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Bound Estimates for the Derivative of Driving Point Impedance Functions

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Abstract. In this paper, a boundary analysis is carried out for the derivative of driving point impedance functions, which is mainly used for synthesis of networks containing *RL*, *RC* and *RLC* circuits. It is known that driving point impedance function, $Z(s)$, is an analytic function defined on the right half of the s-plane. In this study, we derive inequalities for the modulus of derivative of driving point impedance function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s = 0$ on the imaginary axis and the sharpness of these inequalities is proved. Furthermore, an equation for the driving point impedance function, $Z(s)$, is obtained as a natural result of the proved theorem in this study.

1. Introduction

Driving point impedance (DPI) functions are given as positive real functions (PRF) and they depend on the complex frequency parameter, s. A driving point impedance function is physically realizable if it satisfies the properties of positive real functions which are given below [21]:

1-) $Z(s)$ is analytic and single valued in $\Re s \geq 0$ except possibly for poles on the axis of imaginaries,

2-) $Z(\overline{s}) = \overline{Z(s)}$

3-) $\Re Z(s) \geq 0$, in $\Re s \geq 0$

The aim of this study is to obtain a new bound for the derivative of positive real functions. As an application of complex analysis, we investigate bound estimates for the modulus of the derivative of DPI functions which are commonly used in electrical engineering [7–9, 17, 20, 21].

Let f be an analytic function in the unit disc $E = \{z : |z| < 1\}$, $f(0) = 0$ and $\left|f(z)\right| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc E, we have $\left|f(z)\right| \leq |z|$ and $\left|f'(0)\right| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ ([6], p.329). The Schwarz lemma is one of the most important results in the classical complex analysis,which has become a crucial theme in many branches of mathematical research for over a hundred years [14, 16]. Also, in [15] ,they gave simple proofs of various versions of the Schwarz lemma for real-valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly (HQR) mappings with the strip codomain.

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Consider the function

$$f(z) = \frac{Z(s) - Z(1)}{Z(s) + Z(1)},\ z = \frac{s-1}{s+1}.$$

Note that Z(1) is a real and positive function. Also, $Z(s)$ is analytic from right half plane into itself and real on real axis since it is a positive real function as defined at the beginning of the introduction section.

Here, $f(z)$ is an analytic function in E, $f(0) = 0$ and $\left|f(z)\right| < 1$ for $|z| < 1$. Applying the Schwarz lemma for the function $f(z)$, we obtain

$$|Z'(1)| \leq Z(1). \tag{1.1}$$

The inequality (1.1) is sharp with equality for the function

$$Z(s) = sZ(1).$$

In this case, we get the following lemma.

Lemma 1.1. *Let $Z(s)$ be a Positive Real Function. Then*

$$|Z'(1)| \leq Z(1).$$

The result is sharp and the extremal function is

$$Z(s) = sZ(1).$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $\left|f(b)\right| = 1$ and $f'(b)$ exists, then $\left|f'(b)\right| \geq 1$, which is known as the Schwarz lemma on the boundary. This result of the Schwarz Lemma and its generalization are described as the Boundary Schwarz Lemma in the literature. This improvement was obtained by Helmut Unkelbach in [22] and rediscovered by R. Osserman in [18] 60 years later. Boas, Chelst, Burns and Krantz ([2–4]) studied the Schwarz lemma at the boundary of the unit disk, respectively. V. N. Dubinin [5] strengthened the inequality $\left|f'(b)\right| \geq 1$ by involving zeros of the function $f(z)$. In the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [5], [10], [11], [12], [18], [19] and references therein). In addition, M. Mateljević has given a review about some properties of hyperbolic metrics and various versions of Schwarz lemma in [13]. Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $\left|f(b)\right| = 1$ condition of the boundary of the unit circle.

The contribution of this study is twofold. First, a novel boundary estimate analysis for the DPI functions of electrical engineering is presented. In contrary to previous studies, here, we consider the bound estimate of the DPI function from below. Secondly, the presented theorem has been exemplified as a circuit synthesis problem and corresponding DPI function has been derived. In addition, system that use the presented theorem has been illustrated.

The rest of the manuscript is organized as follows: In Section 2, we present the new theorem for bound estimate of DPI function from below and the sharpness of the obtained inequality is proven in this section. At the end, an application example is given using the DPI function obtained in our theorem.

2. Main Results

In this section, boundary analysis results for the derivative of driving point impedance function are presented. From the definition of PRFs, we can state that $Z(s)$ is analytic and single valued on the right half of the s-plane. By assuming $Z(s)$ is also analytic at the boundary point $s = 0$ on the imaginer axis, we shall give an estimate for $|Z'(0)|$ from below using Taylor expansion coefficients. The sharpness of this inequality is also proved within the manuscript.

Let $H^+ = \{s \in C : \Re s > 0\}$ be the right half plane in $\mathbb{C}$.

Motivated by the results of the work presented in [1], the following result has been obtained.

Theorem 2.1. *Let $Z(s)$ be a positive real function, $Z(s_1) = Z(1)$, $s_1 \in H^+$ and it is also an analytic function at the point $s = 0$ of the imaginary axis with $Z(0) = 0$. Then*

$$|Z'(0)| \geq Z(1)\Big(1 + \tfrac{\Re s_1}{|s_1|^2} + \tfrac{Z(1)|s_1-1|-|Z'(1)||s_1+1|}{Z(1)|s_1-1|+|Z'(1)||s_1+1|} \tag{2.1}$$

$$\times\left[1 + \frac{|Z(1)|^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|-|Z(1)||Z'(s_1)|\Re s_1-|Z(1)||Z'(1)|}{|Z(1)|^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|+|Z(1)||Z'(s_1)|\Re s_1+|Z(1)||Z'(1)|}\frac{\Re s_1}{|s_1|^2}\right]\Big)$$

The inequality (2.1) *is sharp, with equality for each possible values $Z'(1)$ and $Z'(s_1)$.*

Proof. Let

$$q(z) = \frac{z - z_1}{1 - \overline{z_1}z}.$$

Also, let $h : E \to E$ be an analytic function and a point $z_1 \in E$ in order to satisfy

$$\left|\frac{h(z) - h(z_1)}{1 - \overline{h(z_1)}h(z)}\right| \leq \left|\frac{z - z_1}{1 - \overline{z_1}z}\right| = \left|q(z)\right|$$

and

$$|h(z)| \leq \frac{|h(z_1)| + \left|q(z)\right|}{1 + |h(z_1)|\left|q(z)\right|} \tag{2.2}$$

by Schwarz-pick lemma [6]. That is, $h(z)$ satisfies the conditions of Schwarz-Pick lemma since it is analytic in E and it is not equal to 0. It is clearly seen that $h(z_1) \neq 0$ from the $h(z)$ function defined below.

If $v : E \to E$ is an analytic function and $0 < |z_1| < 1$, letting

$$h(z) = \frac{v(z) - v(0)}{z\left(1 - \overline{v(0)}v(z)\right)}$$

in (2.2), we obtain

$$\left|\frac{v(z) - v(0)}{z\left(1 - \overline{v(0)}v(z)\right)}\right| \leq \frac{\left|\frac{v(z_1)-v(0)}{z_1\left(1-\overline{v(0)}v(z_1)\right)}\right| + \left|q(z)\right|}{1 + \left|\frac{v(z_1)-v(0)}{z_1\left(1-\overline{v(0)}v(z_1)\right)}\right|\left|q(z)\right|}$$

and

$$|v(z)| \leq \frac{|v(0)| + |z|\,\frac{|C|+\left|q(z)\right|}{1+|C|\left|q(z)\right|}}{1 + |v(0)|\,|z|\,\frac{|C|+\left|q(z)\right|}{1+|C|\left|q(z)\right|}}, \tag{2.3}$$

where

$$C = \frac{v(z_1) - v(0)}{z_1\left(1 - \overline{v(0)}v(z_1)\right)}.$$

If we take

$$v(z)=\frac{f(z)}{z\frac{z-z_1}{1-\overline{z_1}z}},$$

where

$$v(z_1)=\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1},\quad v(0)=\frac{f'(0)}{-z_1}.$$

Then

$$C=\frac{\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}+\frac{f'(0)}{z_1}}{z_1\left(1+\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\overline{\frac{f'(0)}{z_1}}\right)},$$

where $|C|\leq 1$. Let $|v(0)|=\alpha$ and

$$\mathrm{D}=\frac{\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{f'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{f'(0)}{z_1}\right|\right)}.$$

From (2.3), we get

$$\left|f(z)\right|\leq |z|\left|q(z)\right|\frac{\alpha+|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}}{1+\alpha|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}}$$

and

$$\frac{1-\left|f(z)\right|}{1-|z|}\geq\frac{1+\alpha|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}-\alpha|z|\left|q(z)\right|-\left|q(z)\right||z|^2\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}}{(1-|z|)\left(1+\alpha|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}\right)}=\varrho(z). \tag{2.4}$$

Let $\kappa(z)=1+\alpha|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}$ and $\tau(z)=1+\mathrm{D}\left|q(z)\right|$. Then

$$\varrho(z)=\frac{1-|z|^2\left|q(z)\right|^2}{(1-|z|)\,\kappa(z)\tau(z)}+\mathrm{D}\left|q(z)\right|\frac{1-|z|^2}{(1-|z|)\,\kappa(z)\tau(z)}+|z|\,\mathrm{D}\alpha\frac{1-\left|q(z)\right|^2}{(1-|z|)\,\kappa(z)\tau(z)}.$$

Since

$$\lim_{z\to-1}\kappa(z)=\lim_{z\to-1}1+\alpha|z|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}=1+\alpha,$$

$$\lim_{z\to-1}\tau(z)=\lim_{z\to-1}1+\mathrm{D}\left|q(z)\right|=1+\mathrm{D}$$

and

$$1-\left|q(z)\right|^2=1-\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^2=\frac{\left(1-|z_1|^2\right)\left(1-|z|^2\right)}{\left|1-\overline{z_1}z\right|^2}, \tag{2.5}$$

passing to the limit in (2.4) and using (2.5) gives

$$\begin{aligned}\left|f'(-1)\right| &\geq \frac{2}{(1+\alpha)(1+\mathrm{D})}\left(1+\frac{1-|z_1|^2}{|1+z_1|^2}+\mathrm{D}+\alpha\mathrm{D}\frac{1-|z_1|^2}{|1+z_1|^2}\right)\\ &= 1+\frac{1-|z_1|^2}{|1+z_1|^2}+\frac{1-\alpha}{1+\alpha}\left(1+\frac{1-\mathrm{D}}{1+\mathrm{D}}\frac{1-|z_1|^2}{|1+z_1|^2}\right).\end{aligned}$$

Moreover, since

$$\begin{aligned}\frac{1-\alpha}{1+\alpha} &= \frac{1-|v(0)|}{1+|v(0)|}=\frac{1-\left|\frac{f'(0)}{z_1}\right|}{1+\left|\frac{f'(0)}{z_1}\right|}=\frac{|z_1|-\left|f'(0)\right|}{|z_1|+\left|f'(0)\right|}\\ &= \frac{\left|\frac{s_1-1}{s_1+1}\right|-\left|\frac{Z'(1)}{Z(1)}\right|}{\left|\frac{s_1-1}{s_1+1}\right|+\left|\frac{Z'(1)}{Z(1)}\right|}\end{aligned}$$

$$\frac{1-\mathrm{D}}{1+\mathrm{D}}=\frac{1-\frac{\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{f'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{f'(0)}{z_1}\right|\right)}}{1+\frac{\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{f'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{f'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{f'(0)}{z_1}\right|\right)}}=\frac{1-\frac{\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|+\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|\right)}}{1+\frac{\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|+\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|\right)}},$$

$$\begin{aligned}\frac{1-\mathrm{D}}{1+\mathrm{D}} &= \frac{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|\right)-\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|-\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|\right)+\left|\frac{\frac{(s_1+1)^2}{4}\frac{Z'(s_1)}{Z(1)}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\frac{s_1-1}{s_1+1}}\right|+\left|\frac{\frac{Z'(1)}{Z(1)}}{\frac{s_1-1}{s_1+1}}\right|}\\ &= \frac{\left|\frac{s_1-1}{s_1+1}\right|^2+\frac{|s_1+1|^2}{4}\left|\frac{Z'(s_1)}{Z(1)}\right|\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)\left|\frac{Z'(1)}{Z(1)}\right|-\left|\frac{|s_1+1|^2}{4}\left|\frac{Z'(s_1)}{Z(1)}\right|\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)\right|-\left|\frac{Z'(1)}{Z(1)}\right|}{\left|\frac{s_1-1}{s_1+1}\right|^2+\frac{|s_1+1|^2}{4}\left|\frac{Z'(s_1)}{Z(1)}\right|\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)\left|\frac{Z'(1)}{Z(1)}\right|+\frac{|s_1+1|^2}{4}\left|\frac{Z'(s_1)}{Z(1)}\right|\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)+\left|\frac{Z'(1)}{Z(1)}\right|},\end{aligned}$$

and

$$1-\left|\frac{s_1-1}{s_1+1}\right|^2=\frac{4\Re s_1}{|s_1+1|^2}$$

$$\left|1+\frac{s_1-1}{s_1+1}\right|^2=\frac{4\left|s_1\right|^2}{\left|s_1+1\right|^2},$$

we take

$$\left|f'(-1)\right| \geq 1+\frac{\Re s_1}{|s_1|^2}+\frac{Z(1)|s_1-1|-|Z'(1)||s_1+1|}{Z(1)|s_1-1|+|Z'(1)||s_1+1|}$$

$$\times\left[1+\frac{|Z(1)|^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|-|Z(1)||Z'(s_1)|\Re s_1-|Z(1)||Z'(1)|}{|Z(1)|^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|+|Z(1)||Z'(s_1)|\Re s_1+|Z(1)||Z'(1)|}\frac{\Re s_1}{|s_1|^2}\right]$$

From definition of $f(z)$, we have

$$f'(z)=\frac{\frac{4}{(1-z)^2}Z'(\frac{1+z}{1-z})Z(1)}{\left(Z(\frac{1+z}{1-z})+Z(1)\right)^2}$$

and

$$\left|f'(-1)\right|=\frac{|Z'(0)|}{Z(1)}.$$

Thus, we obtain the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp.

Since $v(z)=\frac{f(z)}{z\frac{z-z_1}{1-\overline{z}_1 z}}$ is an analytic function in the unit disc E and $|v(z)|\leq 1$ for $|z|<1$, we obtain

$$\left|f'(0)\right|\leq |z_1|$$

and

$$\left|f'(z_1)\right|\leq \frac{|z_1|}{1-|z_1|^2}.$$

We take $z_1\in(-1,0)$ and arbitrary two numbers x and y. Let

$$\mathtt{M}=\frac{\frac{x\left(1-|z_1|^2\right)}{z_1}+\frac{x}{z_1}}{z_1\left(1+xy\frac{1-|z_1|^2}{z_1^2}\right)}=\frac{1}{z_1^2}\frac{y\left(1-|z_1|^2\right)+x}{1+xy\frac{1-|z_1|^2}{z_1^2}}.$$

The auxiliary function

$$f(z)=z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-x}{z_1}+z\frac{\mathtt{M}+\frac{z-z_1}{1-\overline{z}_1 z}}{1+\mathtt{M}\frac{z-z_1}{1-\overline{z}_1 z}}}{1-\frac{x}{z_1}z\frac{\mathtt{M}+\frac{z-z_1}{1-\overline{z}_1 z}}{1+\mathtt{M}\frac{z-z_1}{1-\overline{z}_1 z}}} \tag{2.6}$$

is analytic in E and $\left|f(z)\right|<1$ for $z\in E$.

From (2.6), with the simple calculations, we obtain

$$\left|f'(-1)\right| = 1 + \frac{1-z_1^2}{(1+z_1)^2} + \frac{x+z_1}{-x+z_1}\left(1 + \frac{1-z_1^2}{(1+z_1)^2}\frac{1+\mathtt{M}}{1-\mathtt{M}}\right).$$

Therefore, since $0 < \Re s_1 < 1$, $\Im s_1 = 0$, $z_1 = \frac{s_1-1}{s_1+1}$ and choosing suitable signs of the numbers x and y, the last equality shows that (2.1). $\square$

3. Discussion

It is still a challenging problem in electrical engineering to convert a DPI function given in s-domain into a physically realizable circuit particularly when the order of s parameter is high in the DPI function [17]. In this section, DPI functions with high ordered s parameter are considered. An example system representation which use the extremal function given in equation (2.6) is also presented in this section. As an application of the theorem presented in this study, the auxiliary function given in equation (2.6) is used as a DPI function $Z(s)$, that is

$$f(z) = \frac{Z(s)-Z(1)}{Z(s)+Z(1)} = z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-x}{z_1} + z\frac{\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{x}{z_1}z\frac{\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z}}}$$

$$Z(s) = Z(1)\frac{1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z} - \frac{x}{z_1}z\left(\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}\right) + z\frac{z-z_1}{1-\overline{z_1}z}\left(\frac{-x}{z_1}\left(1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z}\right) + zz_1\left(\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}\right)\right)}{1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z} - \frac{x}{z_1}z\left(\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}\right) - z\frac{z-z_1}{1-\overline{z_1}z}\left(\frac{-x}{z_1}\left(1+\mathtt{M}\frac{z-z_1}{1-\overline{z_1}z}\right) + zz_1\left(\mathtt{M}+\frac{z-z_1}{1-\overline{z_1}z}\right)\right)}, \tag{2.7}$$

where $z = \frac{s-1}{s+1}$, $z_1 = \frac{s_1-1}{s_1+1}$ and $s, s_1 \in H^+$.

Considering $Z(s)$ as an DPI function, the system model is given as in Fig. 1.

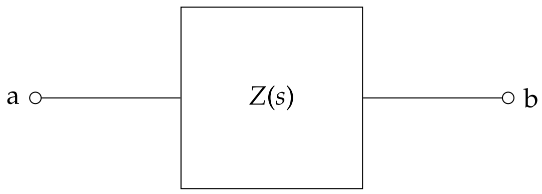

Figure 1: System model for the DPI function given in equation (2.7).

According to given system model in Fig. 1, the system will give different response to different frequency components in the input signal at node a since the parameter s is equal to iw where w is the radial frequency. This block diagram also represents general usage of the DPI function proposed in this study. Considering the literature, it is possible to infer that the boundary analysis for derivative of DPI functions is an important topic for circuit and systems research community. Therefore, it is reasonable to further investigate this topic and present novel mathematical analysis in order to motivate other researchers.

In conclusion, we would like to emphasize again that the obtained $Z(s)$ function above is not arbitrarily selected and it exists as a natural result of Theorem 2.1 proved in Section 2.

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