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Radius of Starlikeness and Hardy Space of Mittag-Leffler Functions

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Abstract. In the present work, Mittag-Leffler functions with its normalization are considered. Several results are obtained so that these functions have certain geometric properties including starlikeness, convexity, close-to-convexity of order α, and radius of starlikeness of order α. Furthermore, we obtain certain condition so that the normalized Mittag-Leffler functions belongs to the Hardy space and to the class of bounded analytic functions. Results obtained are new and their usefulness are depicted by deducing several interesting corollaries and examples.

1. Introduction

A special function of growing importance is the generalized Mittag-Leffler function defined by [30, 31]

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)} \qquad (\lambda, \mu, z \in \mathbb{C}; \Re(\lambda) > 0; \Re(\mu) > 0). \tag{1}$$

The Mittag-Leffler function is an entire functions of order $\rho = 1/\lambda$ and type $\sigma = 1$ [10, Corollary 1.2]. The Mittag-Leffler function is a generalization of the exponential function, to which it reduces for $\lambda = \mu = 1$. Mittag-Leffler functions are important in mathematics as well as in theoretical and applied physics. A primary reason for the recent surge of interest in these functions is their appearance on solving fractional differential and integral equations [12, 18, 25]. The Mittag-Leffler function plays the same role for fractional calculus that the exponential function plays for conventional calculus. The Mittag-Leffler functions are important to investigate fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems [9, 12, 25, 27].

Despite a wealth of analytical information about $E_{\lambda,\mu}$, its behavior as a holomorphic function largely unexplored, also its mapping properties in the complex plane are largely unknown. It is therefore desirable to explore the behavior of $E_{\lambda,\mu}$ for the parameters λ, μ and complex argument z. Given this objective, the present article reports the geometrical behavior of image domain of $E_{\lambda,\mu}$, when z is in open unit disk and $\lambda, \mu \in \mathbb{R}$.

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Note that the function $E_{1,-m}(z) = z^{m+1}e^z$, $m \in \mathbb{Z}_+$, has zero at the origin of multiplicity $m+1$, and $E_{1,1}(z) = e^z$ does not have any zero in $\mathbb{C}$. Except for these two cases the function $E_{\lambda,\mu}(z)$ has an infinite number of zeros. Indeed the zeros of the function $E_{2,3}(z)$ are described by the formula $z_n = -(2\pi n)^2$, $n \in \mathbb{N}$, and have multiplicity 2, which is the unique case of an infinite number of multiple zeros of the function $E_{\lambda,\mu}(z)$. Wiman [31] stated that all zeros of the Mittag-Leffler function $E_{\lambda,1}(z)$ are real, negative, simple, and ordered in the sequence $\{z_n = z_{n,\lambda,1}\}_{n\in\mathbb{N}}$, satisfy the inequalities

$$-\left(\frac{\pi n}{\sin(\pi/\lambda)}\right)^{\lambda} < z_n < -\left(\frac{\pi(n-1)}{\sin(\pi/\lambda)}\right)^{\lambda}, \quad n \in \mathbb{N}.$$

Though Wiman does not given any proof of this result, but Polya [20] proved that all the zeros of $E_{\lambda,1}(z)$ are negative and simple, but only in the case where $\lambda \geq 2$. After a long time gap, Ostrovskii and Peresyolkova [19] proved the negativeness and simpleness of all zeros of the functions $E_{\lambda,1}(z)$ and $E_{\lambda,2}(z)$ for all $\lambda \geq 2$.

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r,\ r > 0\}$ and set $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{H}$ denote the class of analytic functions in $\mathbb{D}$ and $\mathcal{A}$ denote the class of analytic functions f in $\mathbb{D}$ normalized by $f(0) = 0 = f'(0) - 1$, that have Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

For functions $f, g \in \mathcal{A}$, f is given by (2) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or *convolution*) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \qquad z \in \mathbb{D}.$$

Let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike function of order $\alpha\,(0 \leq \alpha < 1)$, class of such functions denoted by $\mathcal{S}^*(\alpha)$, if and only if $\Re\,(zf'(z)/f(z)) > \alpha$, $z \in \mathbb{D}$. Further, the real number

$$r_\alpha^*(f) = \sup\left\{r > 0 : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ \forall\ |z| < r\right\},$$

is called the radius of starlikeness of order α of the function f. Note that $r^*(f) = r_0^*(f)$ is the largest radius such that the image domain $f(\mathbb{D}_{r^*(f)})$ is a starlike domain with respect to the origin. A function $f \in \mathcal{A}$ is called convex function of order $\alpha\,(0 \leq \alpha < 1)$, class of such functions denoted by $\mathcal{K}(\alpha)$, if and only if $1 + \Re\,(zf''(z)/f'(z)) > \alpha$, $z \in \mathbb{D}$. It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$.

A function f belonging to the class $\mathcal{A}$ is said to be in the class $\mathcal{R}(\beta)$ if it satisfies the inequality $\Re(f'(z)) > \beta$ $(z \in \mathbb{D}, \beta < 1)$. Further, a function f belonging to the class $\mathcal{H}$ is said to be in the class $\mathcal{P}(\beta)$ if $f(0) = 1$ and satisfies the inequality $\Re(f(z)) > \beta$ $(z \in \mathbb{D}, \beta < 1)$. For $\beta = 0$, we denote $\mathcal{P}(\beta)$ and $\mathcal{R}(\beta)$ simply by $\mathcal{P}$ and $\mathcal{R}$ respectively.

Let $\mathcal{H}^\infty$ denote the space of all bounded functions on $\mathbb{D}$. This is Banach algebra with respect to the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. For the functions $f \in \mathcal{H}$, set

$$M_p(r,f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f(re^{i\theta})\right|^p \, d\theta\right)^{1/p}, & 0 < p < \infty, \\ \max\limits_{|z|\leq r} |f(z)| & p = \infty. \end{cases} \tag{3}$$

The function f is said to belongs to $\mathcal{H}^p$ $(0 < p \leq \infty)$ and is called Hardy space, if $M_p(r,f)$ is bounded for all $r \in [0,1)$. Clearly, we have [7, p. 2]

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \quad \text{for} \quad 0 < p < q < \infty.$$

For $1 \le p \le \infty$, $\mathcal{H}^p$ is a Banach space with the norm defined by (*cf.* [7, p. 23])

$$\|f\|_p = \lim_{r\to 1^-} M_p(r,f) \qquad (1 \le p \le \infty). \tag{4}$$

Following are two widely known results [14] for the Hardy space $\mathcal{H}^p$:

$$\begin{aligned}\Re(f'(z)) > 0 &\Longrightarrow f' \in \mathcal{H}^p \ \text{ for all } \ p < 1\\ &\Longrightarrow \ f \in \mathcal{H}^{q/(1-q)} \ \text{ for all } \ 0 < q < 1.\end{aligned} \tag{5}$$

Kim and Srivastava [13], also Ponnusamy [26] have studied the Hardy space of hypergeometric functions. By using the same idea Baricz [3] obtained the conditions for generalized Bessel functions to belong to Hardy space, and similarly Yagmur and Orhan [32, 33] studied the same problem for generalized Struve functions and Lommel functions.

For real (or complex numbers) α, β and γ ($\gamma \neq 0, -1, -2, ...$), the Gaussian hypergeometric function is defined by [1, p.333]:

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},$$

where $(x)_k$ is the Pochhamer symbol defined by $(x)_0 = 1;\ (x)_k = x(x+1)\cdots(x+k-1)$ $(k \in \mathbb{N})$. We note that the above series converges absolutely in $\mathbb{D}$, and hence, represents an analytic function in $\mathbb{D}$.

We consider the following two normalizations of the Mittag-Leffler function $E_{\lambda,\mu}(z)$:

$$\begin{aligned}\mathbb{E}_{\lambda,\mu}(z) &= \Gamma(\mu)\, zE_{\lambda,\mu}(z)\\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} z^n \quad (\lambda > 0,\ \mu > 0,\ z \in \mathbb{D})\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(z) &= \Big(\Gamma(\mu)\, z^{\mu} E_{\lambda,\mu}(z)\Big)^{\frac{1}{\mu}}\\ &= z + \frac{\Gamma(\mu)}{\mu\,\Gamma(\lambda+\mu)} z^2 + \cdots \qquad (\lambda > 0,\ \mu > 0,\ z \in \mathbb{D}).\end{aligned} \tag{7}$$

Note that $\mathcal{E}_{\lambda,\mu}(z) = exp\left\{\frac{1}{\mu}\operatorname{Log}\Big(\Gamma(\mu) z^{\mu} E_{\lambda,\mu}(z)\Big)\right\}$, where Log represents the principal branch of the logarithm. Whilst formula (6) and (7) holds for complex-valued λ, μ, however we shall restrict our attention to the case of $\lambda > 0$, $\mu > 0$. We observe that the functions $\mathbb{E}_{\lambda,\mu}(z)$ and $\mathcal{E}_{\lambda,\mu}(z)$ belongs to the class $\mathcal{A}$.

It is important to mention here that in the recent years there was a vivid interest on geometric properties of special functions, like Bessel, Struve, Lommel, hypergeometric, Wright and Mittag-Leffler functions; see the papers [2, 4–6, 17, 22, 23] and the references therein.

In this article, we obtain a sufficient condition for the function $\mathbb{E}_{\lambda,\mu}$ to be starlike function of order α. Also, we obtain radius of starlikeness of order α for the functions $\mathbb{E}_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,\mu}$. Further, we obtain results so that function $\mathbb{E}_{\lambda,\mu}$ belongs to the Hardy spaces $\mathcal{H}^p$ and $\mathcal{H}^\infty$.

2. Key Lemmas

In order to derive our main results, we recall here the following lemmas:

Lemma 2.1. (Popov and Sedletskii [21, Theorem 3.1.1]) *Let $\{z_{n,\lambda,\mu}\}_{n=1}^{\infty}$ denote the set of all zeros of the function $E_{\lambda,\mu}(z)$. For any $\lambda \in (2,\infty)$ and $\mu \in (0, 2\lambda-1]$, all the zeros of the function $E_{\lambda,\mu}(z)$ in $\mathbb{C}$ lie on $(-\infty, 0)$, are simple and being ordered in a sequence $\{z_{n,\lambda,\mu}\}_{n\in\mathbb{N}}$, satisfying the inequalities*

$$\begin{aligned}&-\xi_1^{\lambda}(1/\lambda,\mu) < z_{1,\lambda,\mu} < -\frac{\Gamma(\mu+\lambda)}{\Gamma(\mu)},\\ &-\xi_n^{\lambda}(1/\lambda,\mu) < z_{n,\lambda,\mu} < -\xi_{n-1}^{\lambda}(1/\lambda,\mu), \quad n \ge 2,\end{aligned} \tag{8}$$

where $\xi_n(1/\lambda, \mu)$ *is given by*

$$\xi_n(1/\lambda, \mu) = \frac{\pi(n + \frac{1}{\lambda}(\mu - 1))}{\sin(\pi/\lambda)}.$$

Lemma 2.2. [15, Part 1] *An entire function* $f(z)$ *of finite order* ρ *may be represented in the form*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p\right),$$

where

$$G(u,p) = \begin{cases} 1-u, & p = 0 \\ (1-u)\exp\left\{u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right\}, & p > 0, \end{cases}$$

and $a_1, a_2, \cdots$ *are all nonzero roots of the function* $f(z)$, $p \le \rho$, $P_q(z)$ *is a polynomial in* z *of degree* $q \le \rho$, *and* m *is the multiplicity of the root at the origin.*

Using Lemma 2.2, we obtain following infinite product representation of Mittag-Leffler function:

Lemma 2.3. *Let* $\lambda \in (2, \infty)$ *and* $\mu \in (0, 2\lambda - 1]$. *If* $\{z_{n,\lambda,\mu}\}_{n\in\mathbb{N}}$ *denotes the set of all zeros of* $E_{\lambda,\mu}(z)$, *then the Mittag-Leffler function* $E_{\lambda,\mu}(z)$ *can be represented by*

$$E_{\lambda,\mu}(z) = \frac{1}{\Gamma(\mu)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_{n,\lambda,\mu}^2}\right). \tag{9}$$

Proof. It is well known [10, Corollary 1.2] that $E_{\lambda,\mu}(z)$ is an entire function of order $\rho = 1/\lambda < 1/2$. In view of Lemma 2.2, $P_q(z)$ is a polynomial in z of degree $q \le \rho < 1/2$. Hence q must be zero, i.e. $P_q(z)$ is a constant c(say). In view of Lemma 2.1, zeros $\{z_{n,\lambda,\mu}\}_{n\in\mathbb{N}}$ lies on negative real axis, are simple and being ordered as the inequality (8). Further, the multiplicity of the zeros of $E_{\lambda,\mu}(z)$ at origin is zero, hence $m = 0$. From Lemma 2.1, we observe that

$$\sum_{n=1}^{\infty} \frac{1}{|z_{n,\lambda,\mu}|^2} < \infty.$$

Thus, the Mittag-Leffler function can represented by

$$E_{\lambda,\mu}(z) = e^c \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_{n,\lambda,\mu}^2}\right).$$

When $z \to 0$, $E_{\lambda,\mu}(z) \to 1/\Gamma(\mu)$, hence $e^c = 1/\Gamma(\mu)$. Hence the desired result. □

Lemma 2.4. [29] *For* $\alpha < 1$, $\beta < 1$, *we have* $\mathcal{P}(\alpha) * \mathcal{P}(\beta) \subset \mathcal{P}(\delta)$, *where* $\delta = 1 - 2(1-\alpha)(1-\beta)$. *The value of* δ *is best possible.*

Lemma 2.5. [8] *If the function* f, *convex of order* α $(0 \le \alpha < 1)$, *is not of the form*

$$f(z) = \begin{cases} k + d\,z(1 - ze^{i\gamma})^{2\alpha-1}, & \alpha \ne 1/2, \\ k + d\,\log(1 - ze^{i\gamma}), & \alpha = 1/2, \end{cases}$$

for some complex numbers k *and* d, *and for some real number* γ, *then the following statements hold:*

1. *There exists* $\delta = \delta(f) > 0$ *s.t.* $f' \in \mathcal{H}^{\delta + 1/[2(1-\alpha)]}$;
2. *If* $0 \le \alpha < 1/2$, *then there exists* $\epsilon = \epsilon(f) > 0$ *such that* $f \in \mathcal{H}^{\epsilon + 1/(1-2\alpha)}$
3. *If* $\alpha \ge 1/2$, *then* $f \in \mathcal{H}^{\infty}$.

3. Starlikeness Properties of $\mathbb{E}_{\lambda,\mu}(z)$ and $\mathcal{E}_{\lambda,\mu}(z)$

Theorem 3.1. *Let $0 \leq \alpha < 1$. For all $\lambda \in (2, \infty)$ and $\mu \in (0, 2\lambda - 1]$ the following statements are true:*

a. The radius of starlikeness of order α for the function $\mathbb{E}_{\lambda,\mu}(z)$ is given by $r^*_\alpha(\mathbb{E}_{\lambda,\mu}(z)) = x_{\lambda,\mu,\alpha}$, where $x_{\lambda,\mu,\alpha}$ denotes the smallest positive root of the equation

$$(1-\alpha) + \Gamma(\mu) \sum_{n=1}^{\infty} \frac{(n+1-\alpha)}{\Gamma(\lambda n + \mu)} \, r^n = 0.$$

b. The radius of starlikeness of order α for the function $\mathcal{E}_{\lambda,\mu}(z)$ is given by $r^*_\alpha(\mathcal{E}_{\lambda,\mu}) = y_{\lambda,\mu,\alpha}$, where $y_{\lambda,\mu,\alpha}$ denotes the smallest positive root of the equation

$$(1-\alpha)\mu + \Gamma(\mu) \sum_{n=1}^{\infty} \frac{(n+(1-\alpha)\mu)}{\Gamma(\lambda n + \mu)} \, r^n = 0.$$

Proof. For all $\lambda \in (2, \infty)$ and $\mu \in (0, 2\lambda - 1]$, we need to show that the inequalities

$$\Re\left(\frac{z\mathbb{E}'_{\lambda,\mu}(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{z\mathcal{E}'_{\lambda,\mu}(z)}{\mathcal{E}_{\lambda,\mu}(z)}\right) > \alpha \tag{10}$$

are valid for all $|z| < x_{\lambda,\mu,\alpha}$ and $|z| < y_{\lambda,\mu,\alpha}$, respectively, and the above inequalities does not hold in any larger disk. If $z_{n,\lambda,\mu}$ denotes the set of all zeros of the Mittag-Leffler function $E_{\lambda,\mu}(z)$, then it has infinite product of the form (9), which is uniformly convergent on each compact subset of $\mathbb{C}$. The Logarithmic differentiation of (9) provides

$$\frac{zE'_{\lambda,\mu}(z)}{E_{\lambda,\mu}(z)} = -\sum_{n=1}^{\infty} \frac{2z^2}{z^2_{n,\lambda,\mu} - z^2}. \tag{11}$$

Differentiatiating (6)-(7) and using (11), we obtain

$$\frac{z\mathbb{E}'_{\lambda,\mu}(z)}{\mathbb{E}_{\lambda,\mu}(z)} = 1 - \sum_{n=1}^{\infty} \frac{2z^2}{z^2_{n,\lambda,\mu} - z^2} \quad \text{and} \quad \frac{z\mathcal{E}'_{\lambda,\mu}(z)}{\mathcal{E}_{\lambda,\mu}(z)} = 1 - \frac{1}{\mu}\sum_{n=1}^{\infty} \frac{2z^2}{z^2_{n,\lambda,\mu} - z^2}.$$

Let us denote by $z_{\lambda,\mu}$ the least (in absolute value) real zero of the function $E_{\lambda,\mu}(z)$. Then clearly, $z_{\lambda,\mu} < 0$. Therefore (see [24, p. 550])

$$|z_{\lambda,\mu-\epsilon}| < |z_{\lambda,\mu}|$$

if $\mu > \epsilon > 0$. Under the hypothesis $z_{1,\lambda,\mu} > z_{2,\lambda,\mu} > z_{3,\lambda,\mu} > \cdots$, and the smallest positive zero (in absolute value) is less than $\frac{\Gamma(\mu+\lambda)}{\Gamma(\mu)} = z_{\lambda,\mu}$ (say). This implies that $x_{\lambda,\mu,\alpha} < z_{\lambda,\mu}$ and $y_{\lambda,\mu,\alpha} < z_{\lambda,\mu}$, that is, for all $\alpha < 1$ and $n \in \{2, 3, \cdots\}$ we have $\mathbb{D}_{x_{\lambda,\mu,\alpha}} \subset \mathbb{D}_{z_{\lambda,\mu}} \subset \mathbb{D}_{z_{n,\lambda,\mu}}$ and $\mathbb{D}_{y_{\lambda,\mu,\alpha}} \subset \mathbb{D}_{z_{\lambda,\mu}} \subset \mathbb{D}_{z_{n,\lambda,\mu}}$. One can easily observe that, if $z \in \mathbb{C}$ and $\delta \in \mathbb{R}$ such that $\delta > |z|$, then

$$\frac{|z|}{\delta - |z|} \geq \Re\left(\frac{z}{\delta - z}\right) \geq \frac{-|z|}{\delta + |z|}. \tag{12}$$

Using this inequality, we have

$$\begin{aligned} \Re\left(\frac{z\mathbb{E}'_{\lambda,\mu}(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right) &= 1 - \Re\left(\sum_{n=1}^{\infty} \frac{2z^2}{z^2_{n,\lambda,\mu} - z^2}\right) \\ &\geq 1 - \sum_{n=1}^{\infty} \frac{2|z|^2}{z^2_{n,\lambda,\mu} - |z|^2} = \frac{|z|\,\mathbb{E}'_{\lambda,\mu}(|z|)}{\mathbb{E}_{\lambda,\mu}(|z|)} \qquad (z \in \mathbb{D}_{z,\mu}) \end{aligned}$$

and

$$\begin{aligned}\Re\left(\frac{z\mathcal{E}'_{\lambda,\mu}(z)}{\mathcal{E}_{\lambda,\mu}(z)}\right) &= 1-\frac{1}{\mu}\Re\left(\sum_{n=1}^{\infty}\frac{2z^2}{z^2_{n,\lambda,\mu}-z^2}\right)\\ &\geq 1-\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{2|z|^2}{z^2_{n,\lambda,\mu}-|z|^2}=\frac{|z|\,\mathcal{E}'_{\lambda,\mu}(|z|)}{\mathcal{E}_{\lambda,\mu}(|z|)} \qquad (z\in\mathbb{D}_{z,\mu})\end{aligned}$$

with equality when $z=|z|=r$. The minimum principle for harmonic functions and the above inequality imply that the corresponding inequalities in (10) are valid if and only if we have $|z|<x_{\lambda,\mu,\alpha}$ and $|z|<y_{\lambda,\mu,\alpha}$, where $x_{\lambda,\mu,\alpha}$ and $y_{\lambda,\mu,\alpha}$ are the smallest positive roots of the equations

$$\frac{r\mathbb{E}'_{\lambda,\mu}(r)}{\mathbb{E}_{\lambda,\mu}(r)}=\alpha \quad \text{and} \quad \frac{r\mathcal{E}'_{\lambda,\mu}(r)}{\mathcal{E}_{\lambda,\mu}(r)}=\alpha,$$

respectively, that are equivalent to

$$(1-\alpha)+\Gamma(\mu)\sum_{n=1}^{\infty}\frac{(n+1-\alpha)}{\Gamma(\lambda n+\mu)}\,r^n=0$$

and

$$(1-\alpha)\mu+\Gamma(\mu)\sum_{n=1}^{\infty}\frac{(n+(1-\alpha)\mu)}{\Gamma(\lambda n+\mu)}\,r^n=0,$$

respectively. This completes the proof. □

In particular, when $\alpha=0$, the following results holds.

Corollary 3.2. *For all $\lambda\in(2,\infty)$ and $\mu\in(0,2\lambda-1]$ the following statements are true:*

a. The radius of starlikeness of the function $\mathbb{E}_{\lambda,\mu}(z)$ is $x_{\lambda,\mu,0}$, which is the smallest possitive root of the equation

$$1+\Gamma(\mu)\sum_{n=1}^{\infty}\frac{(n+1)}{\Gamma(\lambda n+\mu)}\,r^n=0.$$

b. The radius of starlikeness of the function $\mathcal{E}_{\lambda,\mu}(z)$ is $y_{\lambda,\mu,0}$, which is the smallest possitive root of the equation

$$\mu+\Gamma(\mu)\sum_{n=1}^{\infty}\frac{(n+\mu)}{\Gamma(\lambda n+\mu)}\,r^n=0.$$

Theorem 3.3. *Let $0\leq\alpha<1$ and $\lambda\geq 1$. If $\mu\geq\mu_1$, where μ_1 is the largest root of*

$$(1-\alpha)(\mu^2-\mu-1)(\mu-1)-\mu(\mu+1)=0,$$

then $\mathbb{E}_{\lambda,\mu}(z)\in\mathcal{S}^*(\alpha)$.

Proof. It is well known [28, p. 110, Theorem 1] that, if a function $f\in\mathcal{A}$ of the form (2) satisfies $\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leq 1-\alpha$ $(0\leq\alpha<1)$, then $f\in\mathcal{S}^*(\alpha)$. Hence to prove $\mathbb{E}_{\lambda,\mu}(z)\in\mathcal{S}^*(\alpha)$, it is sufficient to show that $\Omega(\lambda,\mu,\alpha)\leq 1-\alpha$, where

$$\Omega(\lambda,\mu,\alpha)=\sum_{n=2}^{\infty}\frac{(n-\alpha)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}.$$

For $\lambda \geq 1$, $\mu > 0$, $n \in \mathbb{N}$ and $\nu > 0$, the following inequalities holds

$$(\mu)_n\,\Gamma(\mu) \leq \Gamma(\lambda n + \mu),\ (\nu)_n = \nu(\nu+1)_{n-1},\ \nu^n \leq (\nu)_n,\ \ n+1 \leq 2^n. \tag{13}$$

Hence we get

$$\begin{aligned}
\Omega(\lambda,\mu,\alpha) &= \sum_{n=1}^{\infty} \frac{n\Gamma(\mu)}{\Gamma(\lambda n + \mu)} + (1-\alpha)\sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \\
&\leq \sum_{n=1}^{\infty} \frac{n}{(\mu)_n} + (1-\alpha)\sum_{n=1}^{\infty} \frac{1}{(\mu)_n} \\
&= \frac{1}{\mu}\sum_{n=0}^{\infty} \frac{n+1}{(\mu+1)_n} + \frac{1-\alpha}{\mu}\sum_{n=0}^{\infty} \frac{1}{(\mu+1)_n} \\
&< \frac{1}{\mu}\sum_{n=0}^{\infty} \left(\frac{2}{\mu+1}\right)^n + \frac{1-\alpha}{\mu}\sum_{n=0}^{\infty} \left(\frac{1}{\mu+1}\right)^n \\
&= \frac{1}{\mu}\frac{\mu+1}{\mu-1} + \frac{1-\alpha}{\mu}\frac{\mu+1}{\mu} \leq 1-\alpha,
\end{aligned}$$

which gives the result. □

Remark 3.4. *Observe that on taking $\alpha = 0$ in Theorem 3.3, we obtain that, for $\lambda \geq 1$ and $\mu \geq x$, where $x \approx 3.21432$ is the largest root of $x^3 - 3x^2 - x + 1 = 0$, then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{S}^*$. Incidently, this improve our previous result [5, Theorem 2.2], which state that, for $\lambda \geq 1$ and $\mu \geq (3+\sqrt{17})/2 \approx 3.56155$, the function $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{S}^*$.*

4. Hardy Space of Mittag-Leffler function

Theorem 4.1. *Let $0 \leq \alpha < 1$ and $\lambda \geq 1$, then the following statements are true:*

(a) If $\mu \geq \mu_2$, where μ_2 is the largest root of

$$(1-\alpha)(\mu^2-\mu-1)(\mu^2-4\mu+3) - (2-\alpha)\mu(\mu^2-2\mu-3) - \mu(\mu^2-1) = 0, \tag{14}$$

then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{K}(\alpha)$.

(b) If $\mu > \frac{1+\sqrt{5-4\alpha}}{2(1-\alpha)}$, then $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} \in \mathcal{P}(\alpha)$.

Proof. (a) It is known [28, p. 110, Corollary] that, if function $f \in \mathcal{A}$ of the form (2) satisfies $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha$ $(0 \leq \alpha < 1)$, then $f \in \mathcal{K}(\alpha)$. Hence to prove $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{K}(\alpha)$, it is sufficient to show that $\Delta(\lambda,\mu,\alpha) \leq 1-\alpha$, where

$$\Delta(\lambda,\mu,\alpha) = \sum_{n=2}^{\infty} \frac{n(n-\alpha)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}.$$

Using inequalities in (13) and $(n+1)^2 \leq 4^n$ ($n \in \mathbb{N}$), we have

$$\begin{aligned}
\Delta(\lambda,\mu,\alpha) &= \sum_{n=1}^{\infty} \frac{n^2\,\Gamma(\mu)}{\Gamma(\lambda n+\mu)} + (2-\alpha)\sum_{n=1}^{\infty} \frac{n\,\Gamma(\mu)}{\Gamma(\lambda n+\mu)} + (1-\alpha)\sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \\
&\leq \sum_{n=1}^{\infty} \frac{n^2}{(\mu)_n} + (2-\alpha)\sum_{n=0}^{\infty} \frac{n}{(\mu)_n} + (1-\alpha)\sum_{n=1}^{\infty} \frac{1}{(\mu)_n} \\
&= \frac{1}{\mu}\sum_{n=0}^{\infty} \frac{(n+1)^2}{(\mu+1)_n} + \frac{(2-\alpha)}{\mu}\sum_{n=0}^{\infty} \frac{n+1}{(\mu+1)_n} + \frac{(1-\alpha)}{\mu}\sum_{n=0}^{\infty} \frac{1}{(\mu+1)_n} \\
&< \frac{1}{\mu}\sum_{n=0}^{\infty} \left(\frac{4}{\mu+1}\right)^n + \frac{2-\alpha}{\mu}\sum_{n=0}^{\infty} \left(\frac{2}{\mu+1}\right)^n + \frac{1-\alpha}{\mu}\sum_{n=1}^{\infty} \left(\frac{1}{\mu+1}\right)^n \\
&= \frac{1}{\mu}\,\frac{\mu+1}{\mu-3} + \frac{(2-\alpha)(\mu+1)}{\mu(\mu-1)} + \frac{(1-\alpha)(\mu+1)}{\mu^2} \leq 1-\alpha,
\end{aligned}$$

which show that if $\mu \geq \mu_2$, where μ_2 is the largest root of (14), then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{K}(\alpha)$.

(b) To prove our result, we need to show that $|g(z)-1| < 1$, where $g(z) = \frac{1}{1-\alpha}\left(\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} - \alpha\right)$. Using inequalities (13), we have

$$\begin{aligned}
|g(z)-1| &= \frac{1}{1-\alpha}\left|\sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} z^n\right| \leq \frac{1}{\mu(1-\alpha)}\sum_{n=1}^{\infty} \frac{1}{(\mu+1)_{n-1}} \\
&\leq \frac{1}{\mu(1-\alpha)}\sum_{n=1}^{\infty} \left(\frac{1}{\mu+1}\right)^n = \frac{\mu+1}{\mu^2(1-\alpha)} < 1.
\end{aligned}$$

This implies that under the hypotheis $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} \in \mathcal{P}(\alpha)$. $\square$

If we take $\alpha = 0$ in Theorem 4.1, we get the following Corollary.

Corollary 4.2. *Let $\lambda \geq 1$, then the following statements are true:*

(a) If $\mu \geq \mu_3$, where $\mu_3 \approx 6.18757$ is the largest root of $\mu^4 - 8\mu^3 + 10\mu^2 + 8\mu - 3 = 0$, then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{K}$.

(b) If $\mu > \frac{1+\sqrt{5}}{2}$, then $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} \in \mathcal{P}$.

If we take $\alpha = 1/2$ in Theorem 4.1, we get the following Corollary.

Corollary 4.3. *Let $\lambda \geq 1$, then the following statements are true:*

(a) If $\mu \geq \mu_4$, where $\mu_4 \approx 8.40811$ is the largest root of $\mu^4 - 10\mu^3 + 12\mu^2 + 12\mu - 3 = 0$, then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{K}(1/2)$.

(b) If $\mu > 1 + \sqrt{3}$, then $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} \in \mathcal{P}(1/2)$.

Theorem 4.4. *Let $0 \leq \alpha < 1$ and $\lambda \geq 1$. If $\mu \geq \mu_2$, where μ_2 is the largest root of (14), then*

$$\mathbb{E}_{\lambda,\mu}(z) \in \begin{cases} \mathcal{H}^{1/(1-2\alpha)}, & \alpha \in [0,1/2) \\ \mathcal{H}^{\infty}, & \alpha \geq 1/2. \end{cases}$$

Proof. Note that

$$k+\frac{zd}{(1-ze^{i\gamma})^{1-2\alpha}}=k+z\,d\;{}_2F_1\left(1,1-2\alpha;1;ze^{i\gamma}\right)\quad(\alpha\neq 1/2)$$

and

$$k+d\,\log(1-z\,e^{i\gamma})=k-d\,z\;{}_2F_1(1,1,2;ze^{i\gamma})\quad(\alpha=1/2),$$

which shows that $\mathbb{E}_{\lambda,\mu}(z)$ is not of the forms $k+\dfrac{zd}{(1-ze^{i\gamma})^{1-2\alpha}}$ $(\alpha\neq 1/2)$ and $k+d\,\log(1-z\,e^{i\gamma})$ $(\alpha=1/2)$. We know by part (a) of Theorem 4.1 that, function $\mathbb{E}_{\lambda,\mu}$ is convex of order α. Hence using Lemma 2.5, the desired result holds. □

Theorem 4.5. *Let* $\lambda\geq 1$ *and* $\mu>1+\sqrt{3}$*. If* $f\in\mathcal{R}$ *of the form* (2) *then the convolution* $\mathbb{E}_{\lambda,\mu}*f$ *is in* $\mathcal{H}^\infty\cap\mathcal{R}$*.*

Proof. If $f\in\mathcal{R}$, then $f'\in\mathcal{P}$. Consider $u(z)=\mathbb{E}_{\lambda,\mu}(z)*f(z)$, which is equivalent to

$$u'(z)=\frac{\mathbb{E}_{\lambda,\mu}(z)}{z}*f'(z). \tag{15}$$

We know by part (b) of Corollary 4.3 that, function $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z}\in\mathcal{P}(1/2)$. It follows from Lemma 2.4 that $u'(z)\in\mathcal{P}$. Thus by (5) we have $u'(z)\in\mathcal{H}^q$ for all $q<1$, and hence $u(z)\in\mathcal{H}^{q/(1-q)}$ for all $0<q<1$, or equivalently, $u(z)\in\mathcal{H}^p$ for all $0<p<\infty$.

Using the well-known bound for *Caratheodory functions*, we find that, if $f\in\mathcal{R}$ of the form (2), then $|a_n|\leq 2/n$ $(n\geq 2)$ [16, p. 533, Theorem 1]. Using this fact and first inequality of (13), we find that

$$\begin{aligned}|u(z)| &\leq |z|+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}|a_n||z|^n\leq 1+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{2}{n}\\ &= 1+\sum_{n=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)}\frac{2}{n+1}\leq 1+\sum_{n=1}^{\infty}\frac{2}{(n+1)(\mu)_n}<\infty.\end{aligned}$$

This shows that under the stated condition the power series for $u(z)$ converges absolutely for $|z|=1$. Further, it is well known that [7, P. 42, Theorem 3.11], $u'(z)\in\mathcal{H}^q$ implies continuity of $u(z)$ on $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$. Finally, the continuous functions $u(z)$ on the compact set $\overline{\mathbb{D}}$ are bounded. Hence $u(z)$ is bounded analytic function in $\mathbb{D}$. Therefore $u(z)\in\mathcal{H}^\infty$. This completes the proof. □

Theorem 4.6. *Let* $\lambda\geq 1$ *and* $\mu>\frac{1+\sqrt{5-4\alpha}}{2(1-\alpha)}$*. If* $f\in\mathcal{R}(\beta)$ $(\beta<1)$ *of the form* (2)*, then* $\mathbb{E}_{\lambda,\mu}*f\in\mathcal{R}(\gamma)$ *where* $\gamma=1-2(1-\alpha)(1-\beta)$*.*

Proof. If $f\in\mathcal{R}(\beta)$, then $f'\in\mathcal{P}(\beta)$. Consider $u(z)=\mathbb{E}_{\lambda,\mu}(z)*f(z)$, which is equivalent to

$$u'(z)=\frac{\mathbb{E}_{\lambda,\mu}(z)}{z}*f'(z). \tag{16}$$

We know by part (b) of Theorem 4.1 that, under the stated conditions function $\frac{\mathbb{E}_{\lambda,\mu}(z)}{z}\in\mathcal{P}(\alpha)$. It follows from Lemma 2.4 that $u'(z)\in\mathcal{P}(\gamma)$ or equivalently, $u(z)\in\mathcal{P}(\gamma)$. This completes the proof. □

5. Some observations and concluding remarks

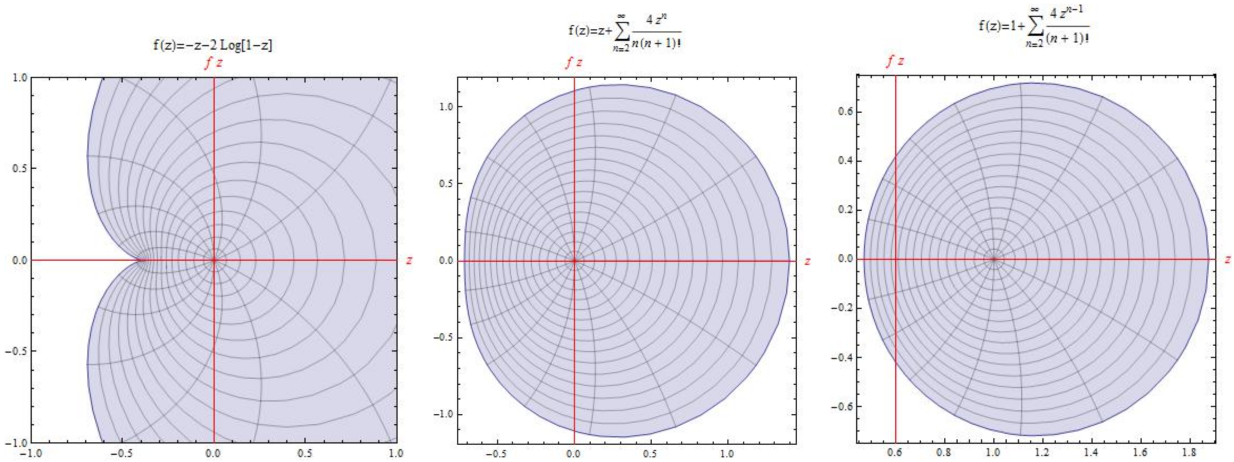
In this section, we examine the geometrical descriptions of image domains of functions in Theorem 4.5. Consider

$$f(z)=-z-2\log(1-z)=z+\sum_{n=2}^{\infty}\frac{2}{n}z^n.$$

We can see easily that $f(z) \in \mathcal{R}$ and Fig. 1(a) showing that $f(z) \notin \mathcal{H}^\infty$. As per Theorem 4.5, taking $\lambda \geq 1$ and $\mu > 1 + \sqrt{3} \approx 2.73205$, the convolution $\mathbb{E}_{\lambda,\mu} * f \in \mathcal{H}^\infty \cap \mathcal{R}$. To see the validity our result, we set $\lambda = 1$ and $\mu = 3$ in Theorem 4.5, we have

$$\begin{aligned} u(z) &= \mathbb{E}_{1,3}(z) * f(z) = \frac{2(e^z - z - 1)}{z} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{2}{n\,(n+1)!} z^n. \end{aligned}$$

As shown in Figure 1(b) below it is true that $u(z) \in \mathcal{H}^\infty$. Further, Figure 1(c) showing that $\Re(u'(z)) > 0$, hence $u(z) \in \mathcal{R}$. Therefore $\mathbb{E}_{1,3} * f \in \mathcal{H}^\infty \cap \mathcal{R}$.



(a) Mapping of $f(z)$ over $\mathbb{D}$ (b) Mapping of $u(z)$ over $\mathbb{D}$ (c) Mapping of $u'(z)$ over $\mathbb{D}$

Figure 1: Mapping of $f(z)$, $u(z)$ and $u'(z)$ over $\mathbb{D}$.

Furthermore, taking $\lambda = 1$ and $\mu = 4$ in Theorem 4.5, we have

$$\begin{aligned} v(z) &= \mathbb{E}_{1,4}(z) * f(z) = \frac{6(e^z - 1 - z) - 3z^2}{z^2} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{12}{n(n+2)!} z^n. \end{aligned}$$

Clearly, as shown in figure 2(a) below it is true that $v(z) \in \mathcal{H}^\infty$. Further, Figure 2(b) showing that $\Re(v'(z)) > 0$, hence $v(z) \in \mathcal{R}$. Therefore $\mathbb{E}_{1,4} * f \in \mathcal{H}^\infty \cap \mathcal{R}$.

We conclude this paper by remarking that, by appropriately selecting parameters and functions, our main results would lead to new results and further applications. These consideration can fruitfully be worked out and we skip the details in this regard.

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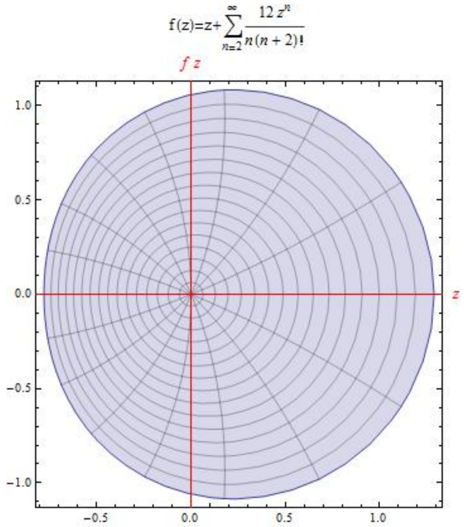
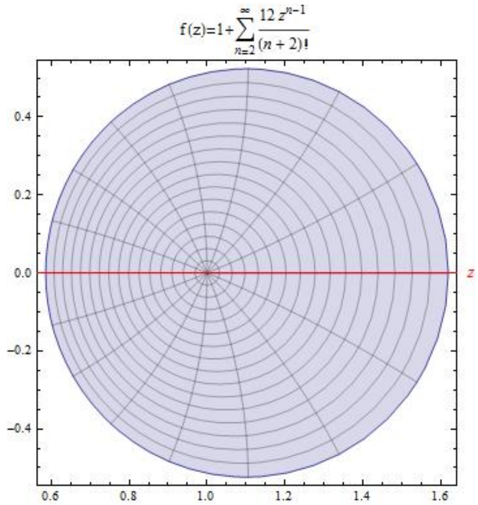


(a) Mapping of $v(z)$ over $\mathbb{D}$



(b) Mapping of $v'(z)$ over $\mathbb{D}$

Figure 2: Mapping of $v(z)$ and $v'(z)$ over $\mathbb{D}$.

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