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A Modified S-type Eigenvalue Localization Set of Tensors Applications

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Abstract. Based on the S-type eigenvalue localization set developed by Li et al. (Linear Algebra Appl. 493 (2016) 469-483) for tensors, a modified S-type eigenvalue localization set for tensors is established in this paper by excluding some sets from the existing S-type eigenvalue localization set developed by Huang et al. (arXiv: 1602.07568v1, 2016). The proposed set containing all eigenvalues of tensors is much sharper compared with that employed by Li et al. and Huang et al. As its applications, a criteria, which can be utilized for identifying the nonsingularity of tensors, is developed. In addition, we provide new upper and lower bounds for the spectral radius of nonnegative tensors and the minimum H-eigenvalue of weakly irreducible strong M-tensors. These bounds are superior to some previous results, which is illustrated by some numerical examples.

1. Introduction

Let n be a positive integer with $n \geq 2$, and $N = \{1, 2, \dots, n\}$. $\mathbb{C}(\mathbb{R})$ stands for the set of all complex (real) numbers. $\mathcal{A} = (a_{i_1 \dots i_m})$ called a complex (real) tensor of order m dimension n, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if $a_{i_1 \dots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j \in N$ for $j = 1, 2, \dots, m$ [19].

The tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called the unit tensor [21], denoted by $\mathcal{I}$, if its entries $\delta_{i_1 \dots i_m} (i_1, \dots, i_m \in N)$ satisfy the following conditions:

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$, $i, j \in N$, $j \neq i$ and a nonempty proper subset S of N, we denote

$$\Delta^N = \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in N \text{ for } j = 2, 3, \dots, m\},$$

$$\Delta^S = \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in S \text{ for } j = 2, 3, \dots, m\}, \ \overline{\Delta^S} = \Delta^N \backslash \Delta^S$$

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and

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m},\ R_{\max}(\mathcal{A}) = \max_{i\in N} R_i(\mathcal{A}),\ R_{\min}(\mathcal{A}) = \min_{i\in N} R_i(\mathcal{A}),$$
$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} |a_{ii_2\ldots i_m}|,\ r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0,\\ \delta_{ji_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| = r_i(\mathcal{A}) - |a_{ij\ldots j}|.$$

Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$, and $x \in \mathbb{C}^n$. Then $\mathcal{A}x^{m-1}$ is a column vector of dimension n and its i-th entry is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2}\cdots x_{i_m},\ i \in N.$$

Qi [31] and Lim [29] independently introduced the following definitions.

Definition 1.1. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenpair of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. Here x^T denotes the transpose of x. Furthermore, we call (λ, x) an H-eigenpair, if λ is a real number and x is a real vector.

Definition 1.2. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenpair of $\mathcal{A}$, if they satisfy the equation

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases}$$

We call (λ, x) a Z-eigenpair, if λ is a real number and x is a real vector.

The generalized matrix eigenvalue problems are important in many applications. In view of this, the definition of the generalized tensor eigenvalue has been developed and is giving by:

Definition 1.3. [6, 11] Let $\mathcal{A}$ and $\mathcal{B}$ be two m-order n-dimensional tensors on $\mathbb{R}$. Assume that both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identical to zero. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenpair of $\mathcal{A}$ relative to $\mathcal{B}$, if the n-system of equations:

$$\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{m-1}$$

possesses a solution. λ is called a $\mathcal{B}$-eigenvalue of $\mathcal{A}$, and x is called a $\mathcal{B}$ eigenvector of $\mathcal{A}$.

As said in [11], the generalized tensor eigenvalue covers the definitions of H-eigenvalues, E-eigenvalues and D-eigenvalues.

Now we turn to introduce some extreme eigenvalues of tensors. The spectral radius $\rho(\mathcal{A})$ of the tensor $\mathcal{A}$ [38] is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Denote by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of the tensor $\mathcal{A}$ [12]. The bounds of $\rho(\mathcal{A})$ and $\tau(\mathcal{A})$ have been concerned by many researchers, and much literature is devoted to presenting some relevant results.

The spectral radius of a tensor is often associated with nonnegative tensors, which are defined as follows:

Definition 1.4. [6, 14, 27, 32, 37, 38] A tensor $\mathcal{A}$ is called nonnegative (resp., positive) if $a_{i_1 i_2\ldots i_m} \geq 0$ (resp., $a_{i_1 i_2\ldots i_m} > 0$) for all $i_1, i_2, \ldots, i_m$.

The definition about symmetry of matrix has been known for us, next we exhibit the definition of symmetry of tensor, which was put forward firstly by Qi [31].

Definition 1.5. [18, 19, 22, 23, 31, 37] A real tensor $\mathcal{A} = (a_{i_1 \dots i_m})$ is called symmetric if its entries satisfy

$$a_{i_1 \dots i_m} = a_{\pi(i_1 \dots i_m)}, \ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices.

M-matrix is an important special matrix and has many beautiful properties [1], and there are many scholars paying their attentions to this matrix. In 2013, the concept of M-matrix has been generalized to M-tensors [10], and many properties of them have been studied.

Definition 1.6. [10, 39, 40] A tensor $\mathcal{A}$ is called a Z-tensor, if all of its off-diagonal entries are non-positive, thus we can get the form of a Z-tensor is $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where s is a real number and $\mathcal{B}$ is a nonnegative tensor ($\mathcal{B} \geq 0$). A Z-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is an M-tensor if $s \geq \rho(\mathcal{B})$, and it is a nonsingular (strong) M-tensor if $s > \rho(\mathcal{B})$.

Eigenvalue problems of tensors play significant roles in many fields, and they have wide practical applications, such as magnetic resonance imaging [34], higher order Markov chains [30], spectral hypergraph theory [8] and so forth. For the past couple of years, many works have been made contribute to spectral properties of tensors, which include estimating the upper bounds on $\rho(\mathcal{A})$ of nonnegative tensors, obtaining the lowers bounds for $\tau(\mathcal{A})$ of M-tensors, and investigating the numerical algorithms for eigenvalues of tensors etc. [4, 6, 7, 10–13, 18, 25–27, 31–33, 36–38].

Further, it is vital that we study the eigenvalue inclusion sets for tensors, as observed in from [19, 21, 24], we can utilize the smallest H-eigenvalue of an even-order real symmetric tensor to determine its positive (semi-)definiteness, but getting the smallest H-eigenvalue of tensors is a task work for us on many occasions. Furthermore, as mentioned in [35], the determinant of the tensor $\mathcal{A}$ denoted by $\det(\mathcal{A})$, is the resultant of the ordered system of homogeneous equations $\mathcal{A}x^{m-1} = 0$ and is closely related to the eigenvalues of $\mathcal{A}$. In general, if $\det(\mathcal{A}) \neq 0$, i.e., 0 is not an eigenvalue of $\mathcal{A}$, then $\mathcal{A}$ is nonsingular. However, it is very difficult to determine the nonsingularity of the tensors by computing their eigenvalues directly. Considering these situations, we shall try to derive a set which contains all eigenvalues of tensors. Some efforts have been made towards this goal recently, we can see [2, 3, 15, 18, 19, 21–24, 31] for more details. A great eigenvalue localizations set is conducive to judge the positive definiteness, the positive semi-definiteness and the nonsingularity of tensors, which stimulates us to establish the new set which contains all eigenvalues of tensors in our paper, this new set is referred to as the modified S-type eigenvalue localization set and it is confirmed to be tighter than those in [17, 19, 22, 31].

Before giving the main results of this paper, we recapitulate some eigenvalue inclusion sets of tensors. For the real supersymmetric tensors, Qi [31] in 2005 gave the Geršgorin eigenvalue localization sets as follows.

Lemma 1.1. [31] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$ and

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \dots i}| \leq r_i(\mathcal{A})\}.$$

This result still stands for general tensors [22, 37]. Subsequently, the following eigenvalue localization set which is more accurate than $\Gamma(\mathcal{A})$ for tensors is developed by Li et al. [22].

Lemma 1.2. [22] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i,j \in N, j \neq i} \mathcal{K}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^j(\mathcal{A}))|z - a_{j \dots j}| \leq |a_{ij \dots j}| r_j(\mathcal{A})\},$$

which is called the Brauer-type eigenvalue localization set. To reduce computations of obtaining $\mathcal{K}(\mathcal{A})$, they also established the S-type eigenvalue localization set as follows.

Lemma 1.3. [22] *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = \left(\bigcup_{i \in S, j \in \bar{S}} \mathcal{K}_{i,j}(\mathcal{A}) \right) \bigcup \left(\bigcup_{i \in \bar{S}, j \in S} \mathcal{K}_{i,j}(\mathcal{A}) \right),$$

where $\mathcal{K}_{i,j}(\mathcal{A})$ ($i \in S$, $j \in \bar{S}$ or $i \in \bar{S}$, $j \in S$) are defined as in Lemma 1.2 and $\bar{S}$ is the complement of S in N.

It is also shown in [22] that when $n \geq 3$, $\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ always true.

Lately, Li et al. in [19] deduced a new eigenvalue localization set. Theorem 6 in [19] confirms that this new set is contained in the sets $\Gamma(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^S(\mathcal{A})$.

Lemma 1.4. [19] *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) = \left(\bigcup_{i \in S, j \in \bar{S}} \Omega^S_{i,j}(\mathcal{A}) \right) \bigcup \left(\bigcup_{i \in \bar{S}, j \in S} \Omega^{\bar{S}}_{i,j}(\mathcal{A}) \right),$$

where

$$\Omega^S_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}|)(|z - a_{j \ldots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A})) \leq r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})\},$$
$$\Omega^{\bar{S}}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}|)(|z - a_{j \ldots j}| - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq r_i(\mathcal{A}) r_j^{\Delta^{\bar{S}}}(\mathcal{A})\},$$

and for $i \in S$,

$$r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}),\ r_i^j(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}) - |a_{ij \ldots j}|,$$

with

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2, \ldots, i_m) \in \Delta^S, \\ \delta_{ii_2 \ldots i_m = 0}}} |a_{ii_2 \ldots i_m}|,\ r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \overline{\Delta^S}} |a_{ii_2 \ldots i_m}|.$$

In the sequel, enlightened by the idea of [24], Huang et al. in [17] proposed a new S-type eigenvalue localization set for tensors as follows, which is better than that in Lemma 1.4.

Lemma 1.5. [17] *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) = \left(\Upsilon^S_{i,j}(\mathcal{A}) \right) \bigcup \left(\Upsilon^{\bar{S}}_{i,j}(\mathcal{A}) \right), \tag{1}$$

where

$$\Upsilon^S_{i,j}(\mathcal{A}) = \left(\bigcup_{i \in S} \hat{\Upsilon}^1_i(\mathcal{A}) \right) \bigcup \left(\bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}^1_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A}) \right) \right),$$
$$\Upsilon^{\bar{S}}_{i,j}(\mathcal{A}) = \left(\bigcup_{i \in \bar{S}} \hat{\Upsilon}^2_i(\mathcal{A}) \right) \bigcup \left(\bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}^2_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A}) \right) \right),$$

with

$$\hat{\Upsilon}^1_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \ldots i}| \leq r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\},\ \hat{\Upsilon}^2_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \ldots i}| \leq r_i^{\overline{\Delta^S}}(\mathcal{A})\},$$
$$\tilde{\Upsilon}^1_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}| - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|z - a_{j \ldots j}| - r_j^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_i^{\Delta^{\bar{S}}}(\mathcal{A}) r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\},$$
$$\tilde{\Upsilon}^2_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}| - r_i^{\overline{\Delta^S}}(\mathcal{A}))(|z - a_{j \ldots j}| - r_j^{\Delta^S}(\mathcal{A})) \leq r_i^{\Delta^S}(\mathcal{A}) r_j^{\overline{\Delta^S}}(\mathcal{A})\}.$$

Motivated by the idea of [18], Huang et al. in [15] newly employed another S-type eigenvalue inclusion set for tensors, which improves those in Theorem 2.2 of [22] and Theorem 6 in [18]. This set is also involved in a nonempty proper set S of N, and we can see that in Lemma 1.6.

Lemma 1.6. [15] *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq \Delta^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in \bar{S}} \Delta_i^j(\mathcal{A})\right) \bigcup \left(\bigcup_{i\in \bar{S}, j\in S} \Delta_i^j(\mathcal{A})\right),$$

where

$$\Delta_i^j(\mathcal{A}) = \left\{z \in \mathbb{C} : |(z - a_{i\ldots i})(z - a_{j\ldots j}) - a_{ij\ldots j}a_{ji\ldots i}| \leq |z - a_{j\ldots j}|r_i^j(\mathcal{A}) + |a_{ij\ldots j}|r_j^i(\mathcal{A})\right\}.$$

In this work, motivated by the ideas of [20, 28, 35], a new set called the modified S-type eigenvalue localization set for tensors is derived, which outperforms those in Lemmas 1.1-1.5. A new criteria for identifying the nonsingularity of tensors, bounds for the spectral radius of nonnegative tensors and the minimum H-eigenvalue of strong M-tensors are established by applying the proposed set. These results perform better than some existing ones. We afford several numerical examples to show the advantages of our results.

2. Preliminaries

In our proofs of main results, we need some results, which are related to tensors and inequality are briefly introduced in this section.

Lemma 2.1. [5] *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is irreducible nonnegative, then $\rho(\mathcal{A})$ is an eigenvalue with an entrywise positive eigenvector x, i.e., $x > 0$, corresponding to it.*

Lemma 2.2. [22] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor. Then $\rho(\mathcal{A}) \geq \max_{i\in N}\{a_{i\ldots i}\}$.*

Lemma 2.3. [36] *Let $\mathcal{A}$ be a nonsingular (strong) M-tensor and $\tau(\mathcal{A})$ denote the minimal value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an H-eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. If $\mathcal{A}$ is a weakly irreducible Z-tensor, then $\tau(\mathcal{A}) > 0$ is the unique eigenvalue with a positive eigenvector.*

The definitions for irreducibility and weakly irreducibility of tensors have been introduced in [18, 38]. For the weakly irreducible strong M-tensors, the following result has been obtained by Wang and Wei in [36].

Lemma 2.4. [36] *Let $\mathcal{A}$ be a weakly irreducible strong M-tensor. Then $\tau(\mathcal{A}) \leq \min_{i\in N}\{a_{i\ldots i}\}$.*

3. Main results

3.1. A modified S-type eigenvalue localization set for tensors

In this section, we construct a modified S-type eigenvalue localization set for tensors, and we also compare the proposed set with those in Lemmas 1.1-1.5.

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\sigma(\mathcal{A}) \subseteq G^S(\mathcal{A}) = \left(G_{i,j}^S(\mathcal{A})\right) \bigcup \left(G_{i,j}^{\bar{S}}(\mathcal{A})\right), \tag{2}$$

where

$$G^{S}_{i,j}(\mathcal{A}) = \left(\bigcup_{i\in S}\hat{\Upsilon}^{1}_{i}(\mathcal{A})\right)\bigcup\left(\bigcup_{i\in S, j\in\bar{S}}\left(\left(\tilde{\Upsilon}^{1}_{i,j}(\mathcal{A})\backslash G^{1}_{i,j}(\mathcal{A})\right)\bigcap\Gamma_i(\mathcal{A})\right)\right),$$

$$G^{\bar{S}}_{i,j}(\mathcal{A}) = \left(\bigcup_{i\in \bar{S}}\hat{\Upsilon}^{2}_{i}(\mathcal{A})\right)\bigcup\left(\bigcup_{i\in \bar{S}, j\in S}\left(\left(\tilde{\Upsilon}^{2}_{i,j}(\mathcal{A})\backslash G^{2}_{i,j}(\mathcal{A})\right)\bigcap\Gamma_i(\mathcal{A})\right)\right),$$

with

$$\begin{aligned}
&\hat{\Upsilon}^{1}_{i}(\mathcal{A}) = \{z\in\mathbb{C} : |z-a_{i\ldots i}| \le r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\},\ \hat{\Upsilon}^{2}_{i}(\mathcal{A}) = \{z\in\mathbb{C} : |z-a_{i\ldots i}| \le r_i^{\overline{\Delta^{S}}}(\mathcal{A})\},\\
&\tilde{\Upsilon}^{1}_{i,j}(\mathcal{A}) = \{z\in\mathbb{C} : (|z-a_{i\ldots i}| - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|z-a_{j\ldots j}| - r_j^{\Delta^{\bar{S}}}(\mathcal{A})) \le r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\},\\
&\tilde{\Upsilon}^{2}_{i,j}(\mathcal{A}) = \{z\in\mathbb{C} : (|z-a_{i\ldots i}| - r_i^{\overline{\Delta^{S}}}(\mathcal{A}))(|z-a_{j\ldots j}| - r_j^{\Delta^{S}}(\mathcal{A})) \le r_i^{\Delta^{S}}(\mathcal{A})r_j^{\overline{\Delta^{S}}}(\mathcal{A})\},\\
&G^{1}_{i,j}(\mathcal{A}) = \{z\in\mathbb{C} : (|z-a_{i\ldots i}| + r_i^{j}(\mathcal{A}))(|z-a_{j\ldots j}| + r_j^{\Delta^{\bar{S}}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))\},\\
&G^{2}_{i,j}(\mathcal{A}) = \{z\in\mathbb{C} : (|z-a_{i\ldots i}| + r_i^{j}(\mathcal{A}))(|z-a_{j\ldots j}| + r_j^{\Delta^{S}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\overline{\Delta^{S}}}(\mathcal{A}))\}.
\end{aligned}$$

Proof. For any $\lambda\in\sigma(\mathcal{A})$, let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n\backslash\{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{3}$$

Let $|x_p| = \max\limits_{i\in S}\{|x_i|\}$ and $|x_q| = \max\limits_{i\in \bar{S}}\{|x_i|\}$. Then, $x_p \neq 0$ or $x_q \neq 0$. Below two cases will be discussed.

(i) $|x_p| \ge |x_q|$, so $|x_p| = \max\limits_{i\in N}\{|x_i|\}$ and $|x_p| > 0$. It follows from the pth equation of (3) that

$$(\lambda - a_{p\ldots p})x_p^{m-1} = \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\Delta^{\bar{S}}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m}. \tag{4}$$

Taking absolute values in Equation (4) and applying the triangle inequality yield

$$\begin{aligned}
&|\lambda - a_{p\ldots p}||x_p|^{m-1}\\
\le\ & \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{(i_2,\ldots,i_m)\in\Delta^{\bar{S}}} |a_{pi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}|\\
\le\ & \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_p|^{m-1} + \sum_{(i_2,\ldots,i_m)\in\Delta^{\bar{S}}} |a_{pi_2\ldots i_m}||x_q|^{m-1}\\
=\ & r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|x_p|^{m-1} + r_p^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1},
\end{aligned}$$

which leads to

$$(|\lambda - a_{p\ldots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_p|^{m-1} \le r_p^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1}. \tag{5}$$

If $|x_q| = 0$, then it follows from (5) that $|\lambda - a_{p\ldots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \le 0$ by $|x_p| > 0$, that is, $|\lambda - a_{p\ldots p}| \le r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$. Evidently, $\lambda\in\hat{\Upsilon}^{1}_{p}(\mathcal{A}) \subseteq \Upsilon^{S}(\mathcal{A})$. Otherwise, $|x_q| > 0$. If $\lambda \notin \bigcup\limits_{i\in S}\hat{\Upsilon}^{1}_{i}(\mathcal{A})$, then for any $i\in S$, we have $|\lambda - a_{i\ldots i}| > r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$. In particular, $|\lambda - a_{p\ldots p}| > r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$ holds true, i.e., $|\lambda - a_{p\ldots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) > 0$. According to (5), it is not difficult to verify $\lambda \in \Gamma_p(\mathcal{A})$. Besides, it follows from (3) that

$$\begin{aligned}
&|\lambda - a_{q\ldots q}||x_q|^{m-1} \\
\leq\ & \sum_{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}}} |a_{qi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| \\
\leq\ & \sum_{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}}} |a_{qi_2\ldots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_q|^{m-1} \\
=\ & r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|x_p|^{m-1} + r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1},
\end{aligned}$$

which is equivalent to

$$(|\lambda - a_{q\ldots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1} \leq r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|x_p|^{m-1}. \tag{6}$$

Note that $|x_p| > 0$ and $|\lambda - a_{p\ldots p}| > r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$, multiplying (5) with (6) results in

$$(|\lambda - a_{p\ldots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|\lambda - a_{q\ldots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A}) r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$$

by $|x_p| \geq |x_q| > 0$, which together with $\lambda \in \Gamma_p(\mathcal{A})$ gives $\lambda \in \left(\tilde{\Upsilon}^1_{p,q}(\mathcal{A}) \bigcap \Gamma_p(\mathcal{A})\right)$.

It follows from (4) that

$$a_{pq\ldots q}x_q^{m-1} = (\lambda - a_{p\ldots p})x_p^{m-1} - \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m}. \tag{7}$$

By taking modulus in both sides of (7) and utilizing the triangle inequality, it has

$$\begin{aligned}
|a_{pq\ldots q}||x_q|^{m-1} &\leq |\lambda - a_{p\ldots p}||x_p|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| \\
&\leq |\lambda - a_{p\ldots p}||x_p|^{m-1} + \sum_{\substack{\delta_{pi_2\ldots i_m}=0,\\ \delta_{qi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_p|^{m-1} = (|\lambda - a_{p\ldots p}| + r_p^q(\mathcal{A}))|x_p|^{m-1}.
\end{aligned} \tag{8}$$

Furthermore, we consider the qth equation of (3) which can be written as

$$a_{qp\ldots p}x_p^{m-1} = (\lambda - a_{q\ldots q})x_q^{m-1} - \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} a_{qi_2\ldots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} a_{qi_2\ldots i_m}x_{i_2}\cdots x_{i_m}. \tag{9}$$

Using the same operations applied in (8) to (9) results in

$$\begin{aligned}
|a_{qp\ldots p}||x_p|^{m-1} &\leq |\lambda - a_{q\ldots q}||x_q|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| \\
&\leq |\lambda - a_{q\ldots q}||x_q|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{pi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}},\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_q|^{m-1} \\
&= |\lambda - a_{q\ldots q}||x_q|^{m-1} + (r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - |a_{qp\ldots p}|)|x_p|^{m-1} + r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1},
\end{aligned}$$

which yields that

$$(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_p|^{m-1} \leq (|\lambda - a_{q\ldots q}| + r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1}. \tag{10}$$

If $|x_q| > 0$, then multiplying (8) with (10) leads to

$$|a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_p|^{m-1}|x_q|^{m-1} \leq (|\lambda - a_{p\ldots p}| + r_p^q(\mathcal{A}))(|\lambda - a_{q\ldots q}| + r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_p|^{m-1}|x_q|^{m-1},$$

and hence

$$|a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq (|\lambda - a_{p\ldots p}| + r_p^q(\mathcal{A}))(|\lambda - a_{q\ldots q}| + r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \tag{11}$$

as $|x_p| \geq |x_q| > 0$. If $|x_q| = 0$, then (10) implies that $2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \leq 0$, and (11) is also valid. (11) means that $\lambda \notin G_{p,q}^1(\mathcal{A})$. Therefore, $\lambda \in \left(\left(\tilde{\Upsilon}_{p,q}^1(\mathcal{A})\backslash G_{p,q}^1(\mathcal{A})\right) \bigcap \Gamma_p(\mathcal{A})\right) \subseteq G^S(\mathcal{A})$.

(ii) $|x_p| \leq |x_q|$, so $|x_q| = \max_{i \in N}\{|x_i|\}$ and $|x_q| > 0$. From qth equation of (3), it has

$$(\lambda - a_{q\ldots q})x_q^{m-1} = \sum_{(i_2,\ldots,i_m)\in\Delta^S} a_{qi_2\ldots i_m}x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{qi_2\ldots i_m}=0}} a_{qi_2\ldots i_m}x_{i_2}\cdots x_{i_m}. \tag{12}$$

Utilizing the similar operations as in (4), we can obtain the following inequality:

$$\begin{aligned} & |\lambda - a_{q\ldots q}||x_q|^{m-1} \\ \leq \; & \sum_{(i_2,\ldots,i_m)\in\Delta^S} |a_{qi_2\ldots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}||x_q|^{m-1} \\ = \; & r_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1} + r_q^{\overline{\Delta^S}}(\mathcal{A})|x_q|^{m-1}, \end{aligned}$$

which yields that

$$(|\lambda - a_{q\ldots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}))|x_q|^{m-1} \leq r_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1}. \tag{13}$$

If $|x_p| = 0$, then it follows from (13) that $|\lambda - a_{q\ldots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) \leq 0$ by $|x_q| > 0$, i.e., $|\lambda - a_{q\ldots q}| \leq r_q^{\overline{\Delta^S}}(\mathcal{A})$, obviously, $\lambda \in \hat{\Upsilon}_q^2(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. Otherwise, $|x_p| > 0$. If $\lambda \notin \bigcup_{i\in\bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, we are easy to see that for any $i \in \bar{S}$, $|\lambda - a_{i\ldots i}| > r_i^{\overline{\Delta^S}}(\mathcal{A})$. In particular, $|\lambda - a_{q\ldots q}| > r_q^{\overline{\Delta^S}}(\mathcal{A})$, i.e., $|\lambda - a_{q\ldots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) > 0$. By (13), we infer that $\lambda \in \Gamma_q(\mathcal{A})$. In addition, it follows from (3) that

$$\begin{aligned} & |\lambda - a_{p\ldots p}||x_p|^{m-1} \\ \leq \; & \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S,\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_p|^{m-1} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta^S}} |a_{pi_2\ldots i_m}||x_q|^{m-1} \\ = \; & r_p^{\Delta^S}(\mathcal{A})|x_p|^{m-1} + r_p^{\overline{\Delta^S}}(\mathcal{A})|x_q|^{m-1}, \end{aligned} \tag{14}$$

Inequality (14) can be simplified to the following inequality

$$(|\lambda - a_{p\ldots p}| - r_p^{\Delta^S}(\mathcal{A}))|x_p|^{m-1} \leq r_p^{\overline{\Delta^S}}(\mathcal{A})|x_q|^{m-1}. \tag{15}$$

Having in mind that $|x_q| > 0$, $|\lambda - a_{q\ldots q}| > r_q^{\overline{\Delta^S}}(\mathcal{A})$ and $|x_q| \geq |x_p| > 0$, multiplying (13) with (15) leads to

$$(|\lambda - a_{p\ldots p}| - r_p^{\Delta^S}(\mathcal{A}))(|\lambda - a_{q\ldots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_p^{\overline{\Delta^S}}(\mathcal{A})r_q^{\Delta^S}(\mathcal{A}).$$

This leads to $\lambda \in \left(\tilde{\Upsilon}_{q,p}^2(\mathcal{A}) \bigcap \Gamma_q(\mathcal{A})\right) \subseteq \Upsilon^S(\mathcal{A})$.

Similar to the proof of Case (i), (12) gives

$$a_{qp\dots p}x_p^{m-1} = (\lambda - a_{q\dots q})x_q^{m-1} - \sum_{\substack{(i_2,\dots,i_m)\in\Delta^S,\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^S},\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m}. \tag{16}$$

By taking modulus in both sides of (16) and utilizing the triangle inequality, it has

$$\begin{aligned}|a_{qp\dots p}||x_p|^{m-1} &= |\lambda - a_{q\dots q}||x_q|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^S,\\ \delta_{pi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^S},\\ \delta_{qi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\le |\lambda - a_{q\dots q}||x_q|^{m-1} + \sum_{\substack{\delta_{qi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_q|^{m-1} = (|\lambda - a_{q\dots q}| + r_q^p(\mathcal{A}))|x_q|^{m-1}. \end{aligned} \tag{17}$$

Besides, consider the pth equation of (3):

$$a_{pq\dots q}x_q^{m-1} = (\lambda - a_{p\dots p})x_p^{m-1} - \sum_{\substack{(i_2,\dots,i_m)\in\Delta^S,\\ \delta_{pi_2\dots i_m}=0}} a_{pi_2\dots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\overline{\Delta^S}},\\ \delta_{qi_2\dots i_m}=0}} a_{pi_2\dots i_m}x_{i_2}\cdots x_{i_m},$$

which results in

$$\begin{aligned}|a_{pq\dots q}||x_q|^{m-1} &= |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^S,\\ \delta_{pi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\overline{\Delta^S}},\\ \delta_{qi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\le |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^S,\\ \delta_{pi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\overline{\Delta^S}},\\ \delta_{qi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_q|^{m-1} \\ &= |\lambda - a_{p\dots p}||x_p|^{m-1} + (r_p^{\overline{\Delta^S}}(\mathcal{A}) - |a_{pq\dots q}|)|x_q|^{m-1} + r_p^{\Delta^S}(\mathcal{A})|x_p|^{m-1}, \end{aligned}$$

i.e.,

$$(2|a_{pq\dots q}| - r_p^{\overline{\Delta^S}}(\mathcal{A}))|x_q|^{m-1} \le (|\lambda - a_{p\dots p}| + r_p^{\Delta^S}(\mathcal{A}))|x_p|^{m-1}. \tag{18}$$

If $|x_p| > 0$, then combining (17) with (18) obtains

$$|a_{qp\dots p}|(2|a_{pq\dots q}| - r_p^{\overline{\Delta^S}}(\mathcal{A}))|x_q|^{m-1}|x_p|^{m-1} \le (|\lambda - a_{q\dots q}| + r_q^p(\mathcal{A}))(|\lambda - a_{p\dots p}| + r_p^{\Delta^S}(\mathcal{A}))|x_q|^{m-1}|x_p|^{m-1}.$$

In view of $|x_q| \ge |x_p| > 0$, we have

$$|a_{qp\dots p}|(2|a_{pq\dots q}| - r_p^{\overline{\Delta^S}}(\mathcal{A})) \le (|\lambda - a_{q\dots q}| + r_q^p(\mathcal{A}))(|\lambda - a_{p\dots p}| + r_p^{\Delta^S}(\mathcal{A})). \tag{19}$$

If $|x_p| = 0$, then it follows from (18) that $2|a_{pq\dots q}| - r_p^{\overline{\Delta^S}}(\mathcal{A}) \le 0$, and (19) also holds true. Thus $\lambda \notin G_{q,p}^2(\mathcal{A})$ follows from (19). By summarizing the above discussions, it holds that $\lambda \in \left(\left(\tilde{\Upsilon}_{q,p}^2(\mathcal{A})\backslash G_{q,p}^2(\mathcal{A})\right)\bigcap \Gamma_q(\mathcal{A})\right) \subseteq G^S(\mathcal{A})$. This completes our proof of Theorem 3.1. □

Remark 3.2. *Note that x_p and x_q do not need to be unique in the proof of Theorem 3.1. If there exist x_{p_1} and x_{p_2} such that $|x_{p_1}| = |x_{p_2}| = \max_{i\in S}\{|x_i|\}$, we can choose one of them, and apply the technique used in Theorem 3.1 to develop the results of Theorem 3.1.*

Remark 3.3. *As mentioned in Remark 3.1 of [17], the number of elements in the sets $\Omega^S(\mathcal{A})$ (see Lemma 1.4) and $\Upsilon^S(\mathcal{A})$ (see Lemma 1.5) are $2|S|(n-|S|)$ and $2|S|(n-|S|)+2n$, respectively, where $|S|$ denotes the cardinality of S. In addition, the set $G^S(\mathcal{A})$ consists of $|S|(n-|S|)$ sets $\tilde{\Upsilon}^1_{i,j}(\mathcal{A})$ and $G^1_{i,j}(\mathcal{A})$, $|S|(n-|S|)$ sets $\tilde{\Upsilon}^2_{i,j}(\mathcal{A})$ and $G^2_{i,j}(\mathcal{A})$, $|S|$ sets $\hat{\Upsilon}^1_i(\mathcal{A})$, $n-|S|$ sets $\hat{\Upsilon}^2_i(\mathcal{A})$ and n sets $\Gamma_i(\mathcal{A})$, then the set $G^S(\mathcal{A})$ contains $4|S|(n-|S|)+2n$ sets. Thus, determining $G^S(\mathcal{A})$ requires more computations than $\Omega^S(\mathcal{A})$ and $\Upsilon^S(\mathcal{A})$, while $G^S(\mathcal{A})$ is sharper than them, as proved in Theorem 3.4.*

Next, we prove the following theorem, which indicates that $G^S(\mathcal{A})$ is better than those in Lemmas 1.1-1.5.

Theorem 3.4. *Let $\mathcal{A}=(a_{i_1\ldots i_m})\in\mathbb{C}^{[m,n]}$, $n\geq 2$ and S be a nonempty proper subset of N. Then*

$$G^S(\mathcal{A})\subseteq\Upsilon^S(\mathcal{A})\subseteq\Omega^S(\mathcal{A})\subseteq\mathcal{K}^S(\mathcal{A})\subseteq\mathcal{K}(\mathcal{A})\subseteq\Gamma(\mathcal{A}).$$

Proof. By Theorem 3.2 in [17], we see that $\Upsilon^S(\mathcal{A})\subseteq\Omega^S(\mathcal{A})\subseteq\mathcal{K}^S(\mathcal{A})\subseteq\mathcal{K}(\mathcal{A})\subseteq\Gamma(\mathcal{A})$ holds. Thus, only $G^S(\mathcal{A})\subseteq\Upsilon^S(\mathcal{A})$ need to be proved.

We first prove that $G^S_{i,j}(\mathcal{A})\subseteq\Upsilon^S_{i,j}(\mathcal{A})$. For any $i\in S$, $j\in\bar{S}$, if $|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))\leq 0$, then $G^1_{i,j}(\mathcal{A})=\emptyset$, and therefore $G^1_{i,j}(\mathcal{A})\subseteq\tilde{\Upsilon}^1_{i,j}(\mathcal{A})$. Now we consider the case that $|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))>0$. Since $r_i^j(\mathcal{A})\geq r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$, it has

$$\begin{aligned}&(|z-a_{i\ldots i}|+r_i^j(\mathcal{A}))(|z-a_{j\ldots j}|+r_j^{\Delta^{\bar{S}}}(\mathcal{A}))-(|z-a_{i\ldots i}|-r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|z-a_{j\ldots j}|-r_j^{\Delta^{\bar{S}}}(\mathcal{A}))\\ =\ &2|z-a_{i\ldots i}|r_j^{\Delta^{\bar{S}}}(\mathcal{A})+|z-a_{j\ldots j}|(r_i^j(\mathcal{A})+r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))+r_j^{\Delta^{\bar{S}}}(\mathcal{A})(r_i^j(\mathcal{A})-r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))\geq 0.\end{aligned}\tag{20}$$

Moreover, it is not difficult to verify that $|a_{ij\ldots j}|\leq r_i^{\Delta^{\bar{S}}}(\mathcal{A})$ and $2|a_{ji\ldots i}|\leq 2r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$, that is $0<2|a_{ji\ldots i}|-r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\leq r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$, which implies that

$$|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))\leq r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),$$

which together with (20) shows that $G^1_{i,j}(\mathcal{A})\subseteq\tilde{\Upsilon}^1_{i,j}(\mathcal{A})$. Thus

$$\left(\left(\tilde{\Upsilon}^1_{i,j}(\mathcal{A})\backslash G^1_{i,j}(\mathcal{A})\right)\bigcap\Gamma_i(\mathcal{A})\right)\subseteq\left(\tilde{\Upsilon}^1_{i,j}(\mathcal{A})\bigcap\Gamma_i(\mathcal{A})\right),$$

and we conclude that $G^S_{i,j}(\mathcal{A})\subseteq\Upsilon^S_{i,j}(\mathcal{A})$.

On the other hand, we prove $G^{\bar{S}}_{i,j}(\mathcal{A})\subseteq\Upsilon^{\bar{S}}_{i,j}(\mathcal{A})$. For any $i\in\bar{S}$, $j\in S$, if $|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^S}}(\mathcal{A}))\leq 0$, then $G^2_{i,j}(\mathcal{A})=\emptyset$, and therefore $G^2_{i,j}(\mathcal{A})\subseteq\tilde{\Upsilon}^2_{i,j}(\mathcal{A})$. If $|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^S}}(\mathcal{A}))>0$, then

$$\begin{aligned}&(|z-a_{i\ldots i}|+r_i^j(\mathcal{A}))(|z-a_{j\ldots j}|+r_j^{\Delta^S}(\mathcal{A}))-(|z-a_{i\ldots i}|-r_i^{\overline{\Delta^S}}(\mathcal{A}))(|z-a_{j\ldots j}|-r_j^{\Delta^S}(\mathcal{A}))\\ =\ &2|z-a_{i\ldots i}|r_j^{\Delta^S}(\mathcal{A})+|z-a_{j\ldots j}|(r_i^j(\mathcal{A})+r_i^{\overline{\Delta^S}}(\mathcal{A}))+r_j^{\Delta^S}(\mathcal{A})(r_i^j(\mathcal{A})-r_i^{\overline{\Delta^S}}(\mathcal{A}))\geq 0\end{aligned}\tag{21}$$

by virtue of $r_i^j(\mathcal{A})\geq r_i^{\overline{\Delta^S}}(\mathcal{A})$. Similar to the above discussions, it can be seen that $|a_{ij\ldots j}|\leq r_i^{\Delta^S}(\mathcal{A})$ and $0<2|a_{ji\ldots i}|-r_j^{\overline{\Delta^S}}(\mathcal{A})\leq r_j^{\overline{\Delta^S}}(\mathcal{A})$, hence

$$|a_{ij\ldots j}|(2|a_{ji\ldots i}|-r_j^{\overline{\Delta^S}}(\mathcal{A}))\leq r_i^{\Delta^S}(\mathcal{A})r_j^{\overline{\Delta^S}}(\mathcal{A}).\tag{22}$$

Combining (21) with (22) proves $G^2_{i,j}(\mathcal{A})\subseteq\tilde{\Upsilon}^2_{i,j}(\mathcal{A})$, then

$$\left(\left(\tilde{\Upsilon}^2_{i,j}(\mathcal{A})\backslash G^2_{i,j}(\mathcal{A})\right)\bigcap\Gamma_i(\mathcal{A})\right)\subseteq\left(\tilde{\Upsilon}^2_{i,j}(\mathcal{A})\bigcap\Gamma_i(\mathcal{A})\right).$$

Consequently, $G^S_{i,j}(\mathcal{A})\subseteq\Upsilon^S_{i,j}(\mathcal{A})$.

We summarize the above proof procedure, and infer that $G^S(\mathcal{A})\subseteq\Upsilon^S(\mathcal{A})$. We finish this proof. □

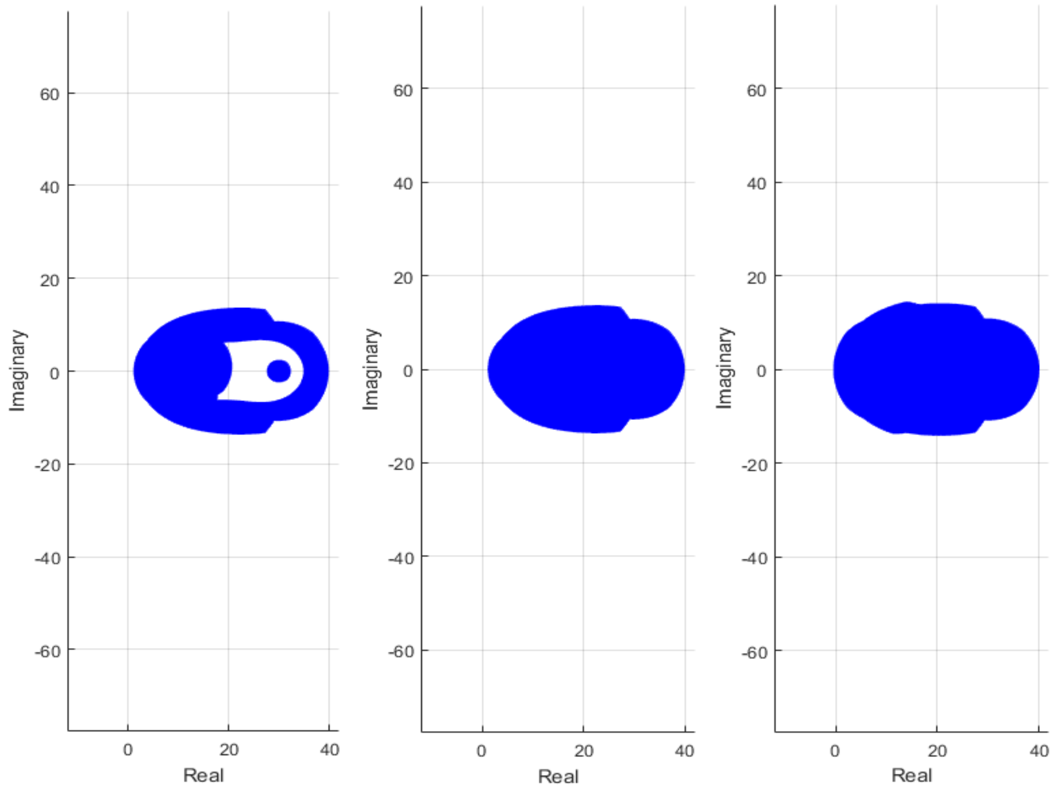

Figure 1: Eigenvalue localization sets $G^S(\mathcal{A})$ (left), $\Upsilon^S(\mathcal{A})$ (middle) and $\Omega^S(\mathcal{A})$ (right).

What follows is an example, which is given to compute the region of Theorem 3.1, and those in Theorem 3.1 of [17] and Theorem 4 of [19]. Moreover, we compare these sets, and depict them in Figure 1.

Example 3.5. *Consider the tensor* $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ *with elements defined as follows:*

$$a_{111} = 11,\ a_{222} = 10,\ a_{333} = 12 + \mathrm{i},\ a_{444} = 30,\ a_{122} = 3 + \mathrm{i},\ a_{144} = 20 - \mathrm{i},$$
$$a_{211} = -2 - \mathrm{i},\ a_{233} = 3 - \mathrm{i},\ a_{322} = 6,\ a_{344} = 2 + \mathrm{i},\ a_{411} = 10,\ a_{422} = 2$$

and other elements of $\mathcal{A}$ *are zeros.*

The localization sets $G^S(\mathcal{A})$, $\Upsilon^S(\mathcal{A})$ and $\Omega^S(\mathcal{A})$ are plotted in Figure 1. It is clear that $G^S(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$, which is in accordance with the result of Theorem 3.4 (see Figure 1).

Example 3.6. *Consider the tensor* $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ *with elements defined as follows:*

$$a_{111} = 2,\ a_{222} = 2 + 10\mathrm{i},\ a_{333} = 2,\ a_{444} = 2,\ a_{122} = 2,\ a_{144} = 2,$$
$$a_{211} = 2,\ a_{233} = 2,\ a_{322} = 3,\ a_{344} = 1,\ a_{411} = 2,\ a_{422} = 2$$

and other elements of $\mathcal{A}$ are zeros.

The localization sets $G^S(\mathcal{A})$ and $\Delta^S(\mathcal{A})$ are plotted in Figure 2. It can be seen that $G^S(\mathcal{A}) \subseteq \Delta^S(\mathcal{A})$ (see Figure 2).

Remark 3.7. *From Example 3.6, we see that the set in Theorem 3.1 is more precise compared with that in Theorem 3.1 of [15] under some circumstances. Here, we want to single out: although the results of some numerical examples illustrate that the set in Theorem 3.1 outperforms the one in Theorem 3.1 of [15], we have not proved the result in Theorem 3.1 is better than that in Theorem 3.1 of [15] in theory now. This problem need to be studied in our future work.*

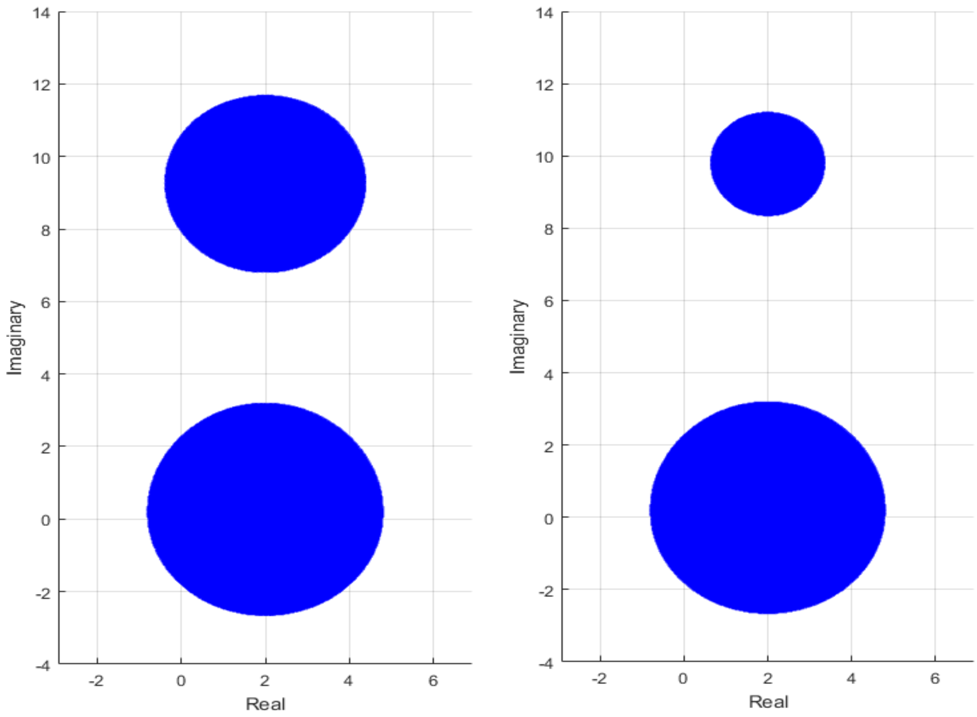

Figure 2: Eigenvalue localization sets $\Delta^S(\mathcal{A})$ (left) and $G^S(\mathcal{A})$ (right).

3.2. *A new sufficient criteria for nonsingularity of tensors*

By applying the set in Section 3.1, in this section, we put out a new sufficient criteria for the nonsingularity of tensors. Furthermore, to illustrate the superiority of this criteria to those derived in [2, 20, 28, 35], a numerical example is implemented.

Theorem 3.8. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$. If there is a nonempty proper subset S of N and the following four statements hold:*
(i) $|a_{i\ldots i}| > r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$ for any $i \in S$;
(ii) $|a_{i\ldots i}| > r_i^{\overline{\Delta^{S}}}(\mathcal{A})$ for any $i \in \bar{S}$;
(iii) For any $i \in S$, $j \in \bar{S}$,

$$(|a_{i\ldots i}| - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|a_{j\ldots j}| - r_j^{\Delta^{\bar{S}}}(\mathcal{A})) > r_i^{\Delta^{\bar{S}}}(\mathcal{A}) r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$$

or

$$(|a_{i\ldots i}| + r_i^{j}(\mathcal{A}))(|a_{j\ldots j}| + r_j^{\Delta^{\bar{S}}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))$$

or $|a_{i\ldots i}| > r_i(\mathcal{A})$;
(iv) For any $i \in \bar{S}$, $j \in S$,

$$(|a_{i\ldots i}| - r_i^{\overline{\Delta^{S}}}(\mathcal{A}))(|a_{j\ldots j}| - r_j^{\Delta^{S}}(\mathcal{A})) > r_i^{\Delta^{S}}(\mathcal{A}) r_j^{\overline{\Delta^{S}}}(\mathcal{A})$$

or

$$(|a_{i\ldots i}| + r_i^{j}(\mathcal{A}))(|a_{j\ldots j}| + r_j^{\Delta^{S}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\overline{\Delta^{S}}}(\mathcal{A}))$$

or $|a_{i\ldots i}| > r_i(\mathcal{A})$, then $\mathcal{A}$ is nonsingular.

Proof. Assume that λ is the eigenvalue of $\mathcal{A}$. From Theorem 3.1, it has $\lambda \in G^S(\mathcal{A})$, which implies that there are $i_0, i_1, i_2 \in S$ and $j_0, j_1, j_2 \in \bar{S}$ such that

$$|\lambda - a_{i_0\dots i_0}| \leq r_{i_0}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \text{ or } |\lambda - a_{j_0\dots j_0}| \leq r_{j_0}^{\overline{\Delta^{S}}}(\mathcal{A})$$

or

$$(|\lambda - a_{i_1\dots i_1}| - r_{i_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|\lambda - a_{j_1\dots j_1}| - r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_{i_1}^{\Delta^{\bar{S}}}(\mathcal{A}) r_{j_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),$$
$$(|z - a_{i_1\dots i_1}| + r_{i_1}^{j_1}(\mathcal{A}))(|z - a_{j_1\dots j_1}| + r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) \geq |a_{i_1 j_1\dots j_1}|(2|a_{j_1 i_1\dots i_1}| - r_{j_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})),\ |\lambda - a_{i_1\dots i_1}| \leq r_{i_1}(\mathcal{A})$$

or

$$(|\lambda - a_{j_2\dots j_2}| - r_{j_2}^{\overline{\Delta^{S}}}(\mathcal{A}))(|\lambda - a_{i_2\dots i_2}| - r_{i_2}^{\Delta^{S}}(\mathcal{A})) \leq r_{j_2}^{\Delta^{S}}(\mathcal{A}) r_{i_2}^{\overline{\Delta^{S}}}(\mathcal{A}),$$
$$(|z - a_{j_2\dots j_2}| + r_{j_2}^{i_2}(\mathcal{A}))(|z - a_{i_2\dots i_2}| + r_{i_2}^{\Delta^{S}}(\mathcal{A})) \geq |a_{j_2 i_2\dots i_2}|(2|a_{i_2 j_2\dots j_2}| - r_{i_2}^{\overline{\Delta^{S}}}(\mathcal{A})),\ |\lambda - a_{j_2\dots j_2}| \leq r_{j_2}(\mathcal{A}).$$

If $\lambda = 0$, then it follows that

$$|\lambda - a_{i_0\dots i_0}| = |a_{i_0\dots i_0}| > r_{i_0}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \text{ and } |\lambda - a_{j_0\dots j_0}| = |a_{j_0\dots j_0}| > r_{j_0}^{\overline{\Delta^{S}}}(\mathcal{A})$$

and

$$\begin{aligned} &(|\lambda - a_{i_1\dots i_1}| - r_{i_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|\lambda - a_{j_1\dots j_1}| - r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) \\ =\ &(|a_{i_1\dots i_1}| - r_{i_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|a_{j_1\dots j_1}| - r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) > r_{i_1}^{\Delta^{\bar{S}}}(\mathcal{A}) r_{j_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \end{aligned}$$

or

$$\begin{aligned} &(|\lambda - a_{i_1\dots i_1}| + r_{i_1}^{j_1}(\mathcal{A}))(|\lambda - a_{j_1\dots j_1}| + r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) \\ =\ &(|a_{i_1\dots i_1}| + r_{i_1}^{j_1}(\mathcal{A}))(|a_{j_1\dots j_1}| + r_{j_1}^{\Delta^{\bar{S}}}(\mathcal{A})) < |a_{i_1 j_1\dots j_1}|(2|a_{j_1 i_1\dots i_1}| - r_{j_1}^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})), \end{aligned}$$

or $|\lambda - a_{i_1\dots i_1}| = |a_{i_1\dots i_1}| > r_{i_1}(\mathcal{A})$; and

$$\begin{aligned} &(|\lambda - a_{j_2\dots j_2}| - r_{j_2}^{\overline{\Delta^{S}}}(\mathcal{A}))(|\lambda - a_{i_2\dots i_2}| - r_{i_2}^{\Delta^{S}}(\mathcal{A})) \\ =\ &(|a_{j_2\dots j_2}| - r_{j_2}^{\overline{\Delta^{S}}}(\mathcal{A}))(|a_{i_2\dots i_2}| - r_{i_2}^{\Delta^{S}}(\mathcal{A})) > r_{j_2}^{\Delta^{S}}(\mathcal{A}) r_{i_2}^{\overline{\Delta^{S}}}(\mathcal{A}) \end{aligned}$$

or

$$\begin{aligned} &(|z - a_{j_2\dots j_2}| + r_{j_2}^{i_2}(\mathcal{A}))(|z - a_{i_2\dots i_2}| + r_{i_2}^{\Delta^{S}}(\mathcal{A})) \\ =\ &(|a_{j_2\dots j_2}| + r_{j_2}^{i_2}(\mathcal{A}))(|a_{i_2\dots i_2}| + r_{i_2}^{\Delta^{S}}(\mathcal{A})) < |a_{j_2 i_2\dots i_2}|(2|a_{i_2 j_2\dots j_2}| - r_{i_2}^{\overline{\Delta^{S}}}(\mathcal{A})), \end{aligned}$$

or $|\lambda - a_{j_2\dots j_2}| = |a_{j_2\dots j_2}| > r_{j_2}(\mathcal{A})$. These lead to a contradiction. Hence, $\lambda \neq 0$ and $\mathcal{A}$ is nonsingular. $\square$

We will verify the advantages of Theorem 3.8 by Example 3.9.

Example 3.9. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ with elements defined as follows:*

$$a_{111} = 11,\ a_{222} = 10,\ a_{333} = 12 + \mathrm{i},\ a_{444} = 20,\ a_{122} = 1 + \mathrm{i},\ a_{133} = 4.7, a_{144} = 26 - \mathrm{i},$$
$$a_{211} = -2 - \mathrm{i},\ a_{233} = 3 - \mathrm{i},\ a_{322} = 2,\ a_{344} = 5 + \mathrm{i},\ a_{411} = 16,\ a_{422} = 2,\ a_{433} = 0.85$$

and other elements of $\mathcal{A}$ are zeros.

After some calculations, we can validate that the tensor $\mathcal{A}$ is not in accordance with the conditions of Corollaries 1 and 3 in [20], Corollaries 3.2 and 3.4 of [2], Corollaries 1 and 2 of [28] and Corollary 2.4 of [35]. Actually, by some calculations, we have

$$
\begin{aligned}
&(|a_{111}| - r_1^{\Delta_1}(\mathcal{A}))|a_{222}| = 110 < 173.4674 = r_1^{\overline{\Delta_1}}(\mathcal{A})r_2(\mathcal{A}),\\
&(|a_{111}| + r_1^2(\mathcal{A}))(|a_{222}| + r_2^{\overline{\Delta_1}}(\mathcal{A})) = 549.1200 > 1.8524 = |a_{122}|(2|a_{211}| - r_2^{\Delta_1}(\mathcal{A})),
\end{aligned}
$$

which means that Corollary 1 of [28] can not be used to determine the nonsingularity of $\mathcal{A}$.

Besides, we can check that

$$
\begin{aligned}
&(|a_{222}| - r_2^{\Delta_2}(\mathcal{A}))|a_{111}| = 110 < 173.4674 = r_2^{\overline{\Delta_2}}(\mathcal{A})r_1(\mathcal{A}),\\
&(|a_{222}| + r_2^1(\mathcal{A}))|a_{111}| = 144.7851 > -65.5280 = (2|a_{122}| - r_1(\mathcal{A}))|a_{211}|,\\
&(|a_{222}| + r_2^3(\mathcal{A}))|a_{333}| = 27.3607 > -9.8 = (2|a_{322}| - r_3(\mathcal{A}))|a_{233}|,\\
&(|a_{222}| + r_2^4(\mathcal{A}))|a_{444}| = 307.9669 > 0 = (2|a_{422}| - r_4(\mathcal{A}))|a_{244}|,
\end{aligned}
$$

then we can not apply Corollary 2 of [28] to identify the nonsingularity of $\mathcal{A}$.

Since $a_{144} = 26 - \mathrm{i} \neq 0$ and

$$
\begin{aligned}
&|a_{111}|^2|a_{444}| = 2420 < 19464 = (r_1(\mathcal{A}))^2 r_4(\mathcal{A}),\\
&|a_{111}||a_{444}|^2 = 4400 < 11418 = r_1(\mathcal{A})(r_4(\mathcal{A}))^2,
\end{aligned}
$$

which shows that the conditions of Corollaries 3.2 and 3.4 of [2] are not satisfied.

Meanwhile, we can verify that

$$
\begin{aligned}
&|a_{111}| = 11 < 32.1334 = r_1(\mathcal{A}),\ |a_{222}| = 10 > -0.9262 = 2|a_{211}| - r_1(\mathcal{A}),\\
&|a_{333}| = 12.0416 > -7.0990 = 2|a_{311}| - r_3(\mathcal{A}),\ |a_{444}| = 20 > 13.15 = 2|a_{411}| - r_4(\mathcal{A}),
\end{aligned}
$$

and see that we can not use Corollary 1 of [20] to determine the nonsingularity of $\mathcal{A}$.

Moreover, it follows from

$$
\begin{aligned}
&(|a_{111}| - r_1^2(\mathcal{A}))|a_{222}| = -197.1922 < 7.6344 = |a_{122}|r_2(\mathcal{A}),\\
&(|a_{111}| + r_1^2(\mathcal{A}))|a_{222}| = 417.1922 > -1.3099 = |a_{122}|(2|a_{211}| - r_2(\mathcal{A})),\\
&(|a_{111}| - r_1^3(\mathcal{A}))|a_{333}| = -36.7463 < 33.3654 = |a_{133}|r_3(\mathcal{A}),\\
&(|a_{111}| + r_1^3(\mathcal{A}))|a_{333}| = 85.9398 > -33.3654 = |a_{133}|(2|a_{311}| - r_3(\mathcal{A}))\\
&(|a_{111}| - r_1^4(\mathcal{A}))|a_{444}| = 97.7157 < 490.4624 = |a_{144}|r_4(\mathcal{A}),\\
&(|a_{111}| + r_1^4(\mathcal{A}))|a_{444}| = 342.2843 > 342.1528 = |a_{144}|(2|a_{411}| - r_4(\mathcal{A}))
\end{aligned}
$$

that Corollary 2.4 of [35] can not be used to identify the nonsingularity of $\mathcal{A}$. What is more, as mentioned in Remark 2.2 of [35], the condition of Corollary 2.4 of [35] is weaker than that in Corollary 3 of [20], thus we also can not apply Corollary 3 of [20] to identify the nonsingularity of $\mathcal{A}$ in this example.

However, we select $S = \{1, 2\}$, $\bar{S} = \{3, 4\}$, and employ Theorem 3.8, then the following results are derived:

$$
\begin{aligned}
&|a_{111}| = 11 > 1.4142 = r_1^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\ |a_{222}| = 10 > 2.2361 = r_2^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
&|a_{333}| = 12.0416 > 5.0990 = r_3^{\overline{\Delta^{S}}}(\mathcal{A}),\ |a_{444}| = 20 > 0.85 = r_4^{\overline{\Delta^{S}}}(\mathcal{A}),\\
&(|a_{111}| - r_1^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|a_{333}| - r_3^{\Delta^{\bar{S}}}(\mathcal{A})) = 66.55 > 61.4384 = r_1^{\Delta^{\bar{S}}}(\mathcal{A})r_3^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
&(|a_{111}| + r_1^4(\mathcal{A}))(|a_{444}| + r_4^{\Delta^{\bar{S}}}(\mathcal{A})) = 356.8314 < 364.2691 = |a_{144}|(2|a_{411}| - r_4^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})),\\
&(|a_{222}| - r_2^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|a_{333}| - r_3^{\Delta^{\bar{S}}}(\mathcal{A})) = 53.9017 > 6.3246 = r_2^{\Delta^{\bar{S}}}(\mathcal{A})r_3^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
&(|a_{222}| - r_2^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|a_{444}| - r_4^{\Delta^{\bar{S}}}(\mathcal{A})) = 194 > 56.9210 = r_2^{\Delta^{\bar{S}}}(\mathcal{A})r_4^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
&(|a_{333}| - r_3^{\overline{\Delta^{S}}}(\mathcal{A}))(|a_{111}| - r_1^{\Delta^{S}}(\mathcal{A})) = 66.55 > 61.4384 = r_3^{\Delta^{S}}(\mathcal{A})r_1^{\overline{\Delta^{S}}}(\mathcal{A}),
\end{aligned}
$$

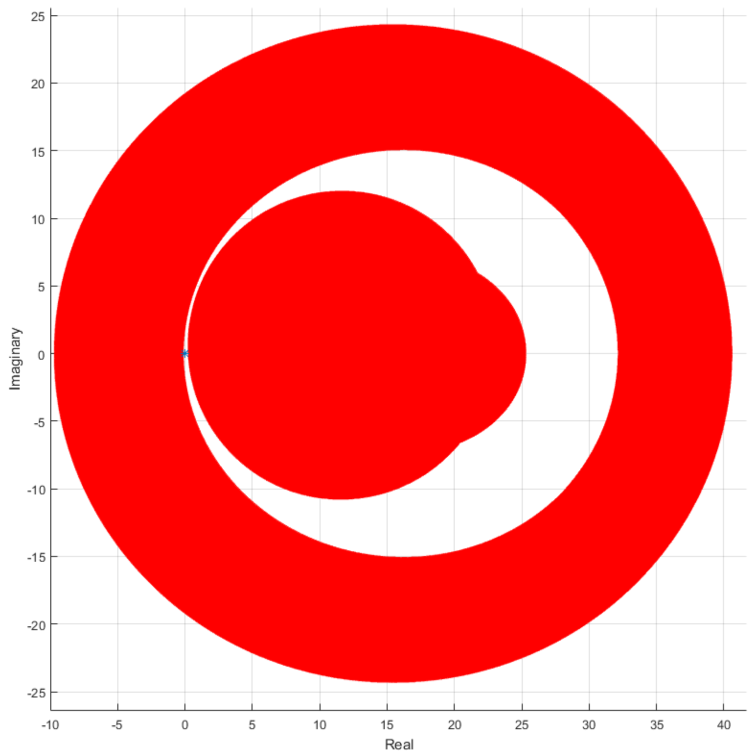

Figure 3: The eigenvalue localization $G^S(\mathcal{A})$.

$$
\begin{aligned}
&(|a_{333}| - r_3^{\overline{\Delta^S}}(\mathcal{A}))(|a_{222}| - r_2^{\Delta^S}(\mathcal{A})) = 53.9017 > 6.3246 = r_3^{\Delta^S}(\mathcal{A}) r_2^{\overline{\Delta^S}}(\mathcal{A}),\\
&(|a_{444}| + r_4^1(\mathcal{A}))(|a_{111}| + r_1^{\Delta^S}(\mathcal{A})) = 280.5621 < 341.1076 = |a_{411}|(2|a_{144}| - r_1^{\overline{\Delta^S}}(\mathcal{A})),\\
&(|a_{444}| - r_4^{\overline{\Delta^S}}(\mathcal{A}))(|a_{222}| - r_2^{\Delta^S}(\mathcal{A})) = 135.0924 > 50.5964 = r_4^{\Delta^S}(\mathcal{A}) r_2^{\overline{\Delta^S}}(\mathcal{A}).
\end{aligned}
$$

Therefore, the tensor $\mathcal{A}$ fulfills the conditions (i)-(iv) of Theorem 3.8, thus $\mathcal{A}$ is nonsingular. This fact can also be seen in Figure 3 since $0 \notin G^S(\mathcal{A})$. Here, the asterisk in Figure 3 denotes the original point.

3.3. *New bounds for the spectral radius of nonnegative tensors*

Founded on the results in Section 3.1, we establish a new lower bound for the spectral radius of nonnegative tensors in this section, which together with Theorem 5.1 of [17] gives the following theorem.

Theorem 3.10. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be irreducible nonnegative with $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\frac{1}{2}\max\{\Gamma_1, \Gamma_2\} = \eta_{\min}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \eta_{\max}(\mathcal{A}) = \max\{\eta_1(\mathcal{A}), \eta_2(\mathcal{A}), \eta_3(\mathcal{A}), \eta_4(\mathcal{A})\},$$

where

$$
\begin{aligned}
\eta_1(\mathcal{A}) &= \max_{i\in S}\{a_{i\ldots i} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\}, \quad \eta_2(\mathcal{A}) = \max_{i\in \bar{S}}\{a_{i\ldots i} + r_i^{\overline{\Delta^{S}}}(\mathcal{A})\},\\
\eta_3(\mathcal{A}) &= \max_{i\in S, j\in\bar{S}} \min\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\},\\
\eta_4(\mathcal{A}) &= \max_{i\in \bar{S}, j\in S} \min\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} + r_i^{\overline{\Delta^{S}}}(\mathcal{A}) + r_j^{\Delta^{S}}(\mathcal{A}) + \Pi_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\},
\end{aligned}
$$

$$\Gamma_1 = \min\left\{\min_{i\in S, j\in \bar{S}}\{a_{i\ldots i} + a_{j\ldots j} - r_i^j(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Psi_{i,j}^{\frac{1}{2}}(\mathcal{A})\}, \min_{i\in \bar{S}, j\in S}\{a_{i\ldots i} + a_{j\ldots j} - r_i^j(\mathcal{A}) - r_j^{\Delta^{S}}(\mathcal{A}) + \Xi_{i,j}^{\frac{1}{2}}(\mathcal{A})\}\right\},$$

$$\Gamma_2 = \min\left\{\min_{i\in S, j\in \bar{S}}\{a_{i\ldots i} + a_{j\ldots j} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{i,j}^{\frac{1}{2}}(\mathcal{A})\}, \min_{i\in \bar{S}, j\in S}\{a_{i\ldots i} + a_{j\ldots j} + r_i^{\overline{\Delta^{S}}}(\mathcal{A}) + r_j^{\Delta^{S}}(\mathcal{A}) + \Pi_{i,j}^{\frac{1}{2}}(\mathcal{A})\}\right\}$$

with

$$\begin{aligned}
\Phi_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
\Pi_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} + r_i^{\overline{\Delta^{S}}}(\mathcal{A}) - r_j^{\Delta^{S}}(\mathcal{A}))^2 + 4r_i^{\Delta^{S}}(\mathcal{A})r_j^{\overline{\Delta^{S}}}(\mathcal{A}),\\
\Psi_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} - r_i^j(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4a_{ij\ldots j}(2a_{ji\ldots i} - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})),\\
\Xi_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} - r_i^j(\mathcal{A}) + r_j^{\Delta^{S}}(\mathcal{A}))^2 + 4a_{ij\ldots j}(2a_{ji\ldots i} - r_j^{\overline{\Delta^{S}}}(\mathcal{A})).
\end{aligned} \tag{23}$$

Here, if $\Psi_{i,j}(\mathcal{A}) < 0$ ($i \in S, j \in \bar{S}$) or $\Xi_{i,j}(\mathcal{A}) < 0$ ($i \in \bar{S}, j \in S$), we assume that $\nabla_{i,j}(\mathcal{A}) = 0$ or $\xi_{i,j}(\mathcal{A}) = 0$, respectively, where $\nabla_{i,j}(\mathcal{A})$ and $\xi_{i,j}(\mathcal{A})$ are defined as in (29) *and* (31), *respectively.*

Proof. Inasmuch as $\mathcal{A}$ is a nonnegative tensor, from Lemma 2.1, we see that $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, then it follows from Theorem 3.1 that $\rho(\mathcal{A}) \in G^S(\mathcal{A}) := \left(G_{i,j}^S(\mathcal{A})\right)\bigcup\left(G_{i,j}^{\bar{S}}(\mathcal{A})\right)$. If $\rho(\mathcal{A}) \in \bigcup\limits_{i\in S}\hat{\Upsilon}_i^1(\mathcal{A})$ or $\rho(\mathcal{A}) \in \bigcup\limits_{i\in\bar{S}}\hat{\Upsilon}_i^2(\mathcal{A})$, it is not difficult to obtain

$$\rho(\mathcal{A}) \le \max\left\{\max_{i\in S}\{a_{i\ldots i} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\}, \max_{i\in\bar{S}}\{a_{i\ldots i} + r_i^{\overline{\Delta^{S}}}(\mathcal{A})\}\right\} \tag{24}$$

in view of the proof of Theorem 5.1 of [17]. If $\rho(\mathcal{A}) \in \bigcup_{i\in S, j\in\bar{S}}\left(\left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A})\backslash G_{i,j}^1(\mathcal{A})\right)\bigcap\Gamma_i(\mathcal{A})\right)$, then there exist $p \in S$ and $q \in \bar{S}$ such that

$$|\rho(\mathcal{A}) - a_{p\ldots p}| \le r_p(\mathcal{A}), \tag{25}$$

$$(|\rho(\mathcal{A}) - a_{p\ldots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|\rho(\mathcal{A}) - a_{q\ldots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \le r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}), \tag{26}$$

$$(|\rho(\mathcal{A}) - a_{p\ldots p}| + r_p^q(\mathcal{A}))(|\rho(\mathcal{A}) - a_{q\ldots q}| + r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \ge |a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})). \tag{27}$$

Combining Lemma 2.2 with (25)-(26), we derive

$$\rho(\mathcal{A}) \le \min\left\{R_p(\mathcal{A}), \frac{1}{2}\{a_{p\ldots p} + a_{q\ldots q} + r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + r_q^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{p,q}^{\frac{1}{2}}(\mathcal{A})\}\right\}, \tag{28}$$

where $\Phi_{p,q}(\mathcal{A}) = (a_{p\ldots p} - a_{q\ldots q} + r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$.

If $|a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \le 0$, then (27) is valid for $\rho(\mathcal{A}) \ge 0$. If $|a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) > 0$, then (27) yields

$$\rho(\mathcal{A}) \ge \frac{1}{2}\{a_{p\ldots p} + a_{q\ldots q} - r_p^q(\mathcal{A}) - r_q^{\Delta^{\bar{S}}}(\mathcal{A}) + \Psi_{p,q}^{\frac{1}{2}}(\mathcal{A})\} := \frac{1}{2}\nabla_{p,q}(\mathcal{A}) \tag{29}$$

with $\Psi_{p,q}(\mathcal{A}) = (a_{p\ldots p} - a_{q\ldots q} - r_p^q(\mathcal{A}) + r_q^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4a_{pq\ldots q}(2a_{qp\ldots p} - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))$.

Combining (28)-(29) gives

$$\begin{aligned}
&\frac{1}{2}\min_{i\in S, j\in\bar{S}}\left\{a_{i\ldots i} + a_{j\ldots j} - r_i^j(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Psi_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}\\
\le\ &\rho(\mathcal{A}) \le \max_{i\in S, j\in\bar{S}}\min\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} + r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\}.
\end{aligned} \tag{30}$$

Furthermore, for the case that $\rho(\mathcal{A}) \in \bigcup_{i\in\bar{S},j\in S}\left(\left(\tilde{\Upsilon}^2_{i,j}(\mathcal{A})\backslash G^2_{i,j}(\mathcal{A})\right)\bigcap \Gamma_i(\mathcal{A})\right)$, in the similar manner applied in Inequality (30), we can derive

$$\begin{aligned}&\frac{1}{2}\min_{i\in\bar{S},j\in S}\{\xi_{i,j}(\mathcal{A})\} := \frac{1}{2}\min_{i\in\bar{S},j\in S}\left\{a_{i\dots i}+a_{j\dots j}-r_i^j(\mathcal{A})-r_j^{\Delta^S}(\mathcal{A})+\Xi^{\frac{1}{2}}_{i,j}(\mathcal{A})\right\}\\ \le\ & \rho(\mathcal{A}) \le \max_{i\in\bar{S},j\in S}\min\left\{\frac{1}{2}\left(a_{i\dots i}+a_{j\dots j}+r_i^{\overline{\Delta^S}}(\mathcal{A})+r_j^{\Delta^S}(\mathcal{A})+\Pi^{\frac{1}{2}}_{i,j}(\mathcal{A})\right), R_i(\mathcal{A})\right\},\end{aligned} \tag{31}$$

where $\Pi_{i,j}(\mathcal{A})$ and $\Xi_{i,j}(\mathcal{A})$ are defined as in (23).

By Lemma 2.1, there exists a vector $y = (y_1,\dots,y_n)^T > 0$ such that

$$\mathcal{A}y^{m-1} = \rho(\mathcal{A})y^{[m-1]}. \tag{32}$$

Let $y_t = \max_{i\in S}\{y_i\}$ and $y_s = \max_{i\in\bar{S}}\{y_i\}$.

If $y_t \le y_s$, then $y_t = \min_{i\in N}\{y_i\}$. It follows from (32) that

$$\begin{aligned}(\rho(\mathcal{A})-a_{t\dots t})y_t^{m-1} &= \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta^{\bar{S}}},\\ \delta_{ti_2\dots i_m}=0}} a_{ti_2\dots i_m}y_{i_2}\cdots y_{i_m} + \sum_{(i_2,\dots,i_m)\in\Delta^{\bar{S}}} a_{ti_2\dots i_m}y_{i_2}\cdots y_{i_m},\\ (\rho(\mathcal{A})-a_{s\dots s})y_s^{m-1} &= \sum_{(i_2,\dots,i_m)\in\overline{\Delta^{\bar{S}}}} a_{si_2\dots i_m}y_{i_2}\cdots y_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta^{\bar{S}},\\ \delta_{si_2\dots i_m}=0}} a_{si_2\dots i_m}y_{i_2}\cdots y_{i_m}.\end{aligned} \tag{33}$$

Apply the technique utilized in the proof of Theorem 3.1 to (33), we obtain

$$(\rho(\mathcal{A})-a_{t\dots t})y_t^{m-1} \ge r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})y_t^{m-1}+r_t^{\Delta^{\bar{S}}}(\mathcal{A})y_s^{m-1},\ (\rho(\mathcal{A})-a_{s\dots s})y_s^{m-1} \ge r_s^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})y_t^{m-1}+r_s^{\Delta^{\bar{S}}}(\mathcal{A})y_s^{m-1},$$

which leads to

$$(\rho(\mathcal{A})-a_{t\dots t}-r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))y_t^{m-1} \ge r_t^{\Delta^{\bar{S}}}(\mathcal{A})y_s^{m-1},\ (\rho(\mathcal{A})-a_{s\dots s}-r_s^{\Delta^{\bar{S}}}(\mathcal{A}))y_s^{m-1} \ge r_s^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})y_t^{m-1},$$

Therefore, by $y_s \ge y_t > 0$, it has

$$(\rho(\mathcal{A})-a_{t\dots t}-r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(\rho(\mathcal{A})-a_{s\dots s}-r_s^{\Delta^{\bar{S}}}(\mathcal{A})) \ge r_t^{\Delta^{\bar{S}}}(\mathcal{A})r_s^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}). \tag{34}$$

By solving Inequality (34), we obtain

$$\begin{aligned}\rho(\mathcal{A}) &\ge \frac{1}{2}\{a_{t\dots t}+a_{s\dots s}+r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})+r_s^{\Delta^{\bar{S}}}(\mathcal{A})+\Phi^{\frac{1}{2}}_{t,s}(\mathcal{A})\}\\ &\ge \min_{i\in S,j\in\bar{S}}\frac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})+r_j^{\Delta^{\bar{S}}}(\mathcal{A})+\Phi^{\frac{1}{2}}_{i,j}(\mathcal{A})\}.\end{aligned} \tag{35}$$

Similarly, we can obtain

$$\rho(\mathcal{A}) \ge \min_{i\in\bar{S},j\in S}\frac{1}{2}\{a_{i\dots i}+a_{j\dots j}+r_i^{\overline{\Delta^S}}(\mathcal{A})+r_j^{\Delta^S}(\mathcal{A})+\Pi^{\frac{1}{2}}_{i,j}(\mathcal{A})\} \tag{36}$$

for the case $y_t \ge y_s > 0$. From Inequalities (24), (30)-(31) and (35)-(36), the conclusion is obtained. □

Remark 3.11. *Using the similar method as Theorem 3.4 in [15], the results of Theorem 3.10 can be extended to general nonnegative tensors; without the condition of irreducibility, compared with Theorem 3.10.*

Remark 3.12. *Remark 5.1 of [17] shows that the upper bound in Theorem 3.10 is better than those of Lemma 5.2 of [37], Theorems 3.3 and 3.4 in [22]. Meanwhile, the numerical results of Example 5.1 in [17] illustrate that the upper bound in Theorem 3.10 is tighter than that in Theorem 13 of [18] for some cases.*

The following example further shows that the new bounds in Theorem 3.10 are superior to those in Theorem 3.3 of [15], Theorem 3.3 of [24] and Theorem 5 of [25] in some cases.

Example 3.13. [17] *Consider the following nonnegative tensor*

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{R}^{[3,3]},$$

where

$$A(1,:,:) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}, A(2,:,:) = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A(3,:,:) = \begin{pmatrix} 15 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now compute the bounds for $\rho(\mathcal{A})$. Let $S = \{1, 2\}$, then $\bar{S} = \{3\}$. Careful manipulations obtains the numerical results, which are listed in Table 1. As observed in Table 1, the bounds in Theorem 3.10 are tighter than those Theorem 3.3 of [15], Theorem 3.3 of [24] and Theorem 5 of [25] in some cases.

Table 1: Some upper and lower bounds for $\rho(\mathcal{A})$.

Theorem 3.3 of [15]	Theorem 3.3 of [24]
$\rho(\mathcal{A}) \leq 20.2250$	$\rho(\mathcal{A}) \leq 25.4711$
Theorem 5 of [25]	Theorem 3.10
$8.0828 \leq \rho(\mathcal{A}) \leq 17.6190$	$9.4372 \leq \rho(\mathcal{A}) \leq 16.3808$

3.4. New upper and lower bounds for the minimum H-eigenvalue of weakly irreducible strong M-tensors

In this section, by combining Theorem 6.1 of [17] and the set in Theorem 3.1, we obtain the bounds for the minimum H-eigenvalue of weakly irreducible strong M-tensors, which are better some known ones.

Theorem 3.14. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible strong M-tensor with $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\pi_{\min}(\mathcal{A}) = \min\{\pi_1(\mathcal{A}), \pi_2(\mathcal{A}), \pi_3(\mathcal{A}), \pi_4(\mathcal{A})\} \leq \tau(\mathcal{A}) \leq \pi_{\max}(\mathcal{A}) = \frac{1}{2}\min\{\varpi_1, \varpi_2\},$$

where

$$\pi_1(\mathcal{A}) = \min_{i\in S}\{a_{i\ldots i} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\}, \quad \pi_2(\mathcal{A}) = \min_{i\in \bar{S}}\{a_{i\ldots i} - r_i^{\overline{\Delta^{S}}}(\mathcal{A})\},$$

$$\pi_3(\mathcal{A}) = \min_{i\in S, j\in\bar{S}} \max\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) - \Theta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\},$$

$$\pi_4(\mathcal{A}) = \min_{i\in S, j\in S} \max\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^{S}}}(\mathcal{A}) - r_j^{\Delta^{S}}(\mathcal{A}) - \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\},$$

$$\varpi_1(\mathcal{A}) = \max\left\{\max_{i\in S, j\in\bar{S}}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) - \mathfrak{R}_{i,j}^{\frac{1}{2}}(\mathcal{A})\}, \max_{i\in\bar{S}, j\in S}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + r_j^{\Delta^{S}}(\mathcal{A}) - \Sigma_{i,j}^{\frac{1}{2}}(\mathcal{A})\}\right\},$$

$$\varpi_2(\mathcal{A}) = \max\left\{\max_{i\in S, j\in\bar{S}}\{a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) - \Theta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}, \max_{i\in\bar{S}, j\in S}\{a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^{S}}}(\mathcal{A}) - r_j^{\Delta^{S}}(\mathcal{A}) - \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\}\right\}$$

with

$$\begin{aligned}
\Theta_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),\\
\Lambda_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} - r_i^{\overline{\Delta^{S}}}(\mathcal{A}) + r_j^{\Delta^{S}}(\mathcal{A}))^2 + 4r_i^{\Delta^{S}}(\mathcal{A})r_j^{\overline{\Delta^{S}}}(\mathcal{A}),\\
\mathfrak{R}_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4a_{ij\ldots j}(2a_{ji\ldots i} + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})),\\
\Sigma_{i,j}(\mathcal{A}) &= (a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}) - r_j^{\Delta^{S}}(\mathcal{A}))^2 + 4a_{ij\ldots j}(2a_{ji\ldots i} + r_j^{\overline{\Delta^{S}}}(\mathcal{A})).
\end{aligned} \tag{37}$$

Here, if $\Re_{i,j}(\mathcal{A}) < 0$ $(i \in S, j \in \bar{S})$ or $\Sigma_{i,j}(\mathcal{A}) < 0$ $(i \in \bar{S}, j \in S)$, we assume that $\kappa_{i,j}(\mathcal{A}) = +\infty$ or $\chi_{i,j}(\mathcal{A}) = +\infty$, respectively, where $\kappa_{i,j}(\mathcal{A})$ and $\chi_{i,j}(\mathcal{A})$ are defined as in (43) *and* (45)*, respectively.*

Proof. From the conditions of this theorem, we see that $\mathcal{A}$ is a strong M-tensor, which is weakly irreducible. From Lemma 2.3, $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$. According to Theorem 3.1, it holds that

$$\tau(\mathcal{A}) \in G^S(\mathcal{A}) := \left(G^S_{i,j}(\mathcal{A})\right) \bigcup \left(G^{\bar{S}}_{i,j}(\mathcal{A})\right).$$

If $\tau(\mathcal{A}) \in \bigcup_{i\in S} \hat{\Upsilon}^1_i(\mathcal{A})$ or $\tau(\mathcal{A}) \in \bigcup_{i\in \bar{S}} \hat{\Upsilon}^2_i(\mathcal{A})$, then by Lemma 2.4 and the proof of Theorem 6.1 of [17], we have

$$\tau(\mathcal{A}) \geq \min\left\{\min_{i\in S}\{a_{i\ldots i} - r_i^{\overline{\Delta^S}}(\mathcal{A})\}, \min_{i\in \bar{S}}\{a_{i\ldots i} - r_i^{\overline{\Delta^S}}(\mathcal{A})\}\right\}. \tag{38}$$

For the case that $\tau(\mathcal{A}) \in \bigcup_{i\in S, j\in \bar{S}} \left(\left(\tilde{\Upsilon}^1_{i,j}(\mathcal{A}) \backslash G^1_{i,j}(\mathcal{A})\right) \bigcap \Gamma_i(\mathcal{A})\right)$, there exist $p \in S$ and $q \in \bar{S}$ such that

$$|\tau(\mathcal{A}) - a_{p\ldots p}| \leq r_p(\mathcal{A}), \tag{39}$$

$$(|\tau(\mathcal{A}) - a_{p\ldots p}| - r_p^{\overline{\Delta^S}}(\mathcal{A}))(|\tau(\mathcal{A}) - a_{q\ldots q}| - r_q^{\Delta^S}(\mathcal{A})) \leq r_p^{\Delta^S}(\mathcal{A}) r_q^{\overline{\Delta^S}}(\mathcal{A}), \tag{40}$$

$$(|\tau(\mathcal{A}) - a_{p\ldots p}| + r_p^q(\mathcal{A}))(|\tau(\mathcal{A}) - a_{q\ldots q}| + r_q^{\Delta^S}(\mathcal{A})) \geq |a_{pq\ldots q}|(2|a_{qp\ldots p}| - r_q^{\overline{\Delta^S}}(\mathcal{A})). \tag{41}$$

Recalling that $\tau(\mathcal{A}) \leq \min_{i\in N}\{a_{i\ldots i}\}$, it follows from (39) and (40) that

$$\tau(\mathcal{A}) \geq \max\left\{\frac{1}{2}\left(a_{p\ldots p} + a_{q\ldots q} - r_p^{\overline{\Delta^S}}(\mathcal{A}) - r_q^{\Delta^S}(\mathcal{A}) - \Theta^{\frac{1}{2}}_{p,q}(\mathcal{A})\right), R_p(\mathcal{A})\right\}, \tag{42}$$

where $\Theta_{p,q}(\mathcal{A}) = (a_{p\ldots p} - a_{q\ldots q} - r_p^{\overline{\Delta^S}}(\mathcal{A}) + r_q^{\Delta^S}(\mathcal{A}))^2 + 4r_p^{\Delta^S}(\mathcal{A}) r_q^{\overline{\Delta^S}}(\mathcal{A})$.

Similar to the proof of Theorem 3.10, it follows from (41) that

$$\tau(\mathcal{A}) \leq \frac{1}{2}\{a_{p\ldots p} + a_{q\ldots q} + r_p^q(\mathcal{A}) + r_q^{\Delta^S}(\mathcal{A}) - \Re^{\frac{1}{2}}_{p,q}(\mathcal{A})\} := \kappa_{p,q}(\mathcal{A}) \tag{43}$$

with $\Re_{p,q}(\mathcal{A}) = (a_{p\ldots p} - a_{q\ldots q} + r_p^q(\mathcal{A}) - r_q^{\Delta^S}(\mathcal{A}))^2 + 4a_{pq\ldots q}(2a_{qp\ldots p} + r_q^{\overline{\Delta^S}}(\mathcal{A}))$. (42) and (43) give

$$\begin{aligned} &\min_{i\in S, j\in \bar{S}} \max\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^S}}(\mathcal{A}) - r_j^{\Delta^S}(\mathcal{A}) - \Theta^{\frac{1}{2}}_{i,j}(\mathcal{A})\right), R_i(\mathcal{A})\right\} \\ &\leq \tau(\mathcal{A}) \leq \frac{1}{2}\max_{i\in S, j\in \bar{S}}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + r_j^{\Delta^S}(\mathcal{A}) - \Re^{\frac{1}{2}}_{i,j}(\mathcal{A})\right\}. \end{aligned} \tag{44}$$

If $\tau(\mathcal{A}) \in \bigcup_{i\in \bar{S}, j\in S} \left(\left(\tilde{\Upsilon}^2_{i,j}(\mathcal{A}) \backslash G^2_{i,j}(\mathcal{A})\right) \bigcap \Gamma_i(\mathcal{A})\right)$ holds, with a almost the same method utilized in the above proof, we can derive

$$\begin{aligned} &\min_{i\in \bar{S}, j\in S} \max\left\{\frac{1}{2}\left(a_{i\ldots i} + a_{j\ldots j} - r_i^{\overline{\Delta^S}}(\mathcal{A}) - r_j^{\Delta^S}(\mathcal{A}) - \Lambda^{\frac{1}{2}}_{i,j}(\mathcal{A})\right), R_i(\mathcal{A})\right\} \\ &\leq \tau(\mathcal{A}) \leq \frac{1}{2}\max_{i\in \bar{S}, j\in S}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + r_j^{\Delta^S}(\mathcal{A}) - \Sigma^{\frac{1}{2}}_{i,j}(\mathcal{A})\right\} := \frac{1}{2}\min_{i\in \bar{S}, j\in S}\{\chi_{i,j}(\mathcal{A})\}, \end{aligned} \tag{45}$$

where $\Lambda_{i,j}(\mathcal{A})$ and $\Sigma_{i,j}(\mathcal{A})$ are defined as in (37).

According to Lemma 2.3, there exists a vector $z = (z_1, \ldots, z_n)^T > 0$ with $z_k = \max_{i\in S}\{z_i\}$ and $z_l = \max_{i\in \bar{S}}\{z_i\}$ such that

$$\mathcal{A}z^{m-1} = \tau(\mathcal{A})z^{[m-1]}. \tag{46}$$

If $z_k \le z_l$, then $z_k = \min_{i\in N}\{z_i\}$. It follows from (46) that

$$(a_{k\ldots k}-\tau(\mathcal{A}))z_k^{m-1} = -\sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar S}},\\ \delta_{ki_2\ldots i_m=0}}} a_{ki_2\ldots i_m}z_{i_2}\cdots z_{i_m} - \sum_{(i_2,\ldots,i_m)\in\Delta^{\bar S}} a_{ki_2\ldots i_m}z_{i_2}\cdots z_{i_m},$$

$$(a_{l\ldots l}-\tau(\mathcal{A}))z_l^{m-1} = -\sum_{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar S}}} a_{li_2\ldots i_m}z_{i_2}\cdots z_{i_m} - \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar S},\\ \delta_{li_2\ldots i_m=0}}} a_{li_2\ldots i_m}z_{i_2}\cdots z_{i_m}.$$

By the analogical proof as in Theorem 3.10, we can obtain

$$(a_{k\ldots k}-\tau(\mathcal{A}))z_k^{m-1} \ge r_k^{\overline{\Delta^{\bar S}}}(\mathcal{A})z_k^{m-1} + r_k^{\Delta^{\bar S}}(\mathcal{A})z_l^{m-1},\ (a_{l\ldots l}-\tau(\mathcal{A}))z_l^{m-1} \ge r_l^{\overline{\Delta^{\bar S}}}(\mathcal{A})z_k^{m-1} + r_l^{\Delta^{\bar S}}(\mathcal{A})z_l^{m-1},$$

and hence

$$(a_{k\ldots k}-\tau(\mathcal{A})-r_k^{\overline{\Delta^{\bar S}}}(\mathcal{A}))z_k^{m-1} \ge r_k^{\Delta^{\bar S}}(\mathcal{A})z_l^{m-1},\ (a_{l\ldots l}-\tau(\mathcal{A})-r_l^{\Delta^{\bar S}}(\mathcal{A}))z_l^{m-1} \ge r_l^{\overline{\Delta^{\bar S}}}(\mathcal{A})z_k^{m-1},$$

which together with $z_l \ge z_k > 0$ leads to

$$(a_{k\ldots k}-\tau(\mathcal{A})-r_k^{\overline{\Delta^{\bar S}}}(\mathcal{A}))(a_{l\ldots l}-\tau(\mathcal{A})-r_l^{\Delta^{\bar S}}(\mathcal{A})) \ge r_k^{\Delta^{\bar S}}(\mathcal{A})r_l^{\overline{\Delta^{\bar S}}}(\mathcal{A}).$$

By direct computations, we have

$$\begin{aligned}\tau(\mathcal{A}) &\le \frac{1}{2}\{a_{k\ldots k}+a_{l\ldots l}-r_k^{\overline{\Delta^{\bar S}}}(\mathcal{A})-r_l^{\Delta^{\bar S}}(\mathcal{A})-\Theta_{k,l}^{\frac12}(\mathcal{A})\}\\ &\le \max_{i\in S,j\in\bar S}\frac{1}{2}\{a_{i\ldots i}+a_{j\ldots j}-r_i^{\overline{\Delta^{\bar S}}}(\mathcal{A})-r_j^{\Delta^{\bar S}}(\mathcal{A})-\Theta_{i,j}^{\frac12}(\mathcal{A})\}.\end{aligned}\tag{47}$$

If $z_k \ge z_l > 0$, then by using the similar method in above, it holds that

$$\tau(\mathcal{A}) \le \max_{i\in\bar S,j\in S}\frac{1}{2}\{a_{i\ldots i}+a_{j\ldots j}-r_i^{\overline{\Delta^S}}(\mathcal{A})-r_j^{\Delta^S}(\mathcal{A})-\Lambda_{i,j}^{\frac12}(\mathcal{A})\}.\tag{48}$$

Taking advantage of Inequalities (38), (44)-(45) and (47)-(48), the results of this theorem are got. □

Remark 3.15. *Analogous to the analysis of Theorem 3.6 in [15], the results of Theorem 3.14 can be generalized to more general cases, that is, the condition "weakly irreducible" in Theorem 3.14 can be removed and the results of that remain true.*

Remark 3.16. *It can be seen from Remark 6.1 of [17] that the lower bound in Theorem 3.14 is an improvement on those in Theorems 2.1 and 2.2 of [12], and it outperforms the one in Theorem 4.5 of [36]. The corresponding numerical results also illustrate these facts.*

The computing results of the following example show the advantage of the new bounds in Theorem 3.14 over the results in Theorem 3.5 of [15], Theorems 3.1-3.2 of [16] and Theorem 2.3 of [9] in some cases.

Example 3.17. [9] *Consider the following weakly irreducible nonsingular* M*-tensor*

$$\mathcal{A} = [A(1,:,:),A(2,:,:),A(3,:,:),A(4,:,:)] \in \mathbb{R}^{[3,4]},$$

where

$$A(1,:,:) = \begin{pmatrix} 37 & -2 & -1 & -4\\ -1 & -3 & -3 & -2\\ -1 & -1 & -3 & -2\\ -2 & -3 & -3 & -3\end{pmatrix},\ A(2,:,:) = \begin{pmatrix} -2 & -4 & -2 & -3\\ -1 & 39 & -2 & -1\\ -3 & -3 & -4 & -2\\ -2 & -3 & -1 & -4\end{pmatrix},$$

$$A(3,:,:) = \begin{pmatrix} -4 & -1 & -1 & -1\\ -1 & 0 & -2 & -3\\ -1 & -1 & 35 & -1\\ -2 & -2 & -4 & -3\end{pmatrix},\ A(4,:,:) = \begin{pmatrix} -2 & -4 & 0 & -1\\ -4 & -4 & -2 & -4\\ -3 & 0 & -3 & -3\\ -3 & -3 & -4 & 49\end{pmatrix}.$$

In Table 2, we contrast the upper and lower bounds in Theorem 3.14 with the ones which have been derived. In this example, we consider $S = \{1, 2\}$ and $\bar{S} = \{3, 4\}$.

Table 2: Some upper and lower bounds for $\tau(\mathcal{A})$.

Theorem 3.1 of [16]	Theorem 3.2 of [16]	Theorem 3.5 of [15]
$2.6604 \le \tau(\mathcal{A}) \le 8.1955$	$2.3993 \le \tau(\mathcal{A}) \le 8.5460$	$2.6604 \le \tau(\mathcal{A}) \le 8.1955$
Theorem 4.5 of [36]	Theorem 2.3 of [9]	Theorem 3.14
$2.2233 \le \tau(\mathcal{A}) \le 8.7447$	$3.5550 \le \tau(\mathcal{A}) \le 7.1629$	$3.6617 \le \tau(\mathcal{A}) \le 7$

From the observations in Table 2, it can be easily viewed that the bounds in Theorem 3.14 are better than those in Theorems 3.1-3.2 of [16], Theorem 3.5 of [15] and Theorem 2.3 of [9].

4. Conclusions

A modified S-type eigenvalue localization set for tensors is developed in this paper, which is more precise compared with those in [17, 19, 22]. By utilizing this new set, a new sufficient criteria which has wider scope of applications compared with those of [2, 20, 28, 35] for the nonsingularity of tensors, and tighter bounds for the spectral radius of nonnegative tensors and the minimum H-eigenvalue of strong M-tensors are obtained.

Besides, we should mention that there are some meaningful problems, which are need to be studied in the future. There are

- The choices of S not only depend on the form of $G^S(\mathcal{A})$, but also rely on the structure of a given tensor. Finding the best choice for S makes that the set $G^S(\mathcal{A})$ is the sharpest.
- Seek new methods to obtain more accurate estimations by partitioning N into three or more parts.
- Investigate S-type eigenvalue localizations for other tensor eigenvalues, such as E-/Z-eigenvalues, generalized tensor eigenvalue and so forth.

Competing interests

The authors declare that they have no competing interests.

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