Filomat 32:18 (2018), 6441–6456
https://doi.org/10.2298/FIL1818441C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Measurable Functions Similar to the Itô Integral and the Paley-Wiener-Zygmund Integral over Continuous Paths

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Abstract. Let $C[0,T]$ denote an analogue of generalized Wiener space, the space of continuous real-valued functions on the interval $[0,T]$. On the space $C[0,T]$, we introduce a finite measure $w_{\alpha,\beta;\varphi}$ and investigate its properties, where φ is an arbitrary finite measure on the Borel class of $\mathbb{R}$. Using the measure $w_{\alpha,\beta;\varphi}$, we also introduce two measurable functions on $C[0,T]$; one of them is similar to the Itô integral and the other is similar to the Paley-Wiener-Zygmund integral. We will prove that if $\varphi(\mathbb{R}) = 1$, then $w_{\alpha,\beta;\varphi}$ is a probability measure with the mean function α and the variance function β, and the two measurable functions are reduced to the Paley-Wiener-Zygmund integral on the analogue of Wiener space $C[0,T]$. As an application of the integrals, we derive a generalized Paley-Wiener-Zygmund theorem which is useful to calculate generalized Wiener integrals on $C[0,T]$. Throughout this paper, we will recognize that the generalized Itô integral is more general than the generalized Paley-Wiener-Zygmund integral.

1. Introduction

Let $C_0[0,T]$ denote the classical Wiener space, the space of continuous real-valued functions x on the interval $[0,T]$ with $x(0) = 0$. In [8], Paley, Wiener and Zygmund defined a stochastic integral which is based on integration by parts and is now called the Paley-Wiener-Zygmund (PWZ) integral. When applied to the classical Wiener space $C_0[0,T]$, it is less general than the Itô integral [6], but the two integrals agree when they are both defined. The PWZ stochastic integrals have been used in various papers, in particular, concerning Feynman integration theories [2, 9]. In particular the PWZ stochastic integrals were used in defining a Banach algebra S of functions on $C_0[0,T]$ which was introduced by Cameron and Storvick in [2]. In [5], Johnson showed that $\mathcal{S}$ is isometrically isomorphic to the Banach algebra of Fresnel integrable functions as given by Albeverio and Høegh-Krohn [1]. Further work for relationships between the Itô integral and the PWZ integral were introduced by Pierce [10] on the generalized Wiener space $C_{\alpha,\beta}[0,T]$ which is a generalized classical Wiener space with the mean function α and the variance function β.

Let $C[0,T]$ denote an analogue of a generalized Wiener space [4, 11, 12], the space of continuous real-valued functions on the interval $[0,T]$. On the space $C[0,T]$, we introduce a finite measure $w_{\alpha,\beta;\varphi}$ and investigate its properties, where $\alpha, \beta : [0,T] \to \mathbb{R}$ are appropriate functions such that β is strictly increasing, and φ is an arbitrary finite measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. Using this finite measure $w_{\alpha,\beta;\varphi}$, we

2010 *Mathematics Subject Classification*. Primary 28C20; Secondary 60A10, 60B12, 60H05
Keywords. analogue of Wiener space, Itô integral, Paley-Wiener-Zygmund integral, Wiener measure, Wiener space
Received:31 May 2018; Accepted: 03 October 2018
Communicated by Miljana Jovanović
This research was supported by Basic Science Research Program through the National Research Foundation(NRF) of Korea funded by the Ministry of Education (2017R1D1A1B03029876).
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also introduce two measurable functions on $C[0,T]$; one of them is similar to the Itô type integral $I_{\alpha,\beta}(g)$ for $g \in L^2_{\alpha,\beta}[0,T]$, where $L^2_{\alpha,\beta}[0,T]$ is the L^2-space with respect to the Lebesgue-Stieltjes measure induced by α and β, and the other is similar to the PWZ integral. And then, we investigate their properties and relationships. In fact, we will prove that if $\varphi(\mathbb{R}) = 1$, then $w_{\alpha,\beta;\varphi}$ is a probability measure with the mean function α and the variance function β, and the two measurable functions are reduced to the PWZ integral. As an application of $I_{\alpha,\beta}$, we derive a generalized PWZ theorem which is useful to calculate generalized Wiener integrals on $C[0,T]$; for a Borel measurable(integrable) function $f : \mathbb{R}^n \to \mathbb{C}$

$$\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x), \ldots, I_{\alpha,\beta}(f_n)(x))dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\Big[u_j - \int_0^T f_j(t)d\alpha(t)\Big]^2\Big\}dm_L^n(\vec{u}),$$

where $\vec{u} = (u_1, \ldots, u_n)$, $\{f_1, \ldots, f_n\}$ is orthonormal in $L^2_{0,\beta}[0,T]$ with $f_j \in L^2_{\alpha,\beta}[0.T]$ for $j = 1, \ldots, n$ and m_L denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. We note that Pierce used the pointwise convergence in $C_{\alpha,\beta}[0,T]$ to define the Itô integral in [10]. Throughout this paper, we will use the $L^2(C[0,T])$-convergence to define $I_{\alpha,\beta}(g)$ on $C[0,T]$ so that we can give an exact proof that $I_{\alpha,\beta}$ is more general than the generalized PWZ integral.

2. An analogue of a generalized Wiener space

In this section, we introduce a finite measure over continuous paths and investigate its properties.

Let $\alpha, \beta : [0,T] \to \mathbb{R}$ be two functions, where β is strictly increasing. Let φ be a positive finite measure on $\mathcal{B}(\mathbb{R})$. For $\vec{t}_n = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le T$, let $J_{\vec{t}_n} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}_n}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. For $\prod_{j=0}^n B_j \in \mathcal{B}(\mathbb{R}^{n+1})$, the subset $J_{\vec{t}_n}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval I and let $\mathcal{I}$ be the set of all such intervals I. Define a premeasure $m_{\alpha,\beta;\varphi}$ on $\mathcal{I}$ by

$$m_{\alpha,\beta;\varphi}(I) = \int_{B_0}\int_{\prod_{j=1}^n B_j} W_n(\alpha,\beta,\vec{t}_n,\vec{u}_n,u_0)dm_L^n(\vec{u}_n)d\varphi(u_0),$$

where for $\vec{u}_n = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}$,

$$W_n(\alpha,\beta,\vec{t}_n,\vec{u}_n,u_0) = \Big[\frac{1}{\prod_{j=1}^n 2\pi[\beta(t_j)-\beta(t_{j-1})]}\Big]^{\frac{1}{2}} \exp\Big\{-\frac{1}{2}\sum_{j=1}^n \frac{[u_j - \alpha(t_j) - u_{j-1} + \alpha(t_{j-1})]^2}{\beta(t_j)-\beta(t_{j-1})}\Big\}.$$

The Borel σ-algebra $\mathcal{B}(C[0,T])$ of $C[0,T]$ with the supremum norm, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for all $I \in \mathcal{I}$. This measure $w_{\alpha,\beta;\varphi}$ is called an analogue of a generalized Wiener measure on $(C[0,T], \mathcal{B}(C[0,T]))$ according to φ [11, 12].

Theorem 2.1. *If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then the following equality holds:*

$$\int_{C[0,T]} f(x(t_0), x(t_1), \ldots, x(t_n))dw_{\alpha,\beta;\varphi}(x) \stackrel{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0,u_1,\ldots,u_n)W_n(\alpha,\beta,\vec{t}_n,\vec{u}_n,u_0)dm_L^n(\vec{u}_n)d\varphi(u_0),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Using the same method as used in the proof of Theorem 3.1 of [3], we can prove the following Lemma.

Lemma 2.2. *For $\lambda > 0$, $a \in \mathbb{R}$ and nonnegative integer n, we have*

$$\Big(\frac{1}{2\pi\lambda}\Big)^{\frac{1}{2}} \int_{\mathbb{R}} u^n \exp\Big\{-\frac{(u-a)^2}{2\lambda}\Big\}dm_L(u) = \sum_{j=0}^{[\frac{n}{2}]} \frac{n!\lambda^j a^{n-2j}}{2^j j!(n-2j)!},$$

where $[\cdot]$ denotes the greatest integer function.

By Theorem 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *If $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$, then we have for nonnegative integers m and n*

$$\begin{aligned}&\int_{C[0,T]}[x(t_2)-x(t_1)]^n[x(t_4)-x(t_3)]^m dw_{\alpha,\beta;\varphi}(x)\\ =\;&\varphi(\mathbb{R})\Big[\sum_{j=0}^{[\frac{n}{2}]}\frac{n![\alpha(t_2)-\alpha(t_1)]^{n-2j}}{2^j j!(n-2j)!}[\beta(t_2)-\beta(t_1)]^j\Big]\Big[\sum_{k=0}^{[\frac{m}{2}]}\frac{m![\alpha(t_4)-\alpha(t_3)]^{m-2k}}{2^k k!(m-2k)!}[\beta(t_4)-\beta(t_3)]^k\Big].\end{aligned}$$

Proof. Suppose that $0 = t_0 < t_1 < t_2 < t_3 < t_4 \le T$. Then we have by Theorem 2.1

$$\begin{aligned}&\int_{C[0,T]}[x(t_2)-x(t_1)]^n[x(t_4)-x(t_3)]^m dw_{\alpha,\beta;\varphi}(x)\\ =\;&\Big[\prod_{j=1}^{4}\frac{1}{2\pi[\beta(t_j)-\beta(t_{j-1})]}\Big]^{\frac12}\int_{\mathbb{R}^5}(u_2-u_1)^n(u_4-u_3)^m\exp\Big\{-\sum_{j=1}^{4}\frac{[u_j-\alpha(t_j)-u_{j-1}+\alpha(t_{j-1})]^2}{2[\beta(t_j)-\beta(t_{j-1})]}\Big\}\\ &dm_L^4(u_1,u_2,u_3,u_4)d\varphi(u_0).\end{aligned}$$

Letting $v_j = u_j - u_{j-1}$ for $j = 1, 2, 3, 4$, we have, by the change of variable theorem,

$$\begin{aligned}&\int_{C[0,T]}[x(t_2)-x(t_1)]^n[x(t_4)-x(t_3)]^m dw_{\alpha,\beta;\varphi}(x)\\ =\;&\varphi(\mathbb{R})\Big[\prod_{j=1}^{4}\frac{1}{2\pi[\beta(t_j)-\beta(t_{j-1})]}\Big]^{\frac12}\int_{\mathbb{R}^4}v_2^n v_4^m\exp\Big\{-\sum_{j=1}^{4}\frac{[v_j-\alpha(t_j)+\alpha(t_{j-1})]^2}{2[\beta(t_j)-\beta(t_{j-1})]}\Big\}dm_L^4(v_1,v_2,v_3,v_4)\\ =\;&\varphi(\mathbb{R})\Big[\frac{1}{(2\pi)^2[\beta(t_4)-\beta(t_3)][\beta(t_2)-\beta(t_1)]}\Big]^{\frac12}\int_{\mathbb{R}^2}v_2^n v_4^m\exp\Big\{-\frac{[v_2-\alpha(t_2)+\alpha(t_1)]^2}{2[\beta(t_2)-\beta(t_1)]}-\frac{[v_4-\alpha(t_4)+\alpha(t_3)]^2}{2[\beta(t_4)-\beta(t_3)]}\Big\}\\ &dm_L^2(v_2,v_4).\end{aligned}$$

By Lemma 2.2, we have the result. For the general case $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$, we can prove the result with minor modifications. □

By Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $t_1, t_2 \in [0, T]$. Then*

1. $\int_{C[0,T]}[x(t_2)-x(t_1)]dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\alpha(t_2)-\alpha(t_1)]$,
2. $\int_{C[0,T]}[x(t_2)-x(t_1)][x(t_4)-x(t_3)]dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\alpha(t_2)-\alpha(t_1)][\alpha(t_4)-\alpha(t_3)]$ *if* $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$, *and*
3. $\int_{C[0,T]}[x(t_2)-x(t_1)]^2 dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[|\beta(t_2)-\beta(t_1)| + [\alpha(t_2)-\alpha(t_1)]^2]$.

Theorem 2.5. *Let $0 \le t_1 \le t_2 \le T$. Then the followings hold:*

1. $\int_{C[0,T]}x(t_1)dw_{\alpha,\beta;\varphi}(x) \stackrel{*}{=} \varphi(\mathbb{R})[\alpha(t_1)-\alpha(0)] + \int_{\mathbb{R}}ud\varphi(u)$.
2. *If* $\int_{\mathbb{R}}u^2 d\varphi(u) < \infty$, *then*

$$\begin{aligned}\int_{C[0,T]}x(t_1)x(t_2)dw_{\alpha,\beta;\varphi}(x) \;=\;& \varphi(\mathbb{R})[[\alpha(t_2)-\alpha(t_1)][\alpha(t_1)-\alpha(0)]+\beta(t_1)-\beta(0)]\\ &+\int_{\mathbb{R}}[\alpha(t_1)-\alpha(0)+u]^2 d\varphi(u)+[\alpha(t_2)-\alpha(t_1)]\int_{\mathbb{R}}ud\varphi(u).\end{aligned}$$

In particular, we have

$$\int_{C[0,T]}[x(t_1)]^2 dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\beta(t_1)-\beta(0)] + \int_{\mathbb{R}}[\alpha(t_1)-\alpha(0)+u]^2 d\varphi(u).$$

Proof. By Theorem 2.1 and Corollary 2.4, we have

$$\int_{C[0,T]} x(t_1)dw_{\alpha,\beta;\varphi}(x) = \int_{C[0,T]}[x(t_1)-x(0)+x(0)]dw_{\alpha,\beta;\varphi}(x) \stackrel{*}{=} \varphi(\mathbb{R})[\alpha(t_1)-\alpha(0)] + \int_{\mathbb{R}} ud\varphi(u).$$

If $0 < t_1 \le T$, then we have by Theorem 2.1

$$\begin{aligned}\int_{C[0,T]}[x(t_1)]^2 dw_{\alpha,\beta;\varphi}(x) &= \Big[\frac{1}{2\pi[\beta(t_1)-\beta(0)]}\Big]^{\frac{1}{2}}\int_{\mathbb{R}^2} u_1^2\exp\Big\{-\frac{[u_1-\alpha(t_1)-u_0+\alpha(0)]^2}{2[\beta(t_1)-\beta(0)]}\Big\}dm_L(u_1)d\varphi(u_0)\\ &= \int_{\mathbb{R}}[\beta(t_1)-\beta(0)+[\alpha(t_1)+u_0-\alpha(0)]^2]d\varphi(u_0)\end{aligned}$$

which also holds for $t_1 = 0$. By Theorem 2.1 and Corollary 2.4, we have

$$\begin{aligned}\int_{C[0,T]} x(t_1)x(t_2)dw_{\alpha,\beta;\varphi}(x) &= \int_{C[0,T]}[x(t_2)-x(t_1)][x(t_1)-x(0)]dw_{\alpha,\beta;\varphi}(x) + \int_{C[0,T]}[x(t_1)]^2 dw_{\alpha,\beta;\varphi}(x)\\ &\quad + \int_{C[0,T]}[x(t_2)-x(t_1)]x(0)dw_{\alpha,\beta;\varphi}(x)\\ &= \varphi(\mathbb{R})[[\alpha(t_2)-\alpha(t_1)][\alpha(t_1)-\alpha(0)]+\beta(t_1)-\beta(0)] + \int_{\mathbb{R}}[\alpha(t_1)-\alpha(0)+u]^2\\ &\quad d\varphi(u) + [\alpha(t_2)-\alpha(t_1)]\int_{\mathbb{R}} ud\varphi(u)\end{aligned}$$

which completes the proof. □

Theorem 2.6. *Let $0 \le t \le T$. Let $X_0(x) = x(0)$ and $X_t(x) = x(t)$ for $x \in C[0,T]$. Let φ_{X_0}, φ_{X_t} and $\varphi_{X_t - X_0}$ be the Fourier-transforms of X_0, X_t and $X_t - X_0$, respectively. Then for $\xi \in \mathbb{R}$*

$$\begin{aligned}\varphi_{X_0}(\xi) &= \int_{\mathbb{R}}\exp\{i\xi u\}d\varphi(u),\\ \varphi_{X_t-X_0}(\xi) &= \varphi(\mathbb{R})\exp\Big\{-\frac{\xi^2}{2}[\beta(t)-\beta(0)]+i\xi[\alpha(t)-\alpha(0)]\Big\}\end{aligned}$$

and

$$\varphi_{X_t}(\xi) = \frac{1}{\varphi(\mathbb{R})}\varphi_{X_t-X_0}(\xi)\varphi_{X_0}(\xi).$$

Proof. For $\xi \in \mathbb{R}$ we have by Theorem 2.1

$$\varphi_{X_t-X_0}(\xi) = \int_{C[0,T]}\exp\{i\xi[x(t)-x(0)]\}dw_{\alpha,\beta;\varphi}(x) = \int_{\mathbb{R}}\exp\Big\{-\frac{1}{2}\xi^2[\beta(t)-\beta(0)]+i\xi[\alpha(t)-\alpha(0)]\Big\}d\varphi(u_0)$$

and

$$\begin{aligned}\varphi_{X_t}(\xi) &= \int_{C[0,T]}\exp\{i\xi x(t)\}dw_{\alpha,\beta;\varphi}(x)\\ &= \int_{C[0,T]}\exp\{i\xi[x(t)-x(0)]+i\xi x(0)\}dw_{\alpha,\beta;\varphi}(x)\\ &= \int_{\mathbb{R}}\exp\Big\{-\frac{1}{2}\xi^2[\beta(t)-\beta(0)]+i\xi[\alpha(t)-\alpha(0)]+i\xi u_0\Big\}d\varphi(u_0).\end{aligned}$$

Now this theorem follows easily. □

By Theorem 2.6, we have the following corollary.

Corollary 2.7. *Suppose that φ is a probability measure on $\mathcal{B}(\mathbb{R})$.*

1. *If $0 \le t_1 < t_2 \le T$ and $X(x) = x(t_2) - x(t_1)$ for $x \in C[0,T]$, then the characteristic function φ_X of X is given by*
$$\varphi_X(\xi) = \exp\Big\{-\frac{1}{2}\xi^2[\beta(t_2) - \beta(t_1)] + i\xi[\alpha(t_2) - \alpha(t_1)]\Big\}$$
for $\xi \in \mathbb{R}$ so that X is normally distributed with the mean $\alpha(t_2) - \alpha(t_1)$ and the variance $\beta(t_2) - \beta(t_1)$.
2. *If $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$, $X_1(x) = x(t_2) - x(t_1)$ and $X_2(x) = x(t_4) - x(t_3)$ for $x \in C[0,T]$, then X_1 and X_2 are independent.*
3. *If $0 \le t \le T$, $X_0(x) = x(0)$ and $X_t(x) = x(t) - x(0)$ for $x \in C[0,T]$, then X_0 and X_t are independent.*

Remark 2.8. *Some results of Corollaries 2.4, 2.7 and Theorems 2.5, 2.6 were proved by Ryu using Theorem 2.1 [11, 12].*

3. An analogue of the Itô integral

In this section, we define a measurable function on $C[0,T]$ that is similar to the Itô integral.

Let α be absolutely continuous on $[0,T]$ and let β be continuous, strictly increasing on $[0,T]$. We observe that the functions α and β induce a Lebesgue-Stieltjes measure $\nu_{\alpha,\beta}$ on $[0,T]$ by $\nu_{\alpha,\beta} = \nu_\alpha + \nu_\beta$, where $\nu_\alpha(E) = \int_E d|\alpha|(t)$ with the total variation $|\alpha|$ of α and $\nu_\beta(E) = \int_E d\beta(t)$ for a Lebesgue measurable subset E of $[0,T]$. Define $L^2_{\alpha,\beta}[0,T]$ to be the space of functions on $[0,T]$ that are square integrable with respect to the measure $\nu_{\alpha,\beta}$ induced by α and β [10]; that is,
$$L^2_{\alpha,\beta}[0,T] = \Big\{f : [0,T] \to \mathbb{R} \Big| \int_0^T [f(t)]^2 d\nu_{\alpha,\beta}(t) < \infty\Big\}.$$
The space $L^2_{\alpha,\beta}[0,T]$ is in fact a Hilbert space (as our notation suggests), and has the obvious inner product
$$\langle f, g\rangle_{\alpha,\beta} = \int_0^T f(t)g(t)d\nu_{\alpha,\beta}(t).$$

Let $S[0,T]$ be the collection of step functions on $[0,T]$ and let $\phi(\in S[0,T])$ have the form $\phi(t) = \sum_{j=1}^n c_j\chi_{I_j}(t)$ for $t \in [0,T]$, where $c_j \in \mathbb{R}$ and the intervals $I_j \subseteq [0,T]$ with endpoints t_{j-1} and t_j are mutually disjoint. For $x \in C[0,T]$, we define $\int_0^T \phi(t)dx(t)$ as the Riemann-Stieltjes integral $\int_0^T \phi(t)dx(t) = \sum_{j=1}^n c_j[x(t_j) - x(t_{j-1})]$. For convenience, the norm on $L^2(C[0,T])$ is denoted by $\|\cdot\|_C$.

Lemma 3.1. *If $\phi \in S[0,T]$, then*

1. $\int_{C[0,T]}\int_0^T \phi(t)dx(t)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\int_0^T \phi(t)d\alpha(t)$,
2. $\|\int_0^T \phi(t)dx(t)\|_C^2 = \varphi(\mathbb{R})[\|\phi\|_{0,\beta}^2 + [\int_0^T \phi(t)d\alpha(t)]^2]$, *and*
3. $\int_0^T \phi(t)dx(t)$ *is normally distributed with the mean* $\int_0^T \phi(t)d\alpha(t)$ *and the variance* $\|\phi\|_{0,\beta}^2$ *if* $\varphi(\mathbb{R}) = 1$.

Proof. Let $\phi(\in S[0,T])$ have the form $\phi(t) = \sum_{j=1}^n c_j\chi_{I_j}(t)$ for $t \in [0,T]$, where $c_j \in \mathbb{R}$ and the intervals $I_j \subseteq [0,T]$ with endpoints t_{j-1} and t_j are mutually disjoint. By Corollary 2.4, we have
$$\int_{C[0,T]}\int_0^T \phi(t)dx(t)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\int_0^T \phi(t)d\alpha(t)$$
and
$$\begin{aligned}\Big\|\int_0^T \phi(t)dx(t)\Big\|_C^2 &= \int_{C[0,T]}\Big[\sum_{j=1}^n c_j[x(t_j) - x(t_{j-1})]\Big]^2 dw_{\alpha,\beta;\varphi}(x)\\ &= \sum_{j=1}^n c_j^2\int_{C[0,T]}[x(t_j) - x(t_{j-1})]^2 dw_{\alpha,\beta;\varphi}(x) + 2\sum_{1\le j<k\le n} c_jc_k\int_{C[0,T]}[x(t_j) - x(t_{j-1})][x(t_k) - x(t_{k-1})]\end{aligned}$$

$$
\begin{aligned}
& dw_{\alpha,\beta;\varphi}(x) \\
= \; & \varphi(\mathbb{R})\Big[\sum_{j=1}^{n} c_j^2[\beta(t_j)-\beta(t_{j-1})] + \sum_{j=1}^{n}\sum_{k=1}^{n} c_jc_k[\alpha(t_j)-\alpha(t_{j-1})][\alpha(t_k)-\alpha(t_{k-1})]\Big] \\
= \; & \varphi(\mathbb{R})\Big[\|\phi\|_{0,\beta}^2 + \Big[\int_0^T \phi(t)d\alpha(t)\Big]^2\Big].
\end{aligned}
$$

If $\varphi(\mathbb{R}) = 1$, then from Corollary 2.7, the characteristic function of $\int_0^T \phi(t)dx(t)$ is given by

$$
\begin{aligned}
\int_{C[0,T]} \exp\Big\{i\xi\int_0^T \phi(t)dx(t)\Big\}dw_{\alpha,\beta;\varphi}(x) &= \int_{C[0,T]} \exp\Big\{i\xi\sum_{j=1}^{n} c_j[x(t_j)-x(t_{j-1})]\Big\}dw_{\alpha,\beta;\varphi}(x) \\
&= \exp\Big\{-\frac{1}{2}\xi^2\sum_{j=1}^{n} c_j^2[\beta(t_j)-\beta(t_{j-1})] + i\xi\sum_{j=1}^{n} c_j[\alpha(t_j)-\alpha(t_{j-1})]\Big\} \\
&= \exp\Big\{-\frac{1}{2}\xi^2\|\phi\|_{0,\beta}^2 + i\xi\int_0^T \phi(t)d\alpha(t)\Big\}
\end{aligned}
$$

for $\xi \in \mathbb{R}$, which completes the proof. □

For $f \in L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}$ be a sequence of the step functions in $S[0,T]$ with $\lim_{n\to\infty}\|\phi_n - f\|_{\alpha,\beta} = 0$. Define $I_{\alpha,\beta}(f)$ by the $L^2(C[0,T])$-limit

$$
I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty}\int_0^T \phi_n(t)dx(t)
$$

for all $x \in C[0,T]$ for which this limit exists. We note that $S[0,T]$ is dense in $L^2_{\alpha,\beta}[0,T]$ so that the sequence $\{\phi_n\}$ in $S[0,T]$ with $\lim_{n\to\infty}\|\phi_n - f\|_{\alpha,\beta} = 0$ exists. Moreover, we have the following lemma.

Lemma 3.2. *If $f \in L^2_{\alpha,\beta}[0,T]$, then $I_{\alpha,\beta}(f)$ is well-defined; that is, $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and is independent of choice of the sequence $\{\phi_n\}$ in $S[0,T]$.*

Proof. For $f \in L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}$ be a sequence of the step functions in $S[0,T]$ with $\lim_{n\to\infty}\|f - \phi_n\|_{\alpha,\beta} = 0$. Then, each $\int_0^T \phi_n(t)dx(t)$ is an element of $L^2(C[0,T])$ by Lemma 3.1. We now have $\phi_n - \phi_m \in S[0,T]$ so that by Lemma 3.1 and the Hölder's inequality,

$$
\begin{aligned}
\Big\|\int_0^T \phi_n(t)dx(t) - \int_0^T \phi_m(t)dx(t)\Big\|_C^2 &= \varphi(\mathbb{R})\Big[\|\phi_n-\phi_m\|_{0,\beta}^2 + \Big[\int_0^T (\phi_n-\phi_m)(t)d\alpha(t)\Big]^2\Big] \\
&\le \varphi(\mathbb{R})[\|\phi_n-\phi_m\|_{0,\beta}^2 + \nu_\alpha([0,T])\|\phi_n-\phi_m\|_{\alpha,0}^2] \\
&\le \varphi(\mathbb{R})\max\{1,\nu_\alpha([0,T])\}\|\phi_n-\phi_m\|_{\alpha,\beta}^2
\end{aligned}
$$

which converges to 0 as m, n approach ∞. From this, we conclude that the sequence $\{\int_0^T \phi_n(t)dx(t)\}$ is Cauchy in $L^2(C[0, T])$ so that $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Moreover, if $\lim_{n\to\infty}\phi_n = f = \lim_{n\to\infty}\psi_n$ in $L^2_{\alpha,\beta}[0,T]$ for sequences $\{\phi_n\}$ and $\{\psi_n\}$ in $S[0,T]$, then by Lemma 3.1, the Hölder's inequality and the Minkowski's inequality, we have

$$
\Big\|\int_0^T \phi_n(t)dx(t) - \int_0^T \psi_n(t)dx(t)\Big\|_C \le [\varphi(\mathbb{R})\max\{1,\nu_\alpha([0,T])\}]^{\frac{1}{2}}[\|\phi_n - f\|_{\alpha,\beta} + \|f - \psi_n\|_{\alpha,\beta}]
$$

which also converges to 0 as n approaches ∞. Now we have

$$
I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty}\int_0^T \phi_n(t)dx(t) = \lim_{n\to\infty}\int_0^T \psi_n(t)dx(t)
$$

in $L^2(C[0,T])$ and conclude that the definition of $I_{\alpha,\beta}(f)$ is essentially independent of choice of the sequence from $S[0,T]$ that is used to define it. $\square$

Theorem 3.3. *Let $f,g \in L^2_{\alpha,\beta}[0,T]$ and $c_1, c_2 \in \mathbb{R}$. Then the followings hold:*

1. *If $f \in S[0,T]$, then $I_{\alpha,\beta}(f)(x) = \int_0^T f(t)dx(t)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.*
2. *$I_{\alpha,\beta}(c_1 f + c_2 g)(x) = c_1 I_{\alpha,\beta}(f)(x) + c_2 I_{\alpha,\beta}(g)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.*
3. *$\int_{C0,T]} I_{\alpha,\beta}(f)(x)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R}) \int_0^T f(t)d\alpha(t)$.*
4. *$\|I_{\alpha,\beta}(f)\|_C^2 = \varphi(\mathbb{R})[\|f\|_{0,\beta}^2 + [\int_0^T f(t)d\alpha(t)]^2]$.*
5. *$\int_{C[0,T]}[I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\langle f,g\rangle_{0,\beta} + [\int_0^T f(t)d\alpha(t)][\int_0^T g(t)d\alpha(t)]]$.*
6. *$I_{\alpha,\beta}(f)$ is a normally distributed random variable with the mean $\int_0^T f(t)d\alpha(t)$ and the variance $\|f\|_{0,\beta}^2$ if $\varphi(\mathbb{R}) = 1$. In this case, $Cov(I_{\alpha,\beta}(f), I_{\alpha,\beta}(g)) = \langle f,g\rangle_{0,\beta}$.*

Proof. The equality in Theorem 3.3.1 is trivial by the definition of $I_{\alpha,\beta}(f)$. Take $\{\phi_n\}$ and $\{\psi_n\}$ to be sequences in $S[0,T]$ with $\lim_{n\to\infty}\phi_n = f$ and $\lim_{n\to\infty}\psi_n = g$ in $L^2_{\alpha,\beta}[0,T]$. Then $c_1\phi_n + c_2\psi_n$ belongs to $S[0,T]$ and $\|c_1\phi_n + c_2\psi_n - (c_1 f + c_2 g)\|_{\alpha,\beta} \le |c_1|\|\phi_n - f\|_{\alpha,\beta} + |c_2|\|\psi_n - g\|_{\alpha,\beta}$ so that $c_1 f + c_2 g = \lim_{n\to\infty}(c_1\phi_n + c_2\psi_n)$ in $L^2_{\alpha,\beta}[0,T]$. Now we have by the linearity of Riemann-Stieltjes integral

$$\|I_{\alpha,\beta}(c_1\phi_n + c_2\psi_n) - [c_1 I_{\alpha,\beta}(f) + c_2 I_{\alpha,\beta}(g)]\|_C \le |c_1|\|I_{\alpha,\beta}(\phi_n) - I_{\alpha,\beta}(f)\|_C + |c_2|\|I_{\alpha,\beta}(\psi_n) - I_{\alpha,\beta}(g)\|_C$$

which converges to 0 as n approaches ∞. By the uniqueness of limit in $L^2(C[0,T])$, we have $I_{\alpha,\beta}(c_1 f + c_2 g)(x) = c_1 I_{\alpha,\beta}(f)(x) + c_2 I_{\alpha,\beta}(g)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, which proves Theorem 3.3.2. We also have by the Hölder's inequality

$$\left|\int_{C[0,T]} I_{\alpha,\beta}(\phi_n)(x)dw_{\alpha,\beta;\varphi}(x) - \int_{C0,T]} I_{\alpha,\beta}(f)(x)dw_{\alpha,\beta;\varphi}(x)\right|^2 \le \varphi(\mathbb{R})\|I_{\alpha,\beta}(\phi_n) - I_{\alpha,\beta}(f)\|_C^2$$

which converges to 0 as n approaches ∞ by the definition of $I_{\alpha,\beta}(f)$. Moreover,

$$\left|\int_0^T \phi_n(t)d\alpha(t) - \int_0^T f(t)d\alpha(t)\right|^2 \le v_a([0,T])\|\phi_n - f\|_{\alpha,\beta}^2$$

which also converges to 0 as n approaches ∞. Now we have by Lemma 3.1

$$\int_{C0,T]} I_{\alpha,\beta}(f)(x)dw_{\alpha,\beta;\varphi}(x) = \lim_{n\to\infty}\int_{C[0,T]} I_{\alpha,\beta}(\phi_n)(x)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\lim_{n\to\infty}\int_0^T \phi_n(t)d\alpha(t) = \varphi(\mathbb{R})\int_0^T f(t)d\alpha(t)$$

which proves Theorem 3.3.3. Since $I_{\alpha,\beta}(f) = \lim_{n\to\infty} I_{\alpha,\beta}(\phi_n)$ in $L^2(C[0,T])$, we have

$$\|I_{\alpha,\beta}(f)\|_C^2 = \lim_{n\to\infty}\|I_{\alpha,\beta}(\phi_n)\|_C^2 = \varphi(\mathbb{R})\lim_{n\to\infty}\left[\|\phi_n\|_{0,\beta}^2 + \left[\int_0^T \phi_n(t)d\alpha(t)\right]^2\right]$$

by Lemma 3.1. Since $0 \le |\|\phi_n\|_{0,\beta} - \|f\|_{0,\beta}| \le \|\phi_n - f\|_{0,\beta} \le \|\phi_n - f\|_{\alpha,\beta}$ and $\lim_{n\to\infty}\phi_n = f$ in $L^2_{\alpha,\beta}[0,T]$, we have $\lim_{n\to\infty}\|\phi_n\|_{0,\beta}^2 = \|f\|_{0,\beta}^2$ so that

$$\|I_{\alpha,\beta}(f)\|_C^2 = \varphi(\mathbb{R})\left[\|f\|_{0,\beta}^2 + \left[\int_0^T f(t)d\alpha(t)\right]^2\right]$$

which proves Theorem 3.3.4. Furthermore, we have by Theorems 3.3.2 and 3.3.4

$$\begin{aligned}\varphi(\mathbb{R})\left[\|f+g\|_{0,\beta}^2 + \left[\int_0^T [f(t)+g(t)]d\alpha(t)\right]^2\right] &= \|I_{\alpha,\beta}(f+g)\|_C^2 \\ &= \|I_{\alpha,\beta}(f)\|_C^2 + \|I_{\alpha,\beta}(g)\|_C^2 + 2\int_{C[0,T]}[I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]dw_{\alpha,\beta;\varphi}(x)\end{aligned}$$

$$= \varphi(\mathbb{R})\Big[\|f\|_{0,\beta}^2 + \|g\|_{0,\beta}^2 + \Big[\int_0^T f(t)d\alpha(t)\Big]^2 + \Big[\int_0^T g(t)d\alpha(t)\Big]^2\Big] + 2\int_{C[0,T]} [I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]dw_{\alpha,\beta;\varphi}(x)$$

so that we have

$$\int_{C[0,T]} [I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\Big[\langle f,g\rangle_{0,\beta} + \Big[\int_0^T f(t)d\alpha(t)\Big]\Big[\int_0^T g(t)d\alpha(t)\Big]\Big]$$

which proves Theorem 3.3.5. Since $\lim_{n\to\infty} I_{\alpha,\beta}(\phi_n) = I_{\alpha,\beta}(f)$ in $L^2(C[0,T])$, take a subsequence $\{I_{\alpha,\beta}(\phi_{n_k})\}_{k=1}^\infty$ of $\{I_{\alpha,\beta}(\phi_n)\}$ with $\lim_{k\to\infty} I_{\alpha,\beta}(\phi_{n_k})(x) = I_{\alpha,\beta}(f)(x)$ pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. If $\varphi(\mathbb{R}) = 1$, then we have for $\xi \in \mathbb{R}$

$$\begin{aligned} E[\exp\{i\xi I_{\alpha,\beta}(f)\}] &= \int_{C[0,T]} \exp\Big\{i\xi \lim_{k\to\infty} I_{\alpha,\beta}(\phi_{n_k})(x)\Big\} dw_{\alpha,\beta;\varphi}(x) \\ &= \lim_{k\to\infty} \int_{C[0,T]} \exp\{i\xi I_{\alpha,\beta}(\phi_{n_k})(x)\} dw_{\alpha,\beta;\varphi}(x) \\ &= \lim_{k\to\infty} \exp\Big\{-\frac{1}{2}\xi^2\|\phi_{n_k}\|_{0,\beta}^2 + i\xi\int_0^T \phi_{n_k}(t)d\alpha(t)\Big\} \\ &= \exp\Big\{-\frac{1}{2}\xi^2\|f\|_{0,\beta}^2 + i\xi\int_0^T f(t)d\alpha(t)\Big\} \end{aligned}$$

by the dominated convergence theorem and Lemma 3.1, so that the final results follow by Theorem 3.3.5. □

The following theorem is useful and the proof of it is motivated by results in [13].

Theorem 3.4. *If f is of bounded variation on $[0,T]$, then $I_{\alpha,\beta}(f)(x) = \int_0^T f(t)dx(t)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $\int_0^T f(t)dx(t)$ denotes the Riemann-Stieltjes integral of f with respect to x.*

Proof. Suppose that f is monotonically increasing on $[0,T]$. If $f(T) = f(0)$, that is, f is a constant function on $[0,T]$, then the result is trivial, so that we assume that $f(T) > f(0)$. Let $M = f(T) - f(0)$ and for $k = 1,\dots,n$, let

$$D_{n,k} = \Big\{t \in [0,T]\Big| f(0) + \frac{k-1}{n}M \le f(t) < f(0) + \frac{k}{n}M\Big\}.$$

Since f is monotonically increasing, $D_{n,k}$ is either an interval or a point or an empty-set. If $D_{n,k}$ is a point, then adjoin the point to an its adjacent interval. In this way we have a decomposition of $[0,T]$ into finitely many disjoint intervals. If necessary, we decompose these intervals so that the lengths of the resulting intervals $J_{n,k}$ ($k = 1,2,\dots,m_n$) with endpoints $t_{n,k-1}$ and $t_{n,k}$ are less than $\frac{T}{n}$. For $t \in [0,T]$, let

$$\phi_n(t) = \sum_{k=1}^{m_n} f(t_{n,k-1}+)\chi_{J_{n,k}}(t)$$

for $n = 1,2,\dots$. Then $\phi_n \in S[0,T]$ and $|\phi_n(t) - f(t)| \le \frac{M}{n}$ for $\nu_{\alpha,\beta}$ a.e. $t \in [0,T]$ so that

$$\lim_{n\to\infty} \|\phi_n - f\|_{\alpha,\beta}^2 \le \lim_{n\to\infty} \frac{M^2}{n^2}\nu_{\alpha,\beta}([0,T]) = 0.$$

Thus $I_{\alpha,\beta}(f) = \lim_{n\to\infty} I_{\alpha,\beta}(\phi_n)$ is in $L^2(C[0,T])$. Now there exists a subsequence $\{I_{\alpha,\beta}(\phi_{n_l})\}_{l=1}^\infty$ of $\{I_{\alpha,\beta}(\phi_n)\}_{n=1}^\infty$ with $\lim_{l\to\infty} I_{\alpha,\beta}(\phi_{n_l})(x) = I_{\alpha,\beta}(f)(x)$ pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$,

$$I_{\alpha,\beta}(f)(x) = \lim_{l\to\infty} I_{\alpha,\beta}(\phi_{n_l})(x) = \lim_{l\to\infty} \sum_{j=1}^{m_{n_l}} f(t_{n_l,j-1}+)[x(t_{n_l,j}) - x(t_{n_l,j-1})] = \int_0^T f(t)dx(t).$$

If f is of bounded variation on $[0,T]$, then there exist monotonically increasing functions f_1 and f_2 on $[0,T]$ with $f = f_1 - f_2$. By Theorem 3.3, $I_{\alpha,\beta}(f)(x) = I_{\alpha,\beta}(f_1)(x) - I_{\alpha,\beta}(f_2)(x) = \int_0^T f_1(t)dx(t) - \int_0^T f_2(t)dx(t) = \int_0^T f(t)dx(t)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, which completes the proof. □

By Theorems 3.3 and 3.4, we have the following corollary.

Corollary 3.5. *Let $f, g \in L^2_{\alpha,\beta}[0,T]$ and each be of bounded variation on $[0,T]$. Then*

$$\int_{C[0,T]} \Big[\int_0^T f(t)dx(t)\Big]\Big[\int_0^T g(t)dx(t)\Big] dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\Big[\langle f, g\rangle_{0,\beta} + \Big[\int_0^T f(t)d\alpha(t)\Big]\Big[\int_0^T g(t)d\alpha(t)\Big]\Big].$$

Theorem 3.6. *Let $\{f_1,\dots,f_n\}$ be a set of functions in $L^2_{\alpha,\beta}[0,T]$ which are independent in $L^2_{0,\beta}[0,T]$, and let $\int_0^T \vec{f}(t)d\alpha(t) = (\int_0^T f_1(t)d\alpha(t),\dots,\int_0^T f_n(t)d\alpha(t))$. If $f : \mathbb{R}^n \to \mathbb{C}$ is Borel measurable, then we have*

$$\begin{aligned} &\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x),\dots,I_{\alpha,\beta}(f_n)(x))dw_{\alpha,\beta;\varphi}(x) \\ \overset{*}{=}\ & \varphi(\mathbb{R})\Big[\frac{1}{(2\pi)^n|M|}\Big]^{\frac12}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac12\Big\langle M^{-1}\Big[\vec{u}-\int_0^T \vec{f}(t)d\alpha(t)\Big], \vec{u}-\int_0^T \vec{f}(t)d\alpha(t)\Big\rangle_{\mathbb{R}^n}\Big\}dm_L^n(\vec{u}), \end{aligned} \tag{1}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the dot product on $\mathbb{R}^n$ and $M = [\langle f_i, f_j\rangle_{0,\beta}]_{n\times n}$ which is positive definite and non-singular. Moreover, if φ is a probability measure on $\mathcal{B}(\mathbb{R})$, then the random vector $(I_{\alpha,\beta}(f_1),\dots,I_{\alpha,\beta}(f_n))$ has the multivariate normal distribution with the mean vector $\int_0^T \vec{f}(t)d\alpha(t)$ and the covariance matrix M.

Proof. Let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})}\varphi$. Then φ_0 is a probability measure on $\mathbb{R}$ so that $w_{\alpha,\beta;\varphi_0}$ is also a probability measure on $C[0,T]$. By Theorem 3.3.6, $I_{\alpha,\beta}(f_i)$ with respect to $w_{\alpha,\beta;\varphi_0}$ is Gaussian with the mean $\int_0^T f_i(t)d\alpha(t)$ and the variance $\|f_i\|^2_{0,\beta}$ for $i = 1,\dots,n$. For $\vec{c} = (c_1,\dots,c_n) \in \mathbb{R}^n$,

$$\begin{aligned} \langle M\vec{c}, \vec{c}\rangle_{\mathbb{R}^n} &= \sum_{i=1}^n\sum_{j=1}^n c_ic_j\langle f_i, f_j\rangle_{0,\beta} \\ &= \sum_{i=1}^n\sum_{j=1}^n c_ic_j\int_{C[0,T]}[I_{\alpha,\beta}(f_i)(x) - E[I_{\alpha,\beta}(f_i)]][I_{\alpha,\beta}(f_j)(x) - E[I_{\alpha,\beta}(f_j)]]dw_{\alpha,\beta;\varphi_0}(x) \\ &= \int_{C[0,T]}\Big[I_{\alpha,\beta}\Big(\sum_{j=1}^n c_jf_j\Big)(x) - E\Big[I_{\alpha,\beta}\Big(\sum_{j=1}^n c_jf_j\Big)\Big]\Big]^2 dw_{\alpha,\beta;\varphi_0}(x) \\ &= \Big\|\sum_{j=1}^n c_jf_j\Big\|^2_{0,\beta} \ge 0 \end{aligned}$$

by Theorem 3.3.2. Moreover, if $\langle M\vec{c},\vec{c}\rangle_{\mathbb{R}^n} = 0$, then $\sum_{j=1}^n c_jf_j(t) = 0$ for ν_β a.e. $t \in [0,T]$, which implies $\vec{c} = \vec{0}$ by the independence of $\{f_1,\dots,f_n\}$ in $L^2_{0,\beta}[0,T]$. Now M is positive definite and symmetric so that M is non-singular and M^{-1} is positive definite. By Theorem 4 of [10], we have (1) for φ_0. Since $w_{\alpha,\beta;\varphi} = \varphi(\mathbb{R})w_{\alpha,\beta;\varphi_0}$ by their definitions, the null sets with respect to $w_{\alpha,\beta;\varphi}$ are equivalent to the null sets with respect to $w_{\alpha,\beta;\varphi_0}$, so that for $f \in L^2_{\alpha,\beta}[0,T]$, $I_{\alpha,\beta}(f)$ with respect to $w_{\alpha,\beta;\varphi}$ is also equivalent to $I_{\alpha,\beta}(f)$ with respect to $w_{\alpha,\beta;\varphi_0}$. Now we have (1) for arbitrary φ, since

$$\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x),\dots,I_{\alpha,\beta}(f_n)(x))dw_{\alpha,\beta;\varphi}(x) \overset{*}{=} \varphi(\mathbb{R})\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x),\dots,I_{\alpha,\beta}(f_n)(x))dw_{\alpha,\beta;\varphi_0}(x),$$

where $I_{\alpha,\beta}(f_i)$ of the left-hand and right-hand sides are taken over $w_{\alpha,\beta;\varphi}$ and $w_{\alpha,\beta;\varphi_0}$, respectively. The remainder of this theorem immediately follows. □

Using characteristic functions and Theorem 3.6, we can prove the following corollary.

Corollary 3.7 (Generalized PWZ Theorem). *Let $\{f_1, \dots, f_n\}$ be a set of functions in $L^2_{\alpha,\beta}[0,T]$, which are nonzero and orthogonal in $L^2_{0,\beta}[0,T]$. Then, for a Borel measurable function $f : \mathbb{R}^n \to \mathbb{C}$,*

$$\begin{aligned}&\int_{C[0,T]} f(I_{\alpha,\beta}(f_1)(x), \dots, I_{\alpha,\beta}(f_n)(x))dw_{\alpha,\beta;\varphi}(x)\\ \overset{*}{=}\ &\varphi(\mathbb{R})\Big[\prod_{j=1}^{n}\frac{1}{2\pi\|f_j\|^2_{0,\beta}}\Big]^{\frac{1}{2}}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{[u_j - \int_0^T f_j(t)d\alpha(t)]^2}{\|f_j\|^2_{0,\beta}}\Big\}dm_L^n(\vec{u}),\end{aligned}$$

where $\vec{u} = (u_1, \dots, u_n)$. Moreover, if φ is a probability measure on $\mathcal{B}(\mathbb{R})$, then $I_{\alpha,\beta}(f_1), \dots, I_{\alpha,\beta}(f_n)$ are independent random variables.

Using Corollary 3.7, we can prove the following corollary suggested by Ryu [12].

Corollary 3.8. *Assume that φ is a probability measure. Let $\{\frac{h_1}{\sqrt{\beta'}}, \dots, \frac{h_n}{\sqrt{\beta'}}\}$ be a set of nonzero orthogonal functions in $L^2[0,T]$ such that $\frac{h_j}{\beta'}(j = 1, \dots, n)$ are of bounded variation on $[0,T]$. For $j = 1, \dots, n$, let $X_j(x) = \int_0^T \frac{h_j(t)}{\beta'(t)}dx(t)$ for $x \in C[0,T]$. Then $X_1, \dots, X_n$ are independent random variables and each X_j has the normal distribution with the mean $\int_0^T \frac{h_j(t)}{\beta'(t)}d\alpha(t)$ and the variance $\|\frac{h_j}{\sqrt{\beta'}}\|^2_{m_L}$. Moreover, if $f : \mathbb{R}^n \to \mathbb{C}$ is Borel measurable, then*

$$\int_{C[0,T]} f(X_1(x), \dots, X_n(x))dw_{\alpha,\beta;\varphi}(x) \overset{*}{=} \Big[\prod_{j=1}^{n}\frac{1}{2\pi\|\frac{h_j}{\sqrt{\beta'}}\|^2_{m_L}}\Big]^{\frac{1}{2}}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{[u_j - \int_0^T \frac{h_j(t)}{\beta'(t)}d\alpha(t)]^2}{\|\frac{h_j}{\sqrt{\beta'}}\|^2_{m_L}}\Big\}dm_L^n(\vec{u}).$$

Proof. By Theorems 3.3 and 3.4, each X_j has the normal distribution with the mean $\int_0^T \frac{h_j(t)}{\beta'(t)}d\alpha(t)$ and the variance $\|\frac{h_j}{\beta'}\|^2_{0,\beta} = \|\frac{h_j}{\sqrt{\beta'}}\|^2_{m_L}$. By the assumption, we also have $\langle\frac{h_l}{\beta'}, \frac{h_j}{\beta'}\rangle_{0,\beta} = \langle\frac{h_l}{\sqrt{\beta'}}, \frac{h_j}{\sqrt{\beta'}}\rangle_{m_L} = 0$ if $l \neq j$; that is, $\{\frac{h_j}{\beta'} : j = 1, \dots, n\}$ is a set of nonzero orthogonal functions in $L^2_{0,\beta}[0,T]$ so that $X_1, \dots, X_n$ are independent random variables and the equality $\overset{*}{=}$ holds by Corollary 3.7. □

Remark 3.9. *Suppose that β' is bounded away from zero. As $\beta' > 0$, ν_β is mutually, absolutely continuous with respect to m_L. We note that ν_α is absolutely continuous with respect to m_L, but that the converse need not hold. Thus $\nu_{\alpha,\beta}$ is absolutely continuous with respect to m_L and $L^2_{\alpha,\beta}[0,T] \subseteq L^2_{0,\beta}[0,T] \subseteq L^2[0,T]$ in general. The inclusions mean that they are continuously embedded as vector spaces, but that they need not be embedded isometrically.*

Theorem 3.10. *$I_{\alpha,\beta}$ is a bounded linear operator from $L^2_{\alpha,\beta}[0,T]$ into $L^2(C[0,T])$ and for all $f \in L^2_{\alpha,\beta}[0,T]$,*

$$\|I_{\alpha,\beta}(f)\|_C \leq [\varphi(\mathbb{R})\max\{1, \nu_\alpha([0,T])\}]^{\frac{1}{2}}\|f\|_{\alpha,\beta}.$$

Moreover, the followings hold:

1. *If β' is bounded away from zero, then $I_{\alpha,\beta}$ is injective.*
2. *If $I_{\alpha,\beta}$ is injective, then the inverse operator $I^{-1}_{\alpha,\beta} : Im(I_{\alpha,\beta}) \to L^2_{\alpha,\beta}[0,T]$ is bounded if and only if $Im(I_{\alpha,\beta})$ itself is a Hilbert space.*
3. *If α is a constant function (or equivalently, $\int_0^T I_{\alpha,\beta}(f)(x)dw_{\alpha,\beta;\varphi}(x) = 0$ for all $f \in S[0,T]$), then for all $f \in L^2_{\alpha,\beta}[0,T]$,*

$$\|I_{\alpha,\beta}(f)\|_C = [\varphi(\mathbb{R})]^{\frac{1}{2}}\|f\|_{\alpha,\beta} \tag{2}$$

so that $I^{-1}_{\alpha,\beta} : Im(I_{\alpha,\beta}) \to L^2_{\alpha,\beta}[0,T]$ is bounded and $Im(I_{\alpha,\beta})$ is a Hilbert space.

4. *If α is a constant function and $\varphi(\mathbb{R}) = 1$, then $I_{\alpha,\beta}$ is an isometric isomorphism between $L^2_{\alpha,\beta}[0,T]$ and $Im(I_{\alpha,\beta})$.*

Proof. Since $[\int_0^T f(t)d\alpha(t)]^2 \le \nu_\alpha([0,T])\|f\|^2_{\alpha,0}$ by the Hölder's inequality, we have the inequality of this theorem by Theorem 3.3.4 so that $I_{\alpha,\beta}$ is a bounded linear operator from $L^2_{\alpha,\beta}[0,T]$ into $L^2(C[0,T])$ by Theorem 3.3.2. Suppose that β' is bounded away from zero and $\|I_{\alpha,\beta}(f)\|_C = 0$ for $f \in L^2_{\alpha,\beta}[0,T]$. By Theorem 3.3.4, $\|f\|^2_{0,\beta} \le \frac{1}{\varphi(\mathbb{R})}\|I_{\alpha,\beta}(f)\|^2_C = 0$ so that $\|f\|_{0,\beta} = 0$ and $\|f\|_{m_L} = 0$ by the above remark. Now we have $\|f\|_{\alpha,0} = 0$ since ν_α is absolutely continuous with respect to m_L. Thus $\|f\|^2_{\alpha,\beta} = \|f\|^2_{\alpha,0} + \|f\|^2_{0,\beta} = 0$; that is, $f = 0$ in $L^2_{\alpha,\beta}[0,T]$, which implies that $I_{\alpha,\beta}$ is injective. Suppose that $I_{\alpha,\beta}$ is injective and the inverse operator $I^{-1}_{\alpha,\beta} : Im(I_{\alpha,\beta}) \to L^2_{\alpha,\beta}[0,T]$ is bounded. Then $Im(I_{\alpha,\beta})$ is a closed subspace of $L^2(C[0,T])$ so that it is a Hilbert space. Conversely, suppose that $Im(I_{\alpha,\beta})$ is a Hilbert space. Then $I_{\alpha,\beta} : L^2_{\alpha,\beta}[0,T] \to Im(I_{\alpha,\beta})$ is open by the open mapping theorem so that $I^{-1}_{\alpha,\beta} : Im(I_{\alpha,\beta}) \to L^2_{\alpha,\beta}[0,T]$ is bounded. If α is a constant function, (2) immediately follows from Theorem 3.3.4. Furthermore, if $\varphi(\mathbb{R}) = 1$, then $I_{\alpha,\beta} : L^2_{\alpha,\beta}[0,T] \to Im(I_{\alpha,\beta})$ is an isometric isomorphism by Theorem 3.3.5. □

Corollary 3.11. *Let f be in $L^2_{\alpha,\beta}[0,T]$ and let $\{f_n\}$ be a sequence of functions in $L^2_{\alpha,\beta}[0,T]$ with $\lim_{n\to\infty}\|f_n - f\|_{\alpha,\beta} = 0$. Then $\{I_{\alpha,\beta}(f_n)\}$ converges to $I_{\alpha,\beta}(f)$ in $L^2(C[0,T])$ so that it converges to $I_{\alpha,\beta}(f)$ in $L^1(C[0,T])$. Moreover, if $I_{\alpha,\beta}(f_n)(x) \to Y(x)$ pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ as $n \to \infty$, then $Y(x) = I_{\alpha,\beta}(f)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ so that $I_{\alpha,\beta}(f_n) \to Y$ in both $L^1(C[0,T])$ and $L^2(C[0,T])$ as $n \to \infty$.*

Proof. Since $I_{\alpha,\beta}$ is a bounded operator by Theorem 3.10, $\{I_{\alpha,\beta}(f_n)\}$ converges to $I_{\alpha,\beta}(f)$ in $L^2(C[0,T])$. We also have by the Hölder's inequality

$$\left[\int_{C[0,T]} |I_{\alpha,\beta}(f_n)(x) - I_{\alpha,\beta}(f)(x)| dw_{\alpha,\beta;\varphi}(x)\right]^2 \le \varphi(\mathbb{R})\|I_{\alpha,\beta}(f_n) - I_{\alpha,\beta}(f)\|^2_C$$

which converges to 0 as $n \to \infty$; that is, $\{I_{\alpha,\beta}(f_n)\}$ converges to $I_{\alpha,\beta}(f)$ in $L^1(C[0,T])$. Moreover, suppose that $I_{\alpha,\beta}(f_n)(x) \to Y(x)$ pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ as $n \to \infty$. Since $\{I_{\alpha,\beta}(f_n)\}$ converges to $I_{\alpha,\beta}(f)$ in $L^2(C[0,T])$, we can take an its subsequence $\{I_{\alpha,\beta}(f_{n_k})\}_{k=1}^\infty$ converging to $I_{\alpha,\beta}(f)$ pointwisely so that $Y(x) = \lim_{k\to\infty} I_{\alpha,\beta}(f_{n_k})(x) = I_{\alpha,\beta}(f)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. □

By Theorems 3.3.5 and 3.3.6, Corollary 3.7 and Theorem 3.10, we have the following corollary.

Corollary 3.12. *Suppose that α is a constant function. Let $\{f_n\}$ be a sequence in $L^2_{\alpha,\beta}[0,T]$. Then, $\{f_n\}$ is orthogonal in $L^2_{\alpha,\beta}[0,T]$ if and only if $\{I_{\alpha,\beta}(f_n)\}$ orthogonal in $Im(I_{\alpha,\beta})$. Moreover, if φ is a probability measure, then we have the followings:*

1. *$\{f_n\}$ is orthogonal in $L^2_{\alpha,\beta}[0,T]$ if and only if $\{I_{\alpha,\beta}(f_n)\}$ is a set of independent random variables on $C[0,T]$.*
2. *$\{f_n\}$ is orthonormal in $L^2_{\alpha,\beta}[0,T]$ if and only if $\{I_{\alpha,\beta}(f_n)\}$ is orthonormal in $Im(I_{\alpha,\beta})$.*
3. *$\{f_n\}$ is completely orthonormal in $L^2_{\alpha,\beta}[0,T]$ if and only if $\{I_{\alpha,\beta}(f_n)\}$ is completely orthonormal in $Im(I_{\alpha,\beta})$.*

By Theorems 3.3 and 3.10, Corollary 3.12 and Proposition 2.3.3 of [7], we have the following theorem.

Theorem 3.13. *Suppose that α is a constant function and $\varphi(\mathbb{R})$ is a probability measure. Let $\{f_n\}$ be completely orthonormal in $L^2_{\alpha,\beta}[0,T]$. Then for $f \in L^2_{\alpha,\beta}[0,T]$,*

$$I_{\alpha,\beta}(f)(x) = \sum_{n=1}^\infty \langle I_{\alpha,\beta}(f), I_{\alpha,\beta}(f_n)\rangle_C I_{\alpha,\beta}(f_n)(x) = \sum_{n=1}^\infty \langle f, f_n\rangle_{\alpha,\beta} I_{\alpha,\beta}(f_n)(x)$$

in $L^2(C[0,T])$ and pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.

4. An analogue of the Paley-Wiener-Zygmund integral

In this section, we define a generalized PWZ integral on $C[0,T]$ and investigate its properties.

Throughout the remainder of this paper, we give additional conditions for α and β; $\frac{\beta'}{|\alpha|'+\beta'}$ is bounded away from zero and $\frac{1}{\sqrt{|\alpha|'+\beta'}}$ is of bounded variation on $[0,T]$. In the case, $L^2_{\alpha,\beta}[0,T] = L^2_{0,\beta}[0,T]$ and the equality means that they are equal as vector spaces and the two norms on them are equivalent so that they have the same topology, but that they need not be equal isometrically. Moreover, we have the following lemma.

Lemma 4.1. *There exists an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of functions of bounded variation in $L^2_{\alpha,\beta}[0,T]$ such that it is orthogonal in $L^2_{0,\beta}[0,T]$.*

Proof. Let $\zeta(t) = |\alpha|'(t) + \beta'(t)$ for $t \in [0,T]$ and let $\{h_j\}_{j=1}^\infty$ be a complete orthonormal set of functions of bounded variation in $L^2[0,T]$. Note that possible such functions are the trigonometric functions on $[0,T]$. For $j \in \mathbb{N}$, let $\phi_j = \frac{h_j}{\sqrt{\zeta}}$. Then we have

$$\langle \phi_l, \phi_j\rangle_{\alpha,\beta} = \int_0^T \frac{h_l(t)h_j(t)}{\zeta(t)} d\nu_{\alpha,\beta}(t) = \langle h_l, h_j\rangle_{m_L} = \delta_{lj},$$

where δ_{lj} denotes the Kronecker delta function, so that $\{\phi_j\}_{j=1}^\infty$ is an orthonormal set in $L^2_{\alpha,\beta}[0,T]$. Since $\frac{\beta'}{\zeta'}$ is bounded away from zero, we have for some $M > 0$

$$M\delta_{lj} = M\langle h_l, h_j\rangle_{m_L} \le \int_0^T \frac{h_l(t)h_j(t)}{\zeta(t)} d\nu_{\beta}(t) \le \langle h_l, h_j\rangle_{m_L} = \delta_{lj},$$

so that if $l \ne j$, then

$$\langle \phi_l, \phi_j\rangle_{0,\beta} = \int_0^T \frac{h_l(t)h_j(t)}{\zeta(t)} d\nu_{\beta}(t) = 0,$$

which implies that $\{\phi_j\}_{j=1}^\infty$ is an orthogonal set in $L^2_{0,\beta}[0,T]$. We also have for $f \in L^2_{\alpha,\beta}[0,T]$

$$\begin{aligned}
\Big\| \sum_{j=1}^n \langle f, \phi_j\rangle_{\alpha,\beta}\phi_j - f\Big\|^2_{\alpha,\beta} &= \int_0^T \Big[\sum_{j=1}^n \frac{h_j(s)}{\sqrt{\zeta(s)}} \int_0^T f(t)\frac{h_j(t)}{\sqrt{\zeta(t)}} d\nu_{\alpha,\beta}(t) - f(s)\Big]^2 d\nu_{\alpha,\beta}(s) \\
&= \int_0^T \Big[\sum_{j=1}^n \langle f\sqrt{\zeta}, h_j\rangle_{m_L} h_j(s) - f(s)\sqrt{\zeta(s)}\Big]^2 dm_L(s) \\
&= \Big\| \sum_{j=1}^n \langle f\sqrt{\zeta}, h_j\rangle_{m_L} h_j - f\sqrt{\zeta}\Big\|^2_{m_L}
\end{aligned}$$

which converges to 0 as $n \to \infty$, since $\{h_j\}_{j=1}^\infty$ is completely orthonormal in $L^2[0,T]$ and $f\sqrt{\zeta} \in L^2[0,T]$. Now, $\{\phi_j\}_{j=1}^\infty$ is a complete orthonormal set of functions in $L^2_{\alpha,\beta}[0,T]$. In addition, if $\frac{1}{\sqrt{\zeta}}$ is of bounded variation on $[0,T]$, then ϕ_j is of bounded variation on $[0,T]$ since it is a product of two functions of bounded variation. $\square$

We note that $\{\phi_j\}_{j=1}^\infty$ in Lemma 4.1 is orthogonal in $L^2_{\alpha,0}[0,T]$ and $L^2_{\alpha,\beta}[0,T]$ is separable. Moreover, $\{\phi_j\}_{j=1}^\infty$ is orthogonal in $L^2_{\alpha,\beta}[0,T]$ if it is orthogonal in both $L^2_{\alpha,0}[0,T]$ and $L^2_{0,\beta}[0,T]$, but that the converse need not hold. For the mean functions α which are needed in several papers, the following example provides existences of $\{\phi_j\}_{j=1}^\infty$ satisfying the conditions of Lemma 4.1.

Example 4.2. 1. *Let* $\frac{1}{\sqrt{|\alpha|'+\beta'}}$ *be of bounded variation,* $|\alpha|' + \beta'$ *be bounded and* β' *be bounded away from zero. Then, for some constants* M_1 *and* M_2*, we have* $0 < M_1 \le \beta'(t) \le (|\alpha|' + \beta')(t) \le M_2$ *for all* $t \in [0,T]$ *so that* $0 < \frac{M_1}{M_2} \le \frac{\beta'(t)}{|\alpha|'(t)+\beta'(t)}$ *for all* $t \in [0,T]$*; that is,* $\frac{\beta'}{|\alpha|'+\beta'}$ *is bounded away from zero. Now,* $\{\phi_j\}_{j=1}^{\infty}$ *satisfying the conditions of Lemma 4.1 exists in* $L^2_{\alpha,\beta}[0,T]$*. In the case,* $L^2_{\alpha,\beta}[0,T] = L^2_{0,\beta}[0,T] = L^2[0,T]$ *and the equalities means that they are equal as vector spaces and the norms on them are equivalent so that they have the same topology, but that they need not be equal isometrically.*

2. *Let* $\frac{1}{\sqrt{\beta'}}$ *be of bounded variation on* $[0,T]$*. It is not difficult to show that* α *(or* $|\alpha|$*) is a constant function on* $[0,T]$ *if and only if* $\|f\|_{\alpha,0} = 0$ *for all* $f \in S[0,T]$ *if and only if* $\langle f, g\rangle_{\alpha,0} = 0$ *for all* $f, g \in S[0,T]$*. In this case,* $\langle f, g\rangle_{\alpha,\beta} = \langle f, g\rangle_{0,\beta}$ *for all* $f, g \in L^2_{0,\beta}[0,T]$ *so that* $L^2_{\alpha,\beta}[0,T] = L^2_{0,\beta}[0,T]$ *isometrically. Now,* $\{\phi_j\}_{j=1}^{\infty}$ *satisfying the conditions of Lemma 4.1 exists in* $L^2_{\alpha,\beta}[0,T]$*.*

3. *Let* $\frac{1}{\sqrt{\beta'}}$ *be of bounded variation on* $[0,T]$*. If for some constant* $c > 0$*,* $|\alpha|(t) = c\beta(t)$ *for all* $t \in [0,T]$ *which is the condition suggested by Yoo, Kim and Kim [14], then* $\langle f, g\rangle_{\alpha,\beta} = \langle f, g\rangle_{0,(1+c)\beta} = (1+c)\langle f, g\rangle_{0,\beta}$ *for all* $f, g \in L^2_{0,(1+c)\beta}[0,T]$ *so that* $L^2_{\alpha,\beta}[0,T] = L^2_{0,(1+c)\beta}[0,T]$ *isometrically. In this case,* $\{\phi_j\}_{j=1}^{\infty}$ *satisfying the conditions of Lemma 4.1 exists in* $L^2_{\alpha,\beta}[0,T]$*.*

Definition 4.3. *Let* $\{\phi_j\}_{j=1}^{\infty}$ *be a sequence in* $L^2_{\alpha,\beta}[0,T]$ *satisfying the conditions of Lemma 4.1. For* $f \in L^2_{\alpha,\beta}[0,T]$*, we define a generalized PWZ integral* $(f, x)_{\alpha,\beta}$ *by the formula*

$$(f, x)_{\alpha,\beta} = \lim_{n\to\infty} \sum_{j=1}^{n} \langle f, \phi_j\rangle_{\alpha,\beta} \int_0^T \phi_j(t)dx(t)$$

pointwisely for all $x \in C[0,T]$ *for which this limit exists.*

If $\varphi(\mathbb{R}) = 1$, $\alpha(t) = 0$ and $\beta(t) = t$ for $t \in [0,T]$, then $(f, x)_{\alpha,\beta}$ is exactly the PWZ integral on the analogue of Wiener space introduced by Im and Ryu [4]. In the followings, we prove that for $f \in L^2_{\alpha,\beta}[0,T]$, the PWZ integral $(f, x)_{\alpha,\beta}$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and it is essentially independent of a particular choice of the complete orthonormal set to define it.

Lemma 4.4. *For* f *in* $L^2_{\alpha,\beta}[0,T]$*, the generalized PWZ integral* $(f, x)_{\alpha,\beta}$ *exists for* $w_{\alpha,\beta;\varphi}$ *a.e.* $x \in C[0,T]$*.*

Proof. Suppose that φ is a probability measure. For each positive integer j, let $X_j(x) = \langle f, \phi_j\rangle_{\alpha,\beta} \int_0^T \phi_j(t)dx(t)$ for $x \in C[0,T]$. By Theorems 3.3, 3.4 and Corollary 3.7, X_j is Gaussian with the mean $\langle f, \phi_j\rangle_{\alpha,\beta} \int_0^T \phi_j(t)d\alpha(t)$ and the variance $\langle f, \phi_j\rangle^2_{\alpha,\beta}\|\phi_j\|^2_{0,\beta}$, and $\{X_j\}_{j=1}^{\infty}$ is a sequence of independent random variables. Furthermore, we have

$$\sum_{j=1}^{\infty} Var[X_j] = \sum_{j=1}^{\infty} \langle f, \phi_j\rangle^2_{\alpha,\beta}\|\phi_j\|^2_{0,\beta} \le \sum_{j=1}^{\infty} \langle f, \phi_j\rangle^2_{\alpha,\beta} = \|f\|^2_{\alpha,\beta} < \infty.$$

By Proposition 2.3.3 of [7], $\sum_{j=1}^{\infty}[X_j(x) - E[X_j]]$ converges pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. For $m \ge n$, we have by the Hölder's inequality

$$\left|\sum_{j=n}^{m} E[X_j]\right|^2 = \left|\sum_{j=n}^{m} \langle f, \phi_j\rangle_{\alpha,\beta} \int_0^T \phi_j(t)d\alpha(t)\right|^2 \le \nu_\alpha([0,T]) \sum_{j=n}^{m} \langle f, \phi_j\rangle^2_{\alpha,\beta},$$

which converges to 0 as $m, n \to \infty$. Now $\sum_{j=1}^{\infty} E[X_j]$ converges so that $\sum_{j=1}^{\infty}(X_j(x) - E[X_j] + E[X_j]) = \sum_{j=1}^{\infty} X_j(x) = (f, x)_{\alpha,\beta}$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. If φ is an arbitrary positive finite measure, let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})}\varphi$ which is a probability measure. By the above argument, $(f, x)_{\alpha,\beta}$ exists for $w_{\alpha,\beta;\varphi_0}$ a.e. $x \in C[0,T]$. Since the null sets with respect to $w_{\alpha,\beta;\varphi}$ are equivalent to the null sets with respect to $w_{\alpha,\beta;\varphi_0}$, we have this lemma. □

Theorem 4.5. *For $f \in L^2_{\alpha,\beta}[0,T]$, we have $(f,x)_{\alpha,\beta} = I_{\alpha,\beta}(f)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and $(f,x)_{\alpha,\beta}$ is independent of a particular choice of the complete orthonormal set as described in Definition 4.3 for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.*

Proof. For $n \in \mathbb{N}$, let $f_n = \sum_{j=1}^n \langle f, \phi_j\rangle_{\alpha,\beta}\phi_j$. Then $\lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0$ so that $\lim_{n\to\infty} \|I_{\alpha,\beta}(f_n) - I_{\alpha,\beta}(f)\|_C = 0$ by Corollary 3.11. By Theorem 3.4 and Lemma 4.4, $I_{\alpha,\beta}(f_n)(x) \to (f,x)_{\alpha,\beta}$ pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ as $n \to \infty$. By Corollary 3.11 again, we have $(f,x)_{\alpha,\beta} = I_{\alpha,\beta}(f)(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Since the above argument does not depend on a particular choice of $\{\phi_j\}_{j=1}^\infty$, we have the second part of this theorem. □

By the linearity of $\langle \cdot, \phi_j\rangle_{\alpha,\beta}$, we have the linearity of the generalized PWZ integral; for $c_1, c_2 \in \mathbb{R}$ and $f, g \in L^2_{\alpha,\beta}[0,T]$,

$$(c_1 f + c_2 g, x)_{\alpha,\beta} = c_1(f,x)_{\alpha,\beta} + c_2(g,x)_{\alpha,\beta}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.

We now have the following theorem by Theorem 3.3.6 and Theorem 4.5.

Theorem 4.6. *For $f \in L^2_{\alpha,\beta}[0,T]$, $(f,\cdot)_{\alpha,\beta}$ is Gaussian with the mean $\int_0^T f(t)d\alpha(t)$ and the variance $\|f\|^2_{0,\beta}$ if $\varphi(\mathbb{R}) = 1$.*

By Theorems 3.4 and 4.5, we have the following theorem.

Theorem 4.7. *Let f be of bounded variation on $[0,T]$. Then for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, we have $(f,x)_{\alpha,\beta} = \int_0^T f(t)dx(t)$, where $\int_0^T f(t)dx(t)$ denotes the Riemann-Stieltjes integral of f with respect to x.*

Remark 4.8. *We can also obtain the results of Theorems 3.3, 3.6, 3.10, 3.13 and Corollaries 3.7, 3.11, 3.12 by Theorem 4.5, with replacing $I_{\alpha,\beta}$ by the generalized PWZ integral. Note that, in order to use the generalized PWZ integral instead of $I_{\alpha,\beta}$, we need additional conditions for α and β; $\frac{\beta'}{|\alpha|'+\beta'}$ is bounded away from zero and $\frac{1}{\sqrt{|\alpha|'+\beta'}}$ is of bounded variation on $[0,T]$.*

5. Applications and examples of the generalized PWZ integral

In this section, we provide examples and applications of the generalized PWZ integral.

Example 5.1. *For $t \in [0,T]$, we have by Theorems 4.5 and 4.7*

$$(\chi_{[0,t]}, x)_{\alpha,\beta} = I_{\alpha,\beta}(\chi_{[0,t]})(x) = \int_0^t dx(s) = x(t) - x(0)$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. In particular, $(1,x)_{\alpha,\beta} = (\chi_{[0,T]}, x)_{\alpha,\beta} = x(T) - x(0)$.

Example 5.2. *If α is a constant function and $\varphi(\mathbb{R}) = 1$, then we can prove Theorem 4.5 by Theorems 3.4 and 3.13. Indeed, for $f \in L^2_{\alpha,\beta}[0,T]$, we have*

$$(f,x)_{\alpha,\beta} = \sum_{j=1}^\infty \langle f, \phi_j\rangle_{\alpha,\beta} \int_0^T \phi_j(t)dx(t) = \sum_{j=1}^\infty \langle f, \phi_j\rangle_{\alpha,\beta} I_{\alpha,\beta}(\phi_j)(x) = I_{\alpha,\beta}(f)(x)$$

pointwisely for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, which does not depend on a particular choice of $\{\phi_j\}_{j=1}^\infty$. In this case, since $\frac{\beta'(t)}{|\alpha|'(t)+\beta'(t)} = 1$ for all $t \in [0,T]$, $\frac{\beta'}{|\alpha|'+\beta'}$ is trivially bounded away from zero.

Example 5.3. *We can prove Theorem 4.6 using the characteristic function of the generalized PWZ integral if $\varphi(\mathbb{R}) = 1$. For $f \in L^2_{\alpha,\beta}[0,T]$ and $n \in \mathbb{N}$, let $f_n = \sum_{j=1}^n \langle f, \phi_j\rangle_{\alpha,\beta}\phi_j$. Then for $\xi \in \mathbb{R}$, we have by Theorems 3.3, 3.4 and the dominated convergence theorem*

$$\begin{aligned}
E[i\xi(f,\cdot)_{\alpha,\beta}] &= \int_{C[0,T]} \exp\Big\{i\xi \lim_{n\to\infty} I_{\alpha,\beta}(f_n)(x)\Big\}\, dw_{\alpha,\beta;\varphi}(x)\\
&= \lim_{n\to\infty}\int_{C[0,T]} \exp\{i\xi I_{\alpha,\beta}(f_n)(x)\} dw_{\alpha,\beta;\varphi}(x)\\
&= \lim_{n\to\infty} \exp\Big\{-\frac{1}{2}\xi^2\|f_n\|^2_{0,\beta} + i\xi\int_0^T f_n(t)d\alpha(t)\Big\}\\
&= \exp\Big\{-\frac{1}{2}\xi^2\|f\|^2_{0,\beta} + i\xi\int_0^T f(t)d\alpha(t)\Big\}
\end{aligned}$$

using the same method as used in the proof of Theorem 3.3.4, which completes the proof.

Theorem 5.4. *Assume that β' is bounded away from zero. If $f \in L^2_{\alpha,\beta}[0,T]$ and x is absolutely continuous on $[0,T]$ with $x' \in L^2[0,T]$, then $(f,x)_{\alpha,\beta}$ exists and it is given by $(f,x)_{\alpha,\beta} = \int_0^T f(t)x'(t)dm_L(t)$.*

Proof. Let f_n be the function as given in Example 5.3. Then we have

$$\int_0^T f_n(t)dx(t) = (L)\int_0^T f_n(t)dx(t),$$

where $(L)\int_0^T f_n(t)dx(t)$ denotes the Lebesgue-Stieltjes integral of f_n with respect to the measure induced by x which is absolutely continuous on $[0,T]$. Since $x' \in L^2[0,T] \subseteq L^1[0,T]$, we have that $x' \in L^1[0,T]$ and

$$\int_0^T f_n(t)dx(t) = (L)\int_0^T f_n(t)dx(t) = \int_0^T f_n(t)x'(t)dm_L(t). \tag{3}$$

Now we have for $g \in L^2_{\alpha,\beta}[0,T]$

$$\|g\|^2_{m_L} \le M\|g\|^2_{0,\beta} \le M\|g\|^2_{\alpha,\beta} \tag{4}$$

for some constant $M > 0$, since β' is bounded away from zero. Thus we have by the Hölder's inequality and (4)

$$\Big[\int_0^T f_n(t)x'(t)dm_L(t) - \int_0^T f(t)x'(t)dm_L(t)\Big]^2 \le \|f_n - f\|^2_{m_L}\|x'\|^2_{m_L} \le M\|f_n - f\|^2_{\alpha,\beta}\|x'\|^2_{m_L},$$

which converges to 0 as $n \to \infty$. Now, by (3), we have $\lim_{n\to\infty}\int_0^T f_n(t)dx(t) = \int_0^T f(t)x'(t)dm_L(t)$ with the existence of the limit. By the definition of $(f,x)_{\alpha,\beta}$, we have $(f,x)_{\alpha,\beta} = \lim_{n\to\infty}\int_0^T f_n(t)dx(t) = \int_0^T f(t)x'(t)dm_L(t)$ as desired. □

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