) recovers [12, Theorem 2.1].

Similarly, observe that Corollary 3.1(b,c) generalizes [3, Theorem 2.2], [10, Theorem 2.2], [9, Theorem 2.3] and [12, Theorem 2.4].

If we assume that A, D and Z are invertible in Corollary 3.1, then $ind(A) = 0$, $A^\pi = 0$, $D^\pi = 0$, $Z^\pi = 0$, S is invertible and so Corollary 3.1 is exactly well-known SMW formula.

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