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Generalizations of Some Conditions for Drazin Inverses of the Sum of Two Matrices

Ramesh Yousefi[a], **Mansour Dana**[a]

[a] *Faculty of Mathematics, University of Kurdistan, Sanandaj, P. O. Box: 416, Islamic Republic of Iran*

Abstract. In this article, we present some formulas of the Drazin inverses of the sum of two matrices under the conditions $P^2QP = 0$, $P^2Q^2 = 0$, $QPQ = 0$ and $PQP^2 = 0$, $PQ^2 = 0$, $QP^3 = 0$ respectively. These conditions are weaker than those used in some literature on this subject. Furthermore, we apply our results to give the representations for the Drazin inverses of block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (A and D are square matrices) with generalized Schur complement is zero.

1. Introduction

Let A be a square complex matrix. As we know, the Drazin inverse [1] of A, denoted by A^d, is the unique matrix satisfying the following three equations

$$A^{k+1}A^d = A^k, \qquad A^dAA^d = A^d, \qquad AA^d = A^dA,$$

where k is the smallest non-negative integer such that $rank(A^{k+1}) = rank(A^k)$, i.e., $k = ind(A)$, the index of A. In the case that $ind(A) = 1$, the Drazin inverse is called the group inverse of A and it is denoted by $A^\sharp$. Clearly, $ind(A) = 0$ if and only if A is nonsingular, and in that case $A^d = A^{-1}$. We denote by A^π the eigenprojection of A corresponding to the eigenvalue 0 that is given by $A^\pi = I - AA^d$.

Suppose $P, Q \in \mathbb{C}^{n\times n}$. In 1958, Drazin (see [7]) studied the problem of finding the formula for $(P+Q)^d$ and he offered the formula $(P+Q)^d = P^d + Q^d$, which is valid when $PQ = QP = 0$. In recent years, many papers focused on the problem under some weaker conditions. According to current literature, there is no formula for $(P+Q)^d$ without any side condition for matrices P and Q, so this problem is still the open one. Formulas for $(P+Q)^d$ can be very useful for deriving formulas for the Drazin inverse of a 2×2 block matrix. Actually, in 1979 Campbell and Meyer [3], posed the problem of finding an explicit representation for the Drazin inverse of a complex block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, in terms of its blocks, where A and D are square matrices, not necessarily of the same size. Until now, there has been no formula for M^d without any side conditions for blocks of matrix M. Some results about the representations of $(P+Q)^d$ and M^d under some conditions were given. Here we list the results below:

(1) Results of the representations of $(P+Q)^d$ under the following conditions respectively:

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Email addresses: Ramesh.yousefi@yahoo.com (Ramesh Yousefi), mdana@uok.ac.ir (Mansour Dana)

(1-1) $PQ = 0$ (see [9]);

(1-2) $P^2Q = 0$, $Q^2 = 0$ (see [10]);

(1-3) $Q^2P = 0$, $PQP = 0$ (see [13]);

(1-4) $P^2Q = 0$, $Q^2P = 0$ (see [2]).

Other results have been studied in [4–6, 12].

(2) Results of the representations of M^d under the following conditions respectively:

(2-1) $CA^\pi = 0$, $A^\pi B = 0$ and $S = 0$ (see [11]);

(2-2) $CA^\pi B = 0$, $AA^\pi B = 0$ and $S = 0$ (see [8]);

(2-3) $CA^\pi B = 0$, $CAA^\pi = 0$ and $S = 0$ (see [8]);

(2-4) $AA^\pi BC = 0$, $CA^\pi BC = 0$ and $S = 0$ (see [13]);

(2-5) $BCAA^\pi = 0$, $BCA^\pi B = 0$ and $S = 0$ (see [13]);

(2-6) $AA^\pi BC = 0$, $ABCA^\pi = 0$ and $S = 0$ (see [2]).

We organize this article in three sections. In section 2, we give the formula of $(P + Q)^d$ under the conditions $P^2QP = 0$, $P^2Q^2 = 0$ and $QPQ = 0$. These result extend the formula (1-1) and (1-3) above. Then we apply the computational formula to give the computational formulas for M^d, under some conditions with generalized Schur complement $S = 0$. Here we list them below:

(a). $AA^\pi BC = 0$, $BCA^\pi B = 0$ and $S = 0$;

(b). $BCAA^\pi = 0$, $CA^\pi BC = 0$, and $S = 0$.

The result (a) generalizes the results (2-1) and (2-2) listed above. The result (b) generalizes the result (2-1), (2-2) and (2-3) listed above.

In section 3, we present our additive formula under conditions $PQP^2 = 0$, $PQ^2 = 0$ and $QP^3 = 0$. And we derive some new representations for M^d, under some conditions with generalized Schur complement $S = 0$. Here we list them below:

(a) $BCA^d = 0$, $BCAA^\pi B = 0$, $CA^2A^\pi = 0$ and $S = 0$;

(b) $A^dBC = 0$, $CAA^\pi BC = 0$, $A^2A^\pi B = 0$ and $S = 0$.

Before giving the main results, we first introduce several lemmas as follows.

Lemma 1.1 ([1]). *Let $P \in \mathbb{C}^{m\times n}$ and $Q \in \mathbb{C}^{n\times m}$. Then $(PQ)^d = P((QP)^d)^2Q$.*

Lemma 1.2 ([9]). *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that $ind(P) = s$ and $ind(Q) = t$. If $PQ = 0$, then*

$$(P + Q)^d = Q^\pi \sum_{i=0}^{t-1} Q^i(P^d)^{i+1} + \sum_{i=0}^{s-1}(Q^d)^{i+1}P^iP^\pi.$$

Lemma 1.3 ([1]). *Let*

$$M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \qquad M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix},$$

where A and B are square matrices with $ind(A) = r$ and $ind(B) = s$. Then

$$M_1^d = \begin{pmatrix} A^d & 0 \\ X & B^d \end{pmatrix}, \qquad M_2^d = \begin{pmatrix} B^d & X \\ 0 & A^d \end{pmatrix},$$

where

$$X = \sum_{i=0}^{r-1}(B^d)^{i+2}CA^iA^\pi + B^\pi \sum_{i=0}^{s-1} B^iC(A^d)^{i+2} - B^dCA^d.$$

Lemma 1.4 ([11]). *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *(A and D are square), if* $S = D - CA^dB = 0$, $A^\pi B = 0$, $CA^\pi = 0$, *then*

$$M^d = \begin{pmatrix} I \\ CA^d \end{pmatrix} ((AW)^d)^2 A \begin{pmatrix} I & A^dB \end{pmatrix}$$

where $W = AA^d + A^dBCA^d$.

2. Additive result (Conditions: $P^2QP = 0$, $P^2Q^2 = 0$, $QPQ = 0$)

Theorem 2.1. *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $P^2QP = 0$, $P^2Q^2 = 0$ *and* $QPQ = 0$, *then*

$$(P+Q)^d = (P+Q)\left(-(Q^d)^2P^d - Q^d(P^d)^2 + \sum_{i=0}^{l-1} Q^\pi Q^i (P^d)^{i+3} + \sum_{i=0}^{r-1} (Q^d)^{i+3} P^\pi P^i\right)(P+Q),$$

where $r = ind(P)$ *and* $l = ind(Q)$.

Proof. From the definition of the Drazin inverse, we have that

$$(P+Q)^d = (P+Q)^2((P+Q)^d)^3 = (P+Q)^2(P^3 + P^2Q + PQP + QPQ + Q^3 + Q^2P + PQ^2 + QP^2)^d.$$

Denote by $F = P^3 + P^2Q + PQP$ and $G = QPQ + Q^3 + Q^2P + PQ^2 + QP^2$. From $P^2QP = 0$, $P^2Q^2 = 0$ and $QPQ = 0$, we get $FG = 0$. Then applying Lemma 1.2, we obtain

$$(P+Q)^d = (P+Q)^2 \underbrace{\left(\sum_{i=0}^{ind(G)-1} G^\pi G^i (F^d)^{i+1} + \sum_{i=0}^{ind(F)-1} (G^d)^{i+1} F^i F^\pi\right)}_{*}. \tag{1}$$

Now, we calculate F^d. Consider the following splitting

$$F = (P^2Q) + (P^3 + PQP) := A + B.$$

According to the condition $P^2QP = 0$, we have $AB = 0$ and $A^2 = 0$. Applying Lemma 1.2, we get

$$(F^d)^n = (B^d)^n + (B^d)^{n+1}A \tag{2}$$

for every $n \in \mathbb{N}$. Notice that $B = P^3 + PQP$. From $P^2QP = 0$, we get $(PQP)^2 = 0$, $(PQP)^d = 0$. Matrices P^3 and PQP satisfy condition of Lemma 1.2. After applying Lemma 1.2, we obtain

$$(B^d)^n = (P^d)^{3n} + PQ(P^d)^{3n+2} \tag{3}$$

for every $n \in \mathbb{N}$. Substituting (3) into (2) we obtain

$$(F^d)^n = \left((P^d)^{3n+1} + PQ(P^d)^{3(n+1)}\right)(P+Q) \tag{4}$$

for every $n \in \mathbb{N}$. Next, we will compute G^d. Consider the following splitting

$$G = (Q^3 + Q^2P + QP^2) + (PQ^2) := S + T.$$

According to the conditions $P^2Q^2 = 0$ and $QPQ = 0$ we have $ST = 0$ and $T^2 = 0$. Applying Lemma 1.2, we get

$$(G^d)^n = (S^d)^n + T(S^d)^{n+1} \tag{5}$$

for every $n \in \mathbb{N}$. Let $S = S_1 + S_2$, where $S_1 = Q^2P + QP^2$ and $S_2 = Q^3$. According to the conditions $P^2Q^2 = 0$, $QPQ = 0$ and $P^2QP = 0$, we have $S_1S_2 = 0$ and $S_1^2 = 0$. Applying Lemma 1.2, we get

$$(S^d)^n = (Q^d)^{3n} + (Q^d)^{3n+2}P^2 + (Q^d)^{3n+1}P \tag{6}$$

for arbitrary $n \in \mathbb{N}$. Substituting (6) into (5) we obtain

$$(G^d)^n = (P+Q)\Big((Q^d)^{3n+2}(P+Q) + (Q^d)^{3(n+1)}P^2\Big) \tag{7}$$

for every $n \in \mathbb{N}$. After computation we get:

$$\begin{cases} F^n = (PQP^{3(n-1)} + P^{3n-1})(P+Q), & \text{if } n \geq 2, \\ G^n = (P+Q)(Q^{3n-2}(P+Q) + Q^{3(n-1)}P^2), & \text{if } n \geq 2. \end{cases}$$

After substituting this expressions, (7) and (4) into $*$ we have

$$\sum_{i=2}^{ind(G)-1} G^\pi G^i (F^d)^{i+1} = \sum_{i=4}^{ind(Q)-1} Q^\pi Q^i (P^d)^{i+4}(P+Q) + \sum_{i=3}^{ind(Q)-1} PQ^\pi Q^i (P^d)^{i+5}(P+Q).$$

Also,

$$G^\pi F^d = \Big(Q^\pi (P^d)^4 + PQ(P^d)^6 - (Q^d)^2(P^d)^2 - Q^d(P^d)^3 - PQ^d(P^d)^4 - P(Q^d)^3(P^d)^2 - P(Q^d)^2(P^d)^3\Big)(P+Q),$$

and

$$G^\pi G(F^d)^2 = \Big(PQ^\pi Q^2(P^d)^7 - PQ^2Q^d(P^d)^6 - PQQ^d(P^d)^5 + \sum_{i=1}^{3} Q^\pi Q^i(P^d)^{i+4}\Big)(P+Q).$$

So,

$$\begin{aligned} \sum_{i=0}^{ind(G)-1} G^\pi G^i (F^d)^{i+1} = \Big(& - Q^d(P^d)^3 - PQ^d(P^d)^4 - P(Q^d)^2(P^d)^3 - P(Q^d)^3(P^d)^2 - (Q^d)^2(P^d)^2 - (P^d)^4 \\ & + \sum_{i=0}^{ind(Q)-1} Q^\pi Q^i (P^d)^{i+4} + \sum_{i=0}^{ind(Q)-1} PQ^\pi Q^i (P^d)^{i+5}\Big)(P+Q). \end{aligned} \tag{8}$$

Now,

$$\sum_{i=2}^{ind(F)-1} (G^d)^{i+1} F^i F^\pi = \Big(\sum_{i=5}^{ind(P)-1} (Q^d)^{i+4} P^\pi P^i + \sum_{i=5}^{ind(P)-1} P(Q^d)^{i+5} P^i P^\pi\Big)(P+Q),$$

on the other hand

$$G^d F^\pi = \Big((Q^d)^4 P^\pi + (Q^d)^5 P^\pi P + P(Q^d)^5 P^\pi + P(Q^d)^6 PP^\pi - (Q^d)^3 P^d - P(Q^d)^4 P^d\Big)(P+Q),$$

and

$$(G^d)^2 FF^\pi = \Big(\sum_{i=2}^{4} (Q^d)^{i+4} P^\pi P^i + \sum_{i=2}^{4} P(Q^d)^{i+5} P^i P^\pi\Big)(P+Q).$$

Hence

$$\sum_{i=0}^{ind(F)-1} (G^d)^{i+1} F^i F^\pi = \Big(-(Q^d)^3 P^d - P(Q^d)^4 P^d + \sum_{i=0}^{ind(P)-1} (Q^d)^{i+4} P^\pi P^i + \sum_{i=0}^{ind(P)-1} P(Q^d)^{i+5} P^i P^\pi\Big)(P+Q). \tag{9}$$

Finally, substituting (8) and (9) into (1), we complete the proof. □

Example 2.2. *Consider the two matrices $P, Q \in \mathbb{C}^{6\times 6}$, where*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

We have

$$P^d = \begin{pmatrix} 0 & 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 \\ 0 & -1 & 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^d = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From $P^2Q \neq 0$ and $Q^2P \neq 0$, formula for $(P+Q)^d$ from [13, Theorem (2.2)] fail to apply. But it satisfies $P^2QP = 0$, $QPQ = 0$ and $P^2Q^2 = 0$, also we have

$$ind(P) = 4, \quad ind(Q) = 2.$$

Applying Theorem 2.1, we get

$$(P+Q)^d = \begin{pmatrix} 1 & 1 & 1 & 1 & 12 & 4 \\ 0 & 1 & 0 & 2 & 3 & 1 \\ 0 & -1 & 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applications to the Drazin inverse of block matrix

We use the formula in Theorem 2.1 to give some representations for the Drazin inverse of some block matrices.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, (A and D are square) with generalized Schur complement $S = D - CA^dB$ is zero. Hartwig et al. [8] extended the results in [11] by replacing the assumptions $CA^\pi = 0$ and $A^\pi B = 0$ with $CA^\pi B = 0$ and $AA^\pi B = 0$. In the following Theorem 2.3, we give the representation for the Drazin inverse of M under the conditions $AA^\pi BC = 0$ and $BCA^\pi B = 0$, the result generalizes the conclusion in [8].

Theorem 2.3. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, (A and D are square) such that $S = D - CA^dB = 0$. If $AA^\pi BC = 0$ and $BCA^\pi B = 0$, then*

$$M^d = M\left[\begin{pmatrix} A^2A^d & B \\ CAA^d & CA^dB \end{pmatrix}\left((Q_1^d)^4 + \sum_{i=1}^{ind(A)} (Q_1^d)^{i+4} \begin{pmatrix} 0 & 0 \\ CA^{i-1}A^\pi & 0 \end{pmatrix}\right)\right]M$$

where

$$(Q_1^d)^n = \begin{pmatrix} I \\ CA^d \end{pmatrix}((AW)^d)^{n+1}A\begin{pmatrix} I & A^dB \end{pmatrix}, \quad W = AA^d + A^dBCA^d, \quad for\, n \geq 1.$$

Proof. Let $P = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$, $Q = \begin{pmatrix} A^2A^d & B \\ CAA^d & CA^dB \end{pmatrix}$, then $M = P + Q$. From P is $(s+1)-$nilpotent, where $s = ind(A)$, so we get $P^d = 0$.
From $AA^{\pi}BC = 0$ and $BCA^{\pi}B = 0$, we get $P^2QP = 0$, $QPQ = 0$ and $P^2Q^2 = 0$, so according to Theorem 2.1, we have

$$M^d = (P+Q)\left(\sum_{i=0}^{ind(P)-1} (Q^d)^{i+3}P^i\right)(P+Q). \tag{10}$$

Let $Q_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}$, $Q_2 = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}$, then we have $Q = Q_1 + Q_2$.
We notice that $Q_1Q_2 = 0$ and Q_2 is nilpotent, thus according to Lemma 1.2, we get

$$Q^d = Q_1^d + Q_2(Q_1^d)^2. \tag{11}$$

The generalized Schur complement of Q_1 is equal to zero, and the matrix Q_1 satisfies

$$(A^2A^d)^{\pi}AA^dB = 0, \quad CAA^d(A^2A^d)^{\pi} = 0,$$

so according to Lemma 1.4, we get

$$(Q_1^d)^n = \begin{pmatrix} I \\ CA^d \end{pmatrix}((AW)^d)^{n+1}A\begin{pmatrix} I & A^dB \end{pmatrix}, \quad W = AA^d + A^dBCA^d, \quad for\, n \geq 1. \tag{12}$$

Substituting (12) into (11), then substituting (11) into (10), we get

$$M^d = (P+Q)\left(\sum_{i=0}^{ind(P)-1} (Q_1^d)^{i+3}P^i + \sum_{i=0}^{ind(P)-1} Q_2(Q_1^d)^{i+4}P^i\right)(P+Q).$$

From $Q_2 = Q - Q_1$, we have the representation of the M^d above can be simplified as follow:

$$M^d = (P+Q)\left[\begin{pmatrix} A^2A^d & B \\ CAA^d & CA^dB \end{pmatrix}\left((Q_1^d)^4 + \sum_{i=1}^{ind(A)} (Q_1^d)^{i+4}\begin{pmatrix} 0 & 0 \\ CA^{i-1}A^{\pi} & 0 \end{pmatrix}\right)\right](P+Q).$$

□

In [8], Hartwig et al. gave the representation for the Drazin inverse of M under the conditions $CA^{\pi}B = 0$ and $CAA^{\pi} = 0$. In the following Theorem 2.4, we give the representation for the Drazin inverse of M under the conditions $CA^{\pi}BC = 0$ and $BCAA^{\pi} = 0$ the result generalizes the conclusion in [8].

Theorem 2.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *(A and D are square) such that* $S = D - CA^dB = 0$. *If* $CA^{\pi}BC = 0$ *and* $BCAA^{\pi} = 0$, *then*

$$M^d = M\left[\left((Q_1^d)^4 + \sum_{i=1}^{ind(A)} \begin{pmatrix} 0 & A^{i-1}A^{\pi}B \\ 0 & 0 \end{pmatrix}(Q_1^d)^{i+4}\right)\begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix}\right]M$$

where

$$(Q_1^d)^n = \begin{pmatrix} I \\ CA^d \end{pmatrix}((AW)^d)^{n+1}A\begin{pmatrix} I & A^dB \end{pmatrix}, \quad W = AA^d + A^dBCA^d, \quad for\, n \geq 1.$$

Proof. We can split matrix M as

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = \begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix} + \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

If we denote by $P = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix}$, we have that matrices P and Q satisfy the symmetrical formulation of Theorem 2.1. Using similar method as in Theorem 2.3, we get that the statement of the theorem is true. □

Example 2.5. *We give an example to demonstrate Theorem 2.4. Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{7\times 7}$ *where*

$$A = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By computing we get that generalized Schur complement $S = D - CA^dB$ *is equal to zero. Since* $CA^\pi \neq 0$ *and* $CA^\pi B \neq 0$ *we know that the conditions of Theorem (4.1) in [8] do not hold. However it satisfies the conditions* $BCAA^\pi = 0$ *and* $CA^\pi BC = 0$ *in Theorem 2.4 in this paper. We have*

$$ind(A) = 3, \quad A^d = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (AW)^d = \begin{pmatrix} 0 & \frac{-1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then according to the formula in Theorem 2.4, we get

$$M^d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{2}{27} & 0 & \frac{11}{81} & \frac{2}{9} & \frac{11}{81} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{27} & \frac{2}{81} & 0 & \frac{11}{243} & \frac{2}{27} & \frac{11}{243} \\ 0 & \frac{-1}{81} & \frac{-2}{243} & 0 & \frac{-11}{729} & \frac{-2}{81} & \frac{-11}{729} \\ 0 & \frac{1}{9} & \frac{2}{27} & 0 & \frac{11}{81} & \frac{2}{9} & \frac{11}{81} \\ 0 & \frac{1}{81} & \frac{2}{243} & 0 & \frac{11}{729} & \frac{2}{81} & \frac{11}{729} \end{pmatrix}.$$

3. Additive result (Conditions: $PQP^2 = 0, PQ^2 = 0, QP^3 = 0$)

Theorem 3.1. *Let* $P, Q \in \mathbb{C}^{n\times n}$*. If* $PQP^2 = 0$, $PQ^2 = 0$ *and* $QP^3 = 0$*, then*

$$\begin{aligned}
(P+Q)^d = &\sum_{i=0}^{k} \left((P^d)^{2i+1} + (Q^d)^{2i+1}\right)\left((QP)^i(QP)^\pi + (PQ)^i(PQ)^\pi\right) \\
&+ \sum_{i=0}^{k-1} \left(P^{2i+1}P^\pi + Q^{2i+1}Q^\pi\right)\left(((QP)^d)^{i+1} + ((PQ)^d)^{i+1}\right) \\
&+ \sum_{i=0}^{k-1} Q^\pi Q^{2i} \left(P^2((QP)^d)^{i+2}Q + QP^2((QP)^d)^{i+2}\right) \\
&+ \sum_{i=0}^{k-1} (Q^d)^{2(i+2)} \left(P^2(QP)^i(QP)^\pi Q + QP^2(QP)^i(QP)^\pi\right) \\
&- P^d - Q^d - Q^dP^2(QP)^d - P(PQ)^d - (Q^d)^2P^2Q(PQ)^d,
\end{aligned}$$

where $k = max\{ind(P^2), ind(Q^2), ind(QP)\}$.

Proof. Using Lemma 1.1, we have

$$(P+Q)^d = \left(\begin{pmatrix} P & Q \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right)^d = \begin{pmatrix} P & Q \end{pmatrix}\left(\begin{pmatrix} P & Q \\ P & Q \end{pmatrix}^2\right)^d \begin{pmatrix} I \\ I \end{pmatrix}$$
$$= \begin{pmatrix} P & Q \end{pmatrix}\begin{pmatrix} P^2+QP & PQ+Q^2 \\ P^2+QP & PQ+Q^2 \end{pmatrix}^d \begin{pmatrix} I \\ I \end{pmatrix}. \tag{13}$$

Let

$$M = \begin{pmatrix} P^2+QP & PQ+Q^2 \\ P^2+QP & PQ+Q^2 \end{pmatrix} := F+G,$$

where

$$F = \begin{pmatrix} QP & PQ \\ QP & PQ \end{pmatrix}, \qquad G = \begin{pmatrix} P^2 & Q^2 \\ P^2 & Q^2 \end{pmatrix}.$$

From $PQP^2 = 0$, $PQ^2 = 0$ and $QP^3 = 0$, we get $FG = 0$. Then applying Lemma 1.2, we have

$$M^d = \sum_{i=0}^{ind(G)-1} G^\pi G^i (F^d)^{i+1} + \sum_{i=0}^{ind(F)-1} (G^d)^{i+1} F^i F^\pi. \tag{14}$$

Now, we calculate F^d. Let $A = \begin{pmatrix} QP & PQ \\ 0 & PQ \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ QP & 0 \end{pmatrix}$, then $F = A + B$, $B^2 = 0$.
From $PQ^2 = 0$, we get $AB = 0$, then we can apply Lemma 1.2 to get the
$F^d = A^d + B(A^d)^2$.
Now by Lemma 1.3, we get $A^d = \begin{pmatrix} (QP)^d & (PQ)^d + (QP)((PQ)^d)^2 \\ 0 & (PQ)^d \end{pmatrix}$.
After computation we get:

$$(F^d)^n = \begin{pmatrix} ((QP)^d)^n & ((PQ)^d)^n + (QP)((PQ)^d)^{n+1} \\ ((QP)^d)^n & ((PQ)^d)^n + (QP)((PQ)^d)^{n+1} \end{pmatrix} \tag{15}$$

for every $n \in \mathbb{N}$. Consider the splitting $G = S + T$, where $S = \begin{pmatrix} P^2 & 0 \\ P^2 & Q^2 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & Q^2 \\ 0 & 0 \end{pmatrix}$. We observe that $ST = 0$ and $T^2 = 0$. Then by Lemma 1.2, we get
$G^d = S^d + T(S^d)^2$.
By Lemma 1.3, $S^d = \begin{pmatrix} (P^d)^2 & 0 \\ (Q^d)^4P^2 + (P^d)^2 & (Q^d)^2 \end{pmatrix}$. we obtain

$$(G^d)^n = \begin{pmatrix} (P^d)^{2n} + (Q^d)^{2(n+1)}P^2 & (Q^d)^{2n} \\ (P^d)^{2n} + (Q^d)^{2(n+1)}P^2 & (Q^d)^{2n} \end{pmatrix} \tag{16}$$

for every $n \in \mathbb{N}$. After computation we get:

$$\begin{cases} G^n = \begin{pmatrix} P^{2n} + (Q)^{2(n-1)}P^2 & Q^{2n} \\ P^{2n} + (Q)^{2(n-1)}P^2 & Q^{2n} \end{pmatrix}, & \text{if } n \geq 2 \\ F^n = \begin{pmatrix} (QP)^n & (PQ)^n + (QP)(PQ)^{n-1} \\ (QP)^n & (PQ)^n + (QP)(PQ)^{n-1} \end{pmatrix}, & \text{if } n \geq 2 \end{cases}.$$

After substituting this expressions, (15) and (16) into (14) we have

$$\sum_{i=2}^{ind(F)-1} (G^d)^{i+1}F^iF^\pi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{17}$$

where

$$a_{11} = \sum_{i=2}^{ind(QP)-1} (P^d)^{2i+2}(QP)^i(QP)^\pi + \sum_{i=2}^{ind(QP)-1} (Q^d)^{2(i+2)}P^2(QP)^i(QP)^\pi + \sum_{i=2}^{ind(QP)-1} (Q^d)^{2i+2}(QP)^i(QP)^\pi,$$

$$a_{12} = \sum_{i=2}^{ind(PQ)-1} (P^d)^{2i+2}(PQ)^i(PQ)^\pi + \sum_{i=2}^{ind(PQ)} (Q^d)^{2i+2}(QP)(PQ)^{i-1}(PQ)^\pi + \sum_{i=2}^{ind(PQ)-1} (Q^d)^{2i+2}(PQ)^i(PQ)^\pi,$$

$$a_{21} = \sum_{i=2}^{ind(QP)-1} (P^d)^{2i+2}(QP)^i(QP)^\pi + \sum_{i=2}^{ind(QP)-1} (Q^d)^{2(i+2)}P^2(QP)^i(QP)^\pi + \sum_{i=2}^{ind(QP)-1} (Q^d)^{2i+2}(QP)^i(QP)^\pi,$$

$$a_{22} = \sum_{i=2}^{ind(PQ)-1} (P^d)^{2i+2}(PQ)^i(PQ)^\pi + \sum_{i=2}^{ind(PQ)} (Q^d)^{2i+2}(QP)(PQ)^{i-1}(PQ)^\pi + \sum_{i=2}^{ind(PQ)-1} (Q^d)^{2i+2}(PQ)^i(PQ)^\pi.$$

On the other hand

$$(G^d)^2FF^\pi = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$b_{11} = (P^d)^4(QP)(QP)^\pi + (Q^d)^6P^2(QP)(QP)^\pi - (Q^d)^3P(QP)(QP)^d + (Q^d)^4(PQ)^\pi(QP)$$

$$b_{12} = (P^d)^4(QP)^\pi(PQ) - (P^d)^3Q(PQ)^d(PQ) + (Q^d)^4(PQ)(PQ)^\pi - (Q^d)^3P(PQ)^d(PQ)$$

$$b_{21} = (P^d)^4(QP)(QP)^\pi + (Q^d)^6P^2(QP)(QP)^\pi - (Q^d)^3P(QP)(QP)^d + (Q^d)^4(PQ)^\pi(QP)$$

$$b_{22} = (P^d)^4(QP)^\pi(PQ) - (P^d)^3Q(PQ)^d(PQ) + (Q^d)^4(PQ)(PQ)^\pi - (Q^d)^3P(PQ)^d(PQ)$$

and

$$G^dF^\pi = \begin{pmatrix} (P^d)^2(QP)^\pi + (Q^d)^4P^2(QP)^\pi - Q^dP(QP)^d & -P^dQ(PQ)^d + (Q^d)^2(PQ)^\pi - Q^dP(PQ)^d \\ (P^d)^2(QP)^\pi + (Q^d)^4P^2(QP)^\pi - Q^dP(QP)^d & -P^dQ(PQ)^d + (Q^d)^2(PQ)^\pi - Q^dP(PQ)^d \end{pmatrix}$$

Thus simplifying (17), we get

$$a_{11} = -(Q^d)^2 + \sum_{i=0}^{ind(QP)-1} \left((P^d)^{2i+2} + (Q^d)^{2(i+2)}P^2 + (Q^d)^{2i+2}\right)(QP)^i(QP)^\pi,$$

$$a_{12} = -(P^d)^2 - (Q^d)^3P - Q^dP(PQ)^d + \sum_{i=0}^{ind(PQ)-1} \left((P^d)^{2i+2} + (Q^d)^{2i+3}P + (Q^d)^{2i+2}\right)(PQ)^i(PQ)^\pi,$$

$$a_{21} = -(Q^d)^2 + \sum_{i=0}^{ind(QP)-1} \left((P^d)^{2i+2} + (Q^d)^{2(i+2)}P^2 + (Q^d)^{2i+2}\right)(QP)^i(QP)^\pi,$$

$$a_{22} = -(P^d)^2 - (Q^d)^3P - Q^dP(PQ)^d + \sum_{i=0}^{ind(PQ)-1} \left((P^d)^{2i+2} + (Q^d)^{2i+3}P + (Q^d)^{2i+2}\right)(PQ)^i(PQ)^\pi.$$

Now,

$$\sum_{i=2}^{ind(G)-1} G^{\pi}G^{i}(F^{d})^{i+1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \tag{18}$$

where

$$c_{11} = \sum_{i=2}^{\infty}\left(P^{\pi}(P)^{2i} + P^{\pi}(Q)^{2i-2}P^{2} - (Q)^{d}(Q)^{2i-1}P^{2} + P^{\pi}(Q)^{2i} - (Q)^{d}(Q)^{2i+1}\right)((QP)^{d})^{i+1},$$

$$c_{12} = \sum_{i=2}^{\infty}\left(P^{\pi}(P)^{2i} + P^{\pi}(Q)^{2i} - (Q)^{2i+1}Q^{d}\right)((PQ)^{d})^{i+1} + \sum_{i=2}^{\infty}\left(P^{\pi}(Q)^{2i+1}P - (Q)^{2i+2}Q^{d}P\right)((PQ)^{d})^{i+2},$$

$$c_{21} = \sum_{i=2}^{\infty}\left(-(P)^{2i+1}(P)^{d} + Q^{\pi}(P)^{2i} + Q^{\pi}(Q)^{2i-2}(P)^{2} + Q^{\pi}(Q)^{2i}\right)((QP)^{d})^{i+1},$$

$$c_{22} = \sum_{i=2}^{\infty}\left(-(P)^{2i+1}P^{d} + Q^{\pi}(P)^{2i} + Q^{\pi}(Q)^{2i}\right)((PQ)^{d})^{i+1} + \sum_{i=2}^{ind((Q)^{2}-1}Q^{\pi}(Q)^{2i+1}P((PQ)^{d})^{i+2}.$$

On the other hand

$$G^{\pi}F^{d} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where

$$d_{11} = P^{\pi}(QP)^{d} - (Q^{d})^{2}(P)^{2}(QP)^{d} - QQ^{d}(QP)^{d}$$

$$d_{12} = P^{\pi}(QP)((PQ)^{d})^{2} + P^{\pi}(PQ)^{d} - QQ^{d}(PQ)^{d} - (Q)^{2}Q^{d}P((PQ)^{d})^{2}$$

$$d_{21} = -PP^{d}(QP)^{d} - (Q^{d})^{2}(P)^{2}(QP)^{d} + Q^{\pi}(QP)^{d}$$

$$d_{22} = -PP^{d}(PQ)^{d} + Q^{\pi}(PQ)^{d} + QQ^{\pi}P((PQ)^{d})^{2}.$$

and

$$G^{\pi}G(F^{d})^{2} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

where

$$e_{11} = (P)^{2}P^{\pi}((QP)^{d})^{2} - QQ^{d}(P)^{2}((QP)^{d})^{2} + (Q)^{2}Q^{\pi}((QP)^{d})^{2}$$

$$e_{12} = (P)^{2}P^{\pi}((PQ)^{d})^{2} + (Q)^{2}Q^{\pi}((PQ)^{d})^{2} + (Q)^{3}Q^{\pi}P((PQ)^{d})^{3}$$

$$e_{21} = (P)^{2}P^{\pi}((QP)^{d})^{2} - QQ^{d}(P)^{2}((QP)^{d})^{2} + (Q)^{2}Q^{\pi}((QP)^{d})^{2}$$

$$e_{22} = (P)^{2}P^{\pi}((PQ)^{d})^{2} + (Q)^{2}Q^{\pi}((PQ)^{d})^{2} + (Q)^{3}Q^{\pi}P((PQ)^{d})^{3}.$$

Thus simplifying (18), we get

$$c_{11} = -((Q)^{d})^{2}(P)^{2}(QP)^{d} + \sum_{i=0}^{\infty}\left(P^{\pi}(P)^{2i} - (Q)^{d}(Q)^{2i+1}\right)((QP)^{d})^{i+1}$$
$$+ \sum_{i=1}^{\infty}\left(P^{\pi}(Q)^{2i} - (Q)^{d}(Q)^{2i-1}P^{2}\right)((QP)^{d})^{i+1} + \sum_{i=2}^{\infty}P^{\pi}(Q)^{2i-2}P^{2}((QP)^{d})^{i+1},$$

$$c_{12} = \sum_{i=0}^{\infty}\left(P^{\pi}(P)^{2i} + (Q)^{2i+1}Q^{d}\right)((PQ)^{d})^{i+1} + \sum_{i=0}^{\infty}\left(P^{\pi}(Q)^{2i+1}P - (Q)^{2i+2}Q^{d}P\right)((PQ)^{d})^{i+2} + \sum_{i=1}^{\infty}P^{\pi}(Q)^{2i}((PQ)^{d})^{i+1},$$

$$c_{21} = -((Q)^{d})^{2}(P)^{2}(QP)^{d} + \sum_{i=0}^{\infty}\left(Q^{\pi}(P)^{2i} - (P)^{2i+1}(P)^{d}\right)((QP)^{d})^{i+1}$$
$$+ \sum_{i=2}^{\infty}Q^{\pi}(Q)^{2i-2}(P)^{2}((QP)^{d})^{i+1} + \sum_{i=1}^{ind((Q)^{2}-1}Q^{\pi}(Q)^{2i}((QP)^{d})^{i+1},$$

$$c_{22} = \sum_{i=0}^{\infty}\left(-(P)^{2i+1}P^{d} + Q^{\pi}(P)^{2i}\right)((PQ)^{d})^{i+1} + \sum_{i=1}^{ind((Q)^{2}-1}Q^{\pi}(Q)^{2i}((PQ)^{d})^{i+1} + \sum_{i=0}^{ind((Q)^{2}-1}Q^{\pi}(Q)^{2i+1}P((PQ)^{d})^{i+2}.$$

After substituting (17) and (18) into (13) we complete the proof. $\square$

Example 3.2. *Consider the two matrices $P, Q \in \mathbb{C}^{6\times 6}$, where*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for every nonzero $a, b, c \in \mathbb{C}$. We have

$$P^{d} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{d} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From $P^{2} \neq 0$ and $QP^{2} \neq 0$, formulas for $(P+Q)^{d}$ from [10, Theorem (2.2)] and [2, Theorem (2.2)] fail to apply. But it satisfies $PQP^{2} = 0$, $QP^{3} = 0$ and $PQ^{2} = 0$, also we have

$$ind(P) = 3, \quad ind(Q) = 2.$$

Applying Theorem 3.1, we get

$$(P+Q)^{d} = \begin{pmatrix} 0 & 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & -1 & 0 & 3 & -3 & -4 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applications to the Drazin inverse of block matrix

We use the formula in Theorem 3.1 to give some representations for the Drazin inverse of block matrix. Now we consider the generalized Schur complement is equal to zero.

Theorem 3.3. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *(A and D are square) such that* $S = D - CA^dB = 0$. *If* $BCA^d = 0$, $BCAA^\pi B = 0$ *and* $CA^2A^\pi = 0$, *then*

$$M^d = \begin{pmatrix} A & I \\ C & CA^d \end{pmatrix} \left\{ \begin{pmatrix} 0 & AA^\pi((BC)^d)^2 \\ 0 & -A^2A^\pi((BC)^d)^2 \end{pmatrix} + \sum_{i=0}^{k-1} A^{2i}A^\pi((BC)^d)^{i+1} + \sum_{i=0}^{k-1}(A^d)^{2(i+2)}(BC)^i(BC)^\pi \right\} \begin{pmatrix} I & 0 \\ AA^\pi & B \end{pmatrix},$$

where $k = max\{ind(A^2), ind(BC)\}$.

Proof. Let $P = \begin{pmatrix} AA^\pi & B \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} A^2A^d & 0 \\ C & CA^dB \end{pmatrix}$, then $M = P + Q$.
Obviously $(AA^\pi)^d = 0$. After using Lemma 1.3, we get $P^d = 0$.
From $BCA^d = 0$, $BCAA^\pi B = 0$ and $CA^2A^\pi = 0$, we get $QP^3 = 0$, $PQP^2 = 0$ and $PQ^2 = 0$. So according to Theorem 3.1, we have

$$\begin{aligned} (P+Q)^d = & \sum_{i=0}^{k}(Q^d)^{2i+1}\left((QP)^i(QP)^\pi + (PQ)^i(PQ)^\pi\right) \\ & + \sum_{i=0}^{k-1}\left(P^{2i+1}P^\pi + Q^{2i+1}Q^\pi\right)\left(((QP)^d)^{i+1} + ((PQ)^d)^{i+1}\right) \\ & + \sum_{i=0}^{k-1} Q^\pi Q^{2i}\left(P^2((QP)^d)^{i+2}Q + QP^2((QP)^d)^{i+2}\right) \\ & + \sum_{i=0}^{k-1}(Q^d)^{2(i+2)}\left(P^2(QP)^i(QP)^\pi Q + QP^2(QP)^i(QP)^\pi\right) \\ & - Q^d - Q^dP^2(QP)^d - P(PQ)^d - (Q^d)^2P^2Q(PQ)^d, \end{aligned} \tag{19}$$

where $k = max\{ind(P^2), ind(Q^2), ind(QP)\}$.
After computation we get:

$$Q^n = \begin{cases} \begin{pmatrix} A^dA^3 & 0 \\ CA^2A^d + CA^dBC & 0 \end{pmatrix}, & \text{if } n = 2, \\ \begin{pmatrix} A^{n+1}A^d & 0 \\ CA^nA^d & 0 \end{pmatrix}, & \text{if } n \geq 3. \end{cases}$$

Furthermore, by Lemma 1.3, we obain

$$(Q^d)^n = \begin{pmatrix} (A^d)^n & 0 \\ C(A^d)^{n+1} & 0 \end{pmatrix}, \qquad Q^\pi = \begin{pmatrix} A^\pi & 0 \\ -CA^d & I \end{pmatrix},$$

for every $n \in \mathbb{N}$. We note that $Q^3Q^\pi = 0$.
After computation we get:

$$P^n = \begin{cases} \begin{pmatrix} AA^\pi & B \\ 0 & 0 \end{pmatrix}, & \text{if } n = 1, \\ \begin{pmatrix} A^nA^\pi & A^{n-1}A^\pi B \\ 0 & 0 \end{pmatrix}, & \text{if } n \geq 2. \end{cases}$$

$$(PQ)^n = \begin{pmatrix} (BC)^n & 0 \\ 0 & 0 \end{pmatrix}, \quad ((PQ)^d)^n = \begin{pmatrix} ((BC)^d)^n & 0 \\ 0 & 0 \end{pmatrix}, \quad (PQ)^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & I \end{pmatrix},$$

After substituting this expressions into (19) we get that the statement of the theorem is valid. □

Theorem 3.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *(A and D are square) such that* $S = D - CA^dB = 0$. *If* $A^2A^\pi B = 0$, $A^dBC = 0$ *and* $CAA^\pi BC = 0$ *then*

$$M^d = \begin{pmatrix} I & AA^\pi \\ 0 & C \end{pmatrix} \left\{ \begin{pmatrix} 0 & 0 \\ ((BC)^d)^2 AA^\pi & -((BC)^d)^2 A^2 A^\pi \end{pmatrix} + \sum_{i=0}^{k-1} (BC)^i (BC)^\pi (A^d)^{2i+2} + \sum_{i=0}^{k-1} ((BC)^d)^{i+1} A^{2i} A^\pi \right\} \begin{pmatrix} A & B \\ I & A^dB \end{pmatrix}.$$

where $k = max\{ind(A^2), ind(BC)\}$.

Proof. We can split matrix M as

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = \begin{pmatrix} A^2A^d & B \\ 0 & CA^dB \end{pmatrix} + \begin{pmatrix} AA^\pi & 0 \\ C & 0 \end{pmatrix}.$$

If we denote by $P = \begin{pmatrix} AA^\pi & 0 \\ C & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} A^2A^d & B \\ 0 & CA^dB \end{pmatrix}$, we have that matrices P and Q satisfy satisfy the symmetrical formulation of Theorem 3.1. Using similar method as in Theorem 3.3, we get that the statement of the theorem is true. □

Example 3.5. *We give an example to demonstrate Theorem 3.3.*
Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{7\times 7}$ *where*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

By computing we get that generalized Schur complement $S = D - CA^dB$ *is equal to zero. Since* $CA^2A^\pi = 0$, $BCAA^\pi = 0$ *and* $BCA^d = 0$, *then it satisfies the conditions of Theorem 3.3. We have*

$$ind(A) = 2, \quad A^d = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (BC)^d = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

then according to the formula in Theorem 3.3 we get

$$M^d = \begin{pmatrix} 1 & 2 & -2 & 1 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -\frac{3}{2} & 1 & 1 & -1 & -1 \\ -1 & -2 & \frac{5}{2} & -1 & -1 & 1 & 1 \end{pmatrix}.$$

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