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Weak Quasi-Entwining Structures

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Abstract. In this paper we introduce the notion of weak quasi-entwining structure as a generalization of quasi-entwining structures and weak entwining structures. Also, we formulate the notions of weak cleft extension, weak Galois extension, and weak Galois extension with normal basis associated to a weak quasi-entwining structure. Moreover, we prove that, under some suitable conditions, there exists an equivalence between weak Galois extensions with normal basis and weak cleft extensions. As particular instances, we recover some results previously proved for Hopf quasigroups, weak Hopf quasigroups and weak Hopf algebras.

1. Introduction

In recent years, there has been a growing interest about the notion of entwining structure and its generalizations. This kind of structures were introduced by Brzeziński and Majid in [12] to understand some symmetry properties of classical principal bundles in non-commutative geometry. In this setting, as was pointed in [13], an entwining structure can be viewed as a symmetry of a non-commutative manifold. From a formal viewpoint, an entwining structure in a category of modules over a commutative ring R, is a triple (A, C, ψ) where A is an algebra, C is a coalgebra and $\psi : C \otimes A \to A \otimes C$ ($\otimes$ denotes the tensor product over R) is a map, called the entwining map, satisfying four conditions. Entwining structures are in one-to-one correspondence with A-coring structures on $A \otimes_R C$ and one of the main examples comes from the Hopf algebra setting because any comodule algebra over a Hopf algebra induces an entwining structure. Moreover, entwining structures are a powerful tool to unify, using its categories of entwining modules, various categories of Hopf modules introduced by several authors in the last decades as, for example, Sweedler Hopf modules [27], [16], Doi and Takeuchi relative Hopf modules [17], [18], [19], Doi-Koppinen modules [20], [21], Yetter-Drinfeld modules [29], etc.

On the other hand, the notion of Galois extension associated to a Hopf algebra H was introduced in 1981 by Kreimer and Takeuchi in the following way: let A be a right H-comodule algebra with coaction $\rho_A(a) = a_{(0)} \otimes a_{(1)}$. An extension $A^{coH} \hookrightarrow A$, where $A^{coH} = \{a \in A\ ;\ \rho_A(a) = a \otimes 1_H\}$ is the subalgebra

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of coinvariant elements, is H-Galois if the canonical morphism $\gamma_A : A \otimes_{A^{coH}} A \to A \otimes H$, defined by $\gamma_A(a \otimes b) = ab_{(0)} \otimes b_{(1)}$, is an isomorphism. This definition has its origin in the approach to Galois theory of groups acting on commutative rings developed by Chase, Harrison and Rosenberg and in the extension of this theory to coactions of a Hopf algebra H acting on a commutative algebra A, developed in 1969 by Chase and Sweedler [16]. An interesting class of H-Galois extensions has been provided by those for which there exists a convolution invertible right H-comodule morphism $h : H \to A$ called the cleaving morphism. These extensions were called cleft and it is well known that, using the notion of normal basis introduced by Kreimer and Takeuchi in [24], Doi and Takeuchi proved in [18] that $A^{coH} \hookrightarrow A$ is a cleft extension if and only if is H-Galois with normal basis, i.e., the extension $A^{coH} \hookrightarrow A$ is H-Galois and A is isomorphic to the tensor product of A^{coH} with H as left A^{coH}-modules and right H-comodules.

The result obtained by Doi and Takeuchi admits a generalization to the entwining structure setting. In [10] Brzeziński proved that if (A, C, ψ) is an entwining structure such that A is an entwined module, the existence of a convolution invertible C-comodule morphism $h : C \to A$ is equivalent to say that A is a Galois extension by the coalgebra C (see [11] for the definition) and A is isomorphic, as left A^{coH}-module and right C-comodule, to the tensor product of the coinvariant subalgebra A^{coC} with C, i.e., the "normal basis condition" holds.

The generalization of entwining structures to the weak setting was proposed, with the name of weak entwining structures, by Caenepeel and De Groot in [14] to treat algebra extensions by weak Hopf algebras [9]. Similarly with the entwining case, there exists a bijective correspondence between weak entwining structures and canonical weak coring structures, in the sense of Wisbauer, on $A \otimes_R C$. Also, weak entwining structures unify the more relevant categories of Hopf modules associated to weak Hopf algebras and, as in the Hopf algebra case, the main family of examples comes from the theory of comodule algebras over weak Hopf algebras. As a consequence, we can apply its properties to obtain results about Hopf-Galois extensions in this context. For example, in [1] the notion of weak cleft extension was defined in a monoidal setting, and Theorem 2.11 of [1] stated that for a weak entwining structure (A, C, ψ) such that A is an entwined module, if the functor $A \otimes -$ preserves coequalizers, A is a weak C-cleft extension of the coinvariants subalgebra if and only if is a weak C-Galois extension and the normal basis property, defined in [1], holds. Then we get the weak entwining version of the result proved by Doi and Takeuchi and, since Galois extensions associated to weak Hopf algebras are examples of weak Galois extensions for weak entwining structures, we obtain that this characterization of weak cleft extensions in terms of weak Galois extensions satisfying the normal basis condition can be applied to them.

A different generalization of the notion of (weak) entwining structure was proposed by Caenepeel and Janssen in [15] with the name of partial entwining structures. In this case the motivation was to introduce a theory of partial actions and coactions of Hopf algebras and then to obtain a Hopf-Galois theory in this context. To this end, these authors introduced the more general notion of lax entwining structure that includes partial and weak entwining structures as special cases. Taking inspiration from the weak entwining case, in [3] we introduced the notions of lax Galois extension with normal basis and lax cleft extension, and we proved, under the same conditions we used in the weak entwining setting, that these notions are equivalent. Using that every partial entwining structure is a lax entwining structure we also obtained in [3] the corresponding result for partial entwining structures.

In the previous cases we always work with associative algebras (Hopf algebras and weak Hopf algebras) but, recently, many Hopf non-associative algebraic structures were introduced generalizing the notions of Hopf algebra and weak Hopf algebra. For example, Hopf quasigroups and weak Hopf quasigroups belong to this family of non-associative Hopf algebra objects. The first ones were introduced by Klim and Majid in [22] to understand the structure and relevant properties of the algebraic 7-sphere and they are particular instances of the notion of unital coassociative H-bialgebra introduced in [26]. As examples, they include the enveloping algebra of a Malcev algebra (see [22] and [25]) and the quasigroup algebra of an I.P. loop. On the other hand, by weakening the unitality and associativity conditions on the Hopf algebra definition, recently we proposed in [5] a new notion called weak Hopf quasigroup, that encompass weak Hopf algebras and Hopf quasigroups. A family of non trivial examples of these algebraic objects can be obtained by working with bigroupoids, i.e., bicategories where every 1-cell is an equivalence and every 2-cell is an isomorphism (see Example 2.3 of [5]).

The first result linking Hopf Galois extensions with normal basis and cleft extensions in a non-associative setting can be found in [6]. More specifically, in [6] we introduce the notion of weak H-cleft extension, for a weak Hopf quasigroup H in a strict monoidal category C with tensor product $\otimes$, which generalizes the one introduced for Hopf quasigroups in [4] with the name of cleft H-comodule algebra. Also, we introduce the definition of H-Galois extension with normal basis, and we proved that, under the suitable conditions, H-cleft extensions are the same that H-Galois extensions with normal basis and such that the inverse of the canonical morphism is almost lineal. Therefore, in [6], we extend the result proved by Doi and Takeuchi in [18] to the weak Hopf quasigroup setting and, as a consequence, for Hopf quasigroups. Of course, if H is a weak Hopf algebra we recover the result proved in [1] for weak Hopf algebras because, in an associative context, the conditions assumed in the main theorem of [6] hold trivially.

As was proved in [7], following the ideas developed in [1] for weak entwining structures and working in a similar setting, it is possible to find the meaning of cleft for Hopf quasigroups in terms of entwinings. To do this, in [7], we propose the notion of quasi-entwining structure. Quasi-entwining structures are triples (A, C, ψ) where A is a unital magma, C is a comonoid and $\psi : C \otimes A \to A \otimes C$ is a morphism satisfying three axioms contained in the classical definition of entwining structure. In a similar way with the previous cases, we get an example of quasi-entwining structure by considering H a Hopf quasigroup and (A, ρ_A) a right H-comodule magma. Then many questions arise if we think about weak Hopf quasigroups in a similar way. For example, is it possible to introduce a "good" notion of entwining structure for weak Hopf quasigroups linked with the notions of weak entwining structure and quasi-entwining structure? If true, is it possible to prove for these general entwinings an equivalence between cleft extensions an Galois extensions with normal basis containing as particular instances the results proved in [1] and [6]? To give an answer to this questions is the main goal of this paper.

Now, we describe the paper in detail. After this introduction, in the second section we introduce the notion of weak quasi-entwining structure proving that any H-comodule magma for a weak Hopf quasigroup H provides an example of these kind of entwining structures. In the third section we propose the definition of weak cleft extension for a weak quasi-entwining structure and we discuss the relations of this new notion with the similar ones that we can find for weak entwining and quasi-entwining structures. Also, in this section we give some examples associated to weak Hopf quasigroups and Hopf quasigroups. Finally, in the last section we introduce the definitions of weak Galois extension and weak Galois extension with normal basis for a weak quasi-entwining structure and we prove that, under suitable conditions, there is no difference between weak Galois extensions with normal basis and cleft extensions for a weak quasi-entwining structure. As a consequence of this result we recover the main theorem proved in [6].

2. Weak quasi-entwining structures

In what follows C denotes a monoidal category with equalizers and coequalizers. With $\otimes$ we will understand the tensor product of C and with K its unit object. Without loss of generality, by the coherence theorems, we can assume the monoidal structure of C strict. Then, in this paper, we omit explicitly the associativity and unit constraints. For each object X in C, $id_X : X \to X$ is the identity morphism of X and, for simplicity of notation, given objects X, Y and Z in C and a morphism $f : X \to Y$ between them, we write $Z \otimes f$ for $id_Z \otimes f$ and $f \otimes Z$ for $f \otimes id_Z$. We also assume that for every object X in C the endofunctors $X \otimes -$ and $- \otimes X$ preserve coequalizers.

Note that the existence of equalizers (or coequalizers) implies that every idempotent morphism in C splits (C is Cauchy complete), i.e., if $g : Y \to Y$ is such that $g = g \circ g$, there exist an object Z, called the image of g, and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $g = i \circ p$ and $p \circ i = id_Z$. Note that Z, p, called the projection associated to g, and i, called the injection associated to g, are unique up to isomorphism.

A magma in C is a pair $A = (A, \mu_A)$ where A is an object in C and $\mu_A : A \otimes A \to A$ (product) is a morphism in C. By a unital magma in C we understand a triple $A = (A, \eta_A, \mu_A)$ where (A, μ_A) is a magma in C and $\eta_A : K \to A$ (unit), is a morphism in C such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. If μ_A is associative, that is, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, the unital magma will be called a monoid in C. Given two unital magmas (monoids) $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \to B$ in C is a morphism of unital magmas (monoids) if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

A comagma in C is a pair $D = (D, \delta_D)$ where D is an object in C and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in C. A counital comagma in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where (D, δ_D) is a comagma in C and $\varepsilon_D : D \to K$ (counit) is a morphism in C such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in C is a counital comagma in C satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct δ_D is coassociative. If D and E are counital comagmas (comonoids) in C, a morphism $f : D \to E$ in C is a morphism of counital comagmas (comonoids) such that $\varepsilon_E \circ f = \varepsilon_D$ and $(f \otimes f) \circ \delta_D = \delta_E \circ f$ hold.

Let A be a magma, let D be comagma and let $f : D \to A$, $g : D \to A$ be morphisms in C. The convolution product of f and g, denoted by $f * g$, is defined by $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$.

Let B be a monoid. The pair (X, ψ_X) is a right B-module if X is an object in C and $\psi_X : X \otimes B \to X$ is a morphism in C satisfying $\psi_X \circ (X \otimes \eta_B) = id_X$, $\psi_X \circ (\psi_X \otimes B) = \psi_X \circ (X \otimes \mu_B)$. Given two right B-modules (X, ψ_X) and (Y, ψ_Y), $f : X \to Y$ is a morphism of right B-modules if $\psi_Y \circ (f \otimes B) = f \circ \psi_X$. In the following, we will denote the category of right B-modules by C_B. In a similar way we can define the notions of left B-modules (we denote the left action by φ_X) and morphism of left B-modules. In this case the category of left B-modules will be denoted by ${}_BC$. Finally, note that K is a monoid and in this case we can identify the categories C_K and ${}_KC$ with C.

If D is a comonoid, the pair (X, ρ_X) is a right D-comodule if X is an object in C and $\rho_X : X \to X \otimes D$ is a morphism in C satisfying $(X \otimes \varepsilon_D) \circ \rho_X = id_X$, $(\rho_X \otimes H) \circ \rho_X = (X \otimes \delta_D) \circ \rho_X$. Given two right H-comodules (X, ρ_X) and (Y, ρ_Y), $f : X \to Y$ is a morphism of right D-comodules if $(f \otimes D) \circ \rho_X = \rho_Y \circ f$. The category of right D-comodules will be denoted by C^D.

Definition 2.1. *A weak quasi-entwining structure in C consists of a triple (A, C, ψ), where A is a unital magma, C a comonoid, and $\psi : C \otimes A \to A \otimes C$ a morphism satisfying the relations:*

(a1) $\psi \circ (C \otimes \eta_A) = (u_\psi \otimes C) \circ \delta_C$,

(a2) $\nabla_{A\otimes C} \circ (\mu_A \otimes C) = (\mu_A \otimes C) \circ (A \otimes \nabla_{A\otimes C})$,

(a3) $(A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A)$,

(a4) $(A \otimes \varepsilon_C) \circ \psi = \mu_A \circ (u_\psi \otimes A)$,

where

$$u_\psi = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A),$$

and

$$\nabla_{A\otimes C} = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (A \otimes C \otimes \eta_A).$$

Note that if in the previous definition $u_\psi = \varepsilon_C \otimes \eta_A$ we obtain that $\nabla_{A\otimes C} = id_{A\otimes C}$. Then condition (a2) adds nothing relevant and we have the notion of quasi-entwining structure introduced in [7]. If A is a monoid and we replace the condition (a2) by

$$\psi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) \tag{1}$$

we get the notion of weak entwining structure introduced by Caenepeel and De Groot in [14] as a generalization of entwining structures defined by Brzeziński and Majid (see [12], [10]). In this associative setting, if (1) holds we obtain (a2). Therefore, weak entwining structures are examples of weak quasi-entwining structures.

Lemma 2.2. *Let (A, C, ψ) be a weak quasi-entwining structure. Then,*

$$u_\psi * u_\psi = u_\psi. \tag{2}$$

Proof. The morphism u_ψ is idempotent for the convolution product because

$u_\psi * u_\psi$

$= (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes u_\psi) \circ \delta_C$ (by (a4) of Definition 2.1)

$= (((A \otimes \varepsilon_C) \circ \psi) \otimes \varepsilon_C) \circ (C \otimes \psi) \circ (\delta_C \otimes \eta_A)$ (by definition of u_ψ)

$= (A \otimes ((\varepsilon_C \otimes \varepsilon_C) \circ \delta_C)) \circ \psi \circ (C \otimes \eta_A)$(by (a3) of Definition 2.1)

$= u_\psi$ (by counit conditions).

□

Lemma 2.3. *Let (A, C, ψ) be a weak quasi-entwining structure. The morphism $\nabla_{A\otimes C}$ is idempotent and the identities*

$$(A \otimes \delta_C) \circ \nabla_{A\otimes C} = (\nabla_{A\otimes C} \otimes C) \circ (A \otimes \delta_C), \tag{3}$$

$$\nabla_{A\otimes C} \circ (u_\psi \otimes C) \circ \delta_C = \psi \circ (C \otimes \eta_A), \tag{4}$$

$$p_{A\otimes C} \circ (u_\psi \otimes C) \circ \delta_C = p_{A\otimes C} \circ (\eta_A \otimes C), \tag{5}$$

hold, where $p_{A\otimes C}$ is the projection associated to $\nabla_{A\otimes C}$.

Proof. Note that, by (a1) of Definition 2.1 we have

$$\nabla_{A\otimes C} = (\mu_A \otimes C) \circ (A \otimes ((u_\psi \otimes C) \circ \delta_C)). \tag{6}$$

Then, using (a2) of Definition 2.1, the coassociativity of δ_C and (2), we obtain that $\nabla_{A\otimes C}$ is idempotent because

$$\nabla_{A\otimes C} \circ \nabla_{A\otimes C} = \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes ((u_\psi \otimes C) \circ \delta_C)) = (\mu_A \otimes C) \circ (A \otimes (((u_\psi * u_\psi) \otimes C) \circ \delta_C)) = \nabla_{A\otimes C}.$$

As a consequence, there exist an object $A\square C$, called the image of $\nabla_{A\otimes C}$, and morphisms $i_{A\otimes C} : A\square C \to A\otimes C$ and $p_{A\otimes C} : A \otimes C \to A\square C$ such that $\nabla_{A\otimes C} = i_{A\otimes C} \circ p_{A\otimes C}$ and $p_{A\otimes C} \circ i_{A\otimes C} = id_{A\square C}$. The morphisms $p_{A\otimes C}$ and $i_{A\otimes C}$ will be called the projection and the injection associated to the idempotent morphism $\nabla_{A\otimes C}$.

The equality (3) follows by (6) and the coassociativity of δ_C. As far as (4),

$\nabla_{A\otimes C} \circ (u_\psi \otimes C) \circ \delta_C)$

$= ((\mu_A \circ (u_\psi \otimes u_\psi)) \otimes C) \circ (C \otimes \delta_C) \circ \delta_C$ (by (6))

$= ((u_\psi * u_\psi) \otimes C) \circ \delta_C$ (by coassociativity of δ_C)

$= (u_\psi \otimes C) \circ \delta_C$ (by (2))

$= \psi \circ (C \otimes \eta_A)$ (by (a1) of Definition 2.1).

Finally, by (6), we have

$$\nabla_{A\otimes C} \circ (\eta_A \otimes C) = (u_\psi \otimes C) \circ \delta_C. \tag{7}$$

Then, (5) follows composing in (7) with $p_{A\otimes C}$. □

Example 2.4. The main family of examples of weak quasi-entwining structures comes from the notion of right H-comodule magma for a weak Hopf quasigroup H. Now we recall the notion of weak Hopf quasigroup in a braided monoidal category $\mathcal{C}$ with braiding c (in this case c^{-1} denotes the inverse of the braiding) introduced in [5]. A weak Hopf quasigroup H in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(b1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(b2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$

$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H)$.

(b3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$

$= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

(b4) There exists $\lambda_H : H \to H$ in C (called the antipode of H) such that, if we denote the morphisms $id_H * \lambda_H$ by Π^L_H (target morphism) and $\lambda_H * id_H$ by Π^R_H (source morphism),

- (b4-1) $\Pi^L_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
- (b4-2) $\Pi^R_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
- (b4-3) $\lambda_H * \Pi^L_H = \Pi^R_H * \lambda_H = \lambda_H$.
- (b4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi^R_H \otimes H)$.
- (b4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi^L_H \otimes H)$.
- (b4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi^L_H)$.
- (b4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi^R_H)$.

Note that, if in the previous definition the triple (H, η_H, μ_H) is a monoid, we obtain the notion of weak Hopf algebra in a braided monoidal category. Then, if C is symmetric, we have the monoidal version of the original definition of weak Hopf algebra introduced by Böhm, Nill and Szlachányi in [9]. On the other hand, under these conditions, if ε_H and δ_H are morphisms of unital magmas (equivalently, η_H, μ_H are morphisms of counital comagmas), $\Pi^L_H = \Pi^R_H = \eta_H \otimes \varepsilon_H$. As a consequence, conditions (b2), (b3), (b4-1)-(b4-3) trivialize, and we get the monoidal notion of Hopf quasigroup defined by Klim and Majid in [22] in a category of vector spaces over a field $\mathbb{F}$.

For any weak Hopf quasigroup the morphisms Π^L_H, Π^R_H are idempotent. Also $\overline{\Pi}^L_H$ and $\overline{\Pi}^R_H$ defined by

$$\overline{\Pi}^L_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H)$$

and

$$\overline{\Pi}^R_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

are idempotent.

Let H be a weak Hopf quasigroup and let A be a unital magma, which is also a right H-comodule with coaction $\rho_A : A \to A \otimes H$. We will say that (A, ρ_A) is a right H-comodule magma if the equality

$$\mu_{A\otimes H} \circ (\rho_A \otimes \rho_A) = \rho_A \circ \mu_A, \tag{8}$$

holds. If (A, ρ_A) is a right H-comodule magma, the following equivalent conditions hold:

(c1) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes (\delta_H \circ \eta_H))$.

(c2) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ ((\rho_A \circ \eta_A) \otimes (\delta_H \circ \eta_H))$.

(c3) $(A \otimes \overline{\Pi}^R_H) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$.

(c4) $(A \otimes \Pi^L_H) \circ \rho_A = ((\mu_A \circ c^{-1}_{A,A}) \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$.

(c5) $(A \otimes \overline{\Pi}^R_H) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

(c6) $(A\otimes \Pi_{H}^{L})\circ \rho_{A}\circ \eta_{A}=\rho_{A}\circ \eta_{A}$.

Indeed, first we will show that if (A,ρ_A) is a right H-comodule magma the equality (c6) holds.

$$
\begin{aligned}
&\rho_{A}\circ \eta_{A}\\
&=(((A\otimes \varepsilon_{H})\circ \rho_{A}\circ \mu_{A}\circ (A\otimes \eta_{A}))\otimes H)\circ \rho_{A}\circ \eta_{A} \;\text{(by unit and counit properties)}\\
&=(((A\otimes \varepsilon_{H})\circ \mu_{A\otimes H}\circ (\rho_{A}\otimes \rho_{A})\circ (A\otimes \eta_{A}))\otimes H)\circ \rho_{A}\circ \eta_{A} \;\text{(by (8))}\\
&=(\mu_{A}\otimes (((\varepsilon_{H}\circ \mu_{H})\otimes H)\circ (H\otimes c_{H,H})\circ (\delta_{H}\otimes H)))\circ (A\otimes c_{H,A}\otimes H)\circ ((\rho_{A}\circ \eta_{A})\otimes (\rho_{A}\circ \eta_{A}))\\
&\quad\text{(by the right H-comodule condition for A and naturality of c)}\\
&=(\mu_{A}\otimes (\mu_{H}\circ (H\otimes \Pi_{H}^{L})))\circ (A\otimes c_{H,A}\otimes H)\circ ((\rho_{A}\circ \eta_{A})\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by (7) of [5])}\\
&=(A\otimes \mu_{H})\circ (\mu_{A\otimes H}\otimes H)\circ (\rho_{A}\otimes \rho_{A}\otimes \lambda_{H})\circ (\eta_{A}\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by the right H-comodule condition for A and (b4-6))}\\
&=(A\otimes \mu_{H})\circ ((\rho_{A}\circ \mu_{A})\otimes \lambda_{H})\circ (\eta_{A}\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by (8))}\\
&=(A\otimes \Pi_{H}^{L})\circ \rho_{A}\circ \eta_{A} \;\text{(by the right H-comodule condition for A, unit properties and (b4-6))}.
\end{aligned}
$$

On the other hand, note that

$$
\begin{aligned}
&(A\otimes (\mu_{H}\circ c_{H,H}^{-1})\otimes H)\circ ((\rho_{A}\circ \eta_{A})\otimes (\delta_{H}\circ \eta_{H}))\\
&=(A\otimes ((\mu_{H}\otimes H)\circ (H\otimes c_{H,H})\circ ((\delta_{H}\circ \eta_{H})\otimes H)))\circ \rho_{A}\circ \eta_{A} \;\text{(by naturality of c)}\\
&=(A\otimes ((H\otimes \Pi_{H}^{L})\circ \delta_{H}))\circ \rho_{A}\circ \eta_{A} \;\text{(by (15) of [5])}\\
&=(\rho_{A}\otimes \Pi_{H}^{L})\circ \rho_{A}\circ \eta_{A} \;\text{(by the right H-comodule condition for A)}.
\end{aligned}
$$

Then, we obtain that (c1) $\Longleftrightarrow$ (c6). Similarly, by (18) of [5] and the comodule condition for A, we prove that

$$(A\otimes \mu_{H}\otimes H)\circ ((\rho_{A}\circ \eta_{A})\otimes (\delta_{H}\circ \eta_{H}))=(\rho_{A}\otimes \overline{\Pi}_{H}^{R})\circ \rho_{A}\circ \eta_{A},$$

and then (c2) $\Longleftrightarrow$ (c5). Also, by (c1) and (34) of [5], we obtain

$$(\rho_{A}\otimes \Pi_{H}^{L}))\circ \rho_{A}\circ \eta_{A}=(\rho_{A}\otimes \overline{\Pi}_{H}^{R})\circ \rho_{A}\circ \eta_{A}. \tag{9}$$

Thus,

$$(A\otimes \Pi_{H}^{L}))\circ \rho_{A}\circ \eta_{A}=(A\otimes \overline{\Pi}_{H}^{R})\circ \rho_{A}\circ \eta_{A} \tag{10}$$

and, using the equivalence (c1) $\Longleftrightarrow$ (c6), we prove that (c1) implies (c5). In the same way, by (33) of [5] and (c2) we obtain (9) and (10). Therefore, by the equivalence (c2) $\Longleftrightarrow$ (c5), we get (c6). Trivially, (c3)$\Longrightarrow$(c5) and (c4)$\Longrightarrow$(c6). Then, (c3)$\Longrightarrow$(c1) and (c4)$\Longrightarrow$(c2). Finally, (c1)$\Longrightarrow$(c3) and (c2)$\Longrightarrow$(c4) because

$$
\begin{aligned}
&(\mu_{A}\otimes H)\circ (A\otimes (\rho_{A}\circ \eta_{A}))\\
&=(((A\otimes \varepsilon_{H})\circ \rho_{A}\circ \mu_{A})\otimes H)\circ (A\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by counit properties)}\\
&=(\mu_{A}\otimes (\varepsilon_{H}\circ \mu_{H})\otimes H)\circ (A\otimes c_{H,A}\otimes H\otimes H)\circ (\rho_{A}\otimes ((\rho_{A}\otimes H)\circ \rho_{A}\circ \eta_{A})) \;\text{(by (8))}\\
&=(\mu_{A}\otimes (\varepsilon_{H}\circ \mu_{H})\otimes H)\circ (A\otimes c_{H,A}\otimes H\otimes H)\circ (\rho_{A}\otimes ((A\otimes (\mu_{H}\circ c_{H,H}^{-1})\otimes H)\circ ((\rho_{A}\circ \eta_{A})\otimes (\delta_{H}\circ \eta_{H})))) \;\text{(by (c1))}\\
&=(\mu_{A}\otimes (((\varepsilon_{H}\circ \mu_{H})\otimes \Pi_{H}^{L})\circ (H\otimes \delta_{H})))\circ (A\otimes c_{H,A}\otimes H)\circ (\rho_{A}\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by naturality of c and (15) of [5])}\\
&=(\mu_{A}\otimes (((\varepsilon_{H}\circ \mu_{H})\otimes (\overline{\Pi}_{H}^{R}\circ \Pi_{H}^{L}))\circ (H\otimes \delta_{H})))\circ (A\otimes c_{H,A}\otimes H)\circ (\rho_{A}\otimes (\rho_{A}\circ \eta_{A})) \;\text{(by (34) of [5])}
\end{aligned}
$$

$= (\mu_A \otimes (((\varepsilon_H \circ \mu_H) \otimes \overline{\Pi}_H^R) \circ (H \otimes \delta_H))) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A))$ (by the right H-comodule condition for A and (c6))

$= (\mu_A \otimes (\overline{\Pi}_H^R \circ \mu_H \circ (\Pi_H^R \otimes H))) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A))$ (by (10) of [5])

$= (\mu_A \otimes (\overline{\Pi}_H^R \circ \mu_H)) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A))$ (by (30) of [5])

$= (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \mu_A \circ (A \otimes \eta_A)$ (by (8))

$= (A \otimes \overline{\Pi}_H^R) \circ \rho_A$ (by unit properties)

and

$((\mu_A \circ c_{A,A}^{-1})) \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$

$= (((A \otimes \varepsilon_H) \circ \rho_A \circ \mu_A \circ c_{A,A}^{-1}) \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$ (by counit properties)

$= ((\mu_A \circ c_{A,A}^{-1}) \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H) \circ (A \otimes c_{H,A}^{-1} \otimes H \otimes H) \circ (\rho_A \otimes ((\rho_A \otimes H) \circ \rho_A \circ \eta_A))$ (by naturality of c and (8)

$= ((\mu_A \circ c_{A,A}^{-1}) \otimes (((\varepsilon_H \circ \mu_H) \otimes \Pi_H^L) \circ (H \otimes \delta_H) \circ c_{H,H}^{-1})) \circ (A \otimes c_{H,A}^{-1} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A))$ (by naturality of c, (15) of [5], (c2) and (b2))

$= ((\mu_A \circ c_{A,A}^{-1}) \otimes (\Pi_H^L \circ \mu_H \circ (\overline{\Pi}_H^R \otimes H) \circ c_{H,H}^{-1})) \circ (A \otimes c_{H,A}^{-1} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A))$ (by (10) of [5])

$= (\mu_A \otimes (\Pi_H^L \circ \mu_H)) \circ (A \otimes c_{H,A} \otimes H) \circ (((A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A) \otimes \rho_A)$ (by naturality of c)

$= (\mu_A \otimes (\Pi_H^L \circ \mu_H)) \circ (A \otimes c_{H,A} \otimes H) \circ ((\rho_A \circ \eta_A) \otimes \rho_A)$ (by (c5))

$= (A \otimes \Pi_H^L) \circ \rho_A \circ \mu_A \circ (\eta_A \otimes A)$ (by (8))

$= (A \otimes \Pi_H^L) \circ \rho_A$ (by unit properties).

Taking into account the level of generality of weak Hopf quasigroups, as a consequence of the above identities, if H is a Hopf quasigroup (Hopf algebra) and (A, ρ_A) is a right H-comodule magma (monoid) the identity $\rho_A \circ \eta_A = \eta_A \otimes \eta_H$ is a consequence of (8). Also, if H is a weak Hopf quasigroup (weak Hopf algebra) and (A, ρ_A) is a right H-comodule magma (monoid), the equality $\rho_A \circ \eta_A = (A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A$ follows by (8).

Let (A, ρ_A) be a right H-comodule magma. Then, the triple

$$(A, H, \psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A)) \tag{11}$$

is a weak quasi-entwining structure. Indeed, the condition (a1) of Definition 2.1 holds because:

$(u_\psi \otimes H) \circ \delta_H$

$= (A \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (c_{H,A} \otimes c_{H,H}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes (\rho_A \circ \eta_A))$ (by naturality of c)

$= (A \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A))$ (by naturality of c)

$= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes ((A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A))$ (by (7) of [5])

$= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A))$ (by (c6)).

On the other hand, by (c3) and the naturality of c we have

$$\nabla_{A \otimes H} = (A \otimes (\mu_H \circ c_{H,H}^{-1})) \circ (((A \otimes \overline{\Pi}_H^R) \circ \rho_A) \otimes H). \tag{12}$$

Then,

$\nabla_{A\otimes H} \circ (\mu_A \otimes H)$

$= (\mu_A \otimes (\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H)) \otimes (\mu_H \circ c_{H,H}^{-1})) \circ (((A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A)) \otimes (\delta_H \circ \eta_H) \otimes H)$

(by (8) and (18) of [5])

$= (\mu_A \otimes (\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)) \otimes (\mu_H \circ c_{H,H}^{-1})) \circ (((A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A)) \otimes (\delta_H \circ \eta_H) \otimes H)$ (by (b2))

$= (\mu_A \otimes (\varepsilon_H \circ \mu_H) \otimes (\mu_H \circ c_{H,H}^{-1})) \circ (A \otimes c_{H,A} \otimes ((H \otimes \overline{\Pi}_H^R) \circ \delta_H) \otimes H) \circ (\rho_A \otimes \rho_A \otimes H)$ (by (18) of [5])

$= (A \otimes \varepsilon_H \otimes (\mu_H \circ c_{H,H}^{-1} \circ (\overline{\Pi}_H^R \otimes H))) \circ ((\mu_{A\otimes H} \circ (\rho_A \otimes \rho_A)) \otimes H \otimes H) \circ (A \otimes \rho_A \otimes H)$ (by the condition of right H-comodule for A)

$= (((A \otimes \varepsilon_H) \circ \rho_A \circ \mu_A) \otimes H) \circ (A \otimes \nabla_A)$ (by (8))

$= (\mu_A \otimes H) \circ (A \otimes \nabla_A)$ (by the properties of the counit).

Therefore, (a2) of Definition 2.1 holds. Also, by the naturality of c, the comodule condition for A and (b1) we obtain (a3) of Definition 2.1. Finally, (a4) follows by

$\mu_A \circ (u_\psi \otimes A)$

$= ((\varepsilon_H \circ \mu_H) \otimes A) \circ (H \otimes c_{A,H}^{-1}) \circ (H \otimes ((\mu_A \circ c_{A,A}^{-1}) \otimes H) \circ (A \otimes (\rho_A \circ \eta_A)))$ (by naturality of c)

$= ((\varepsilon_H \circ \mu_H) \otimes A) \circ (H \otimes c_{A,H}^{-1}) \circ (H \otimes ((A \otimes \Pi_H^L) \circ \rho_A))$ (by (c4))

$= (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes A) \circ (H \otimes ((A \otimes \Pi_H^L) \circ \rho_A))$ (by naturality of c)

$= (A \otimes \varepsilon_H) \circ \psi$ (by (7) of [5]).

Definition 2.5. *Let (A, C, ψ) be a weak quasi-entwining structure in $\mathcal{C}$. We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples*

$$(M, \phi_M : M \otimes A \to M, \rho_M : M \to M \otimes C),$$

where $id_M = \phi_M \circ (M \otimes \eta_A)$, (M, ρ_M) is a right C-comodule and the equality

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A) \tag{13}$$

holds. The morphisms $f : M \to N$ in $\mathcal{M}_A^C(\psi)$ are morphisms of C-comodules, i.e., $(f \otimes C) \circ \rho_M = \rho_N \circ f$.

The objects of $\mathcal{M}_A^C(\psi)$ will be called weak entwined quasi-modules. Then, $(A, \phi_A = \mu_A, \rho_A : A \to A \otimes C)$ is an object in $\mathcal{M}_A^C(\psi)$ if and only if

$$\rho_A \circ \mu_A = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\rho_A \otimes A). \tag{14}$$

Note that for this particular case we obtain that

$$\nabla_{A\otimes C} \circ \rho_A = \rho_A. \tag{15}$$

Indeed,

$\nabla_{A\otimes C} \circ \rho_A$

$= (\mu_A \otimes C) \circ (A \otimes ((u_\psi \otimes C) \circ \delta_C)) \circ \rho_A$ *(by (6))*

$= ((\mu_A \circ (A \otimes u_\psi) \circ \rho_A) \otimes C) \circ \rho_A$ *(by the right C-comodule condition for A)*

$= (((\mu_A \otimes \varepsilon_C) \circ (A \otimes \psi) \circ (\rho_A \otimes \eta_A)) \otimes C) \circ \rho_A$ *(by the definition of u_ψ)*

$= (((A\otimes \varepsilon_C)\circ \rho_A\otimes \mu_A\circ (A\otimes \eta_A))\otimes C)\circ \rho_A$ *(by (14))*

$= \rho_A$ *(by unit and counit properties).*

Therefore, we have that

$$(A\otimes \varepsilon_C)\circ \nabla_{A\otimes C}\circ \rho_A = id_A. \tag{16}$$

Example 2.6. If H is a weak Hopf quasigroup, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is an object in $\mathcal{M}_H^H(\psi)$ for $\psi = (H\otimes\mu_H)\circ(c_{H,H}\otimes H)\circ(H\otimes\delta_H)$. This case is a particular instance associated to the weak quasi-entwining structure introduced in Example 2.4 because $(H, \rho_H = \delta_H)$ is a right H-comodule magma. If (A, ρ_A) is a right H-comodule magma, the triple $(A, \phi_A = \mu_A, \rho_A)$ is an object in $\mathcal{M}_A^H(\psi)$ because (14) holds.

3. Weak cleft extensions for weak quasi-entwining structures

Let (A, C, ψ) be a weak quasi-entwining structure in C such that there exists a coaction $\rho_A : A \to A\otimes C$ satisfying that (A, μ_A, ρ_A) is in $\mathcal{M}_A^C(\psi)$. We denote by A^{coC} the equalizer object, called the subobject of coinvariants of the morphisms ρ_A and $\zeta_A = (\mu_A\otimes C)\circ(A\otimes(\rho_A\circ\eta_A))$. Then we have an equalizer diagram

$$A^{coC} \xrightarrow{\;i_A\;} A \underset{\zeta_A}{\overset{\rho_A}{\rightrightarrows}} A\otimes C \tag{17}$$

where i_A is the equalizer morphism.

By the unit properties $\zeta_A\circ\eta_A = \rho_A\circ\eta_A$. As a consequence, there exists a unique morphism $\eta_{A^{coC}} : K \to A^{coC}$ such that

$$\eta_A = i_A\circ \eta_{A^{coC}}. \tag{18}$$

On the other hand, if the equalities

$$\mu_A\circ(A\otimes(\mu_A\circ(i_A\otimes A))) = \mu_A\circ((\mu_A\circ(A\otimes i_A))\otimes A) \tag{19}$$

and

$$\rho_A\circ\mu_A\circ(i_A\otimes A) = (\mu_A\otimes C)\circ(i_A\otimes\rho_A) \tag{20}$$

hold, we have

$\rho_A\circ\mu_A\circ(i_A\otimes i_A)$

$= (\mu_A\otimes C)\circ(A\otimes\mu_A\otimes C)\circ(i_A\otimes i_A\otimes(\rho_A\circ\eta_A))$ (by (20) and properties of i_A)

$= \zeta_A\circ\mu_A\circ(i_A\otimes i_A)$ (by (19)).

Therefore, there exists a unique morphism $\mu_{A^{coC}} : A^{coC}\otimes A^{coC} \to A^{coC}$ satisfying

$$\mu_A\circ(i_A\otimes i_A) = i_A\circ\mu_{A^{coC}}. \tag{21}$$

By (18) and (21) we obtain that $(A^{coC}, \eta_{A^{coC}}, \mu_{A^{coC}})$ is a unital magma. Also, by (19), it is possible to prove that $(A^{coC}, \eta_{A^{coC}}, \mu_{A^{coC}})$ is a monoid (the monoid of coinvariants).

Remark 3.1. Let H be a weak Hopf quasigroup, (A, ρ_A) a right H-comodule magma and ψ the morphism introduced in (11). In this case (A, H, ψ) is a weak quasi-entwining structure and (A, μ_A, ρ_A) is in $\mathcal{M}_A^H(\psi)$. Under these conditions the identity (20) holds. The proof in the braided case is the same that the one known in the symmetric setting (see Lemma 3.4 of [6]). Also the equality (20) holds for H-comodule magmas associated to Hopf quasigroups because in this case $\zeta_A = A\otimes\eta_H$.

Definition 3.2. *Let (A, C, ψ) be a weak quasi-entwining structure in $\mathcal{C}$ such that there exists a coaction $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) is in $\mathcal{M}_A^C(\psi)$. We will say that $A^{coC} \hookrightarrow A$ is a weak cleft extension if there exist a morphism of right C-comodules, $h : C \to A$ and a morphism $h^{-1} : C \to A$ such that:*

(d1) $u_\psi * h^{-1} = h^{-1}$,

(d2) $\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) = \mu_A \circ (A \otimes u_\psi)$,

(d3) $\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h \otimes h^{-1}) \circ \delta_C)) = \mu_A \circ (A \otimes (h * h^{-1}))$,

(d4) $\rho_A \circ q_A = \zeta_A \circ q_A$,

where $q_A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A$. The morphism h will be called a cleaving morphism of $A^{coC} \hookrightarrow A$ and h^{-1} the inverse of h. Note that (d2) implies that $h^{-1} * h = u_\psi$.

Example 3.3. Following [7], a quasi-entwining structure in $\mathcal{C}$ consists of a triple (A, C, ψ), where A is a unital magma, C a comonoid, and $\psi : C \otimes A \to A \otimes C$ a morphism satisfying the identities (a3) of Definition 2.1 and

(e1) $\psi \circ (C \otimes \eta_A) = \eta_A \otimes C$,

(e2) $(A \otimes \varepsilon_C) \circ \psi = \varepsilon_C \otimes A$.

As was pointed in the beginning of the previous section, any quasi-entwining structure is an example of weak quasi-entwining structure where $u_\psi = \varepsilon_C \otimes \eta_A$ and, as a consequence, $\nabla_{A\otimes C} = id_{A\otimes C}$. If H is a Hopf quasigroup and (A, ρ_A) is a right H-comodule magma, the triple (11) is an example of quasi-entwining structure. In this setting (see [7]) we can define the category of entwined quasi-modules as in Definition 2.5. By Proposition 1.4 of [7] we know that if A is a unital magma, C is a comonoid, and $\psi : C \otimes A \to A \otimes C$ is a morphism such that there exists a morphism $e : K \to C$ satisfying the identities $\delta_C \circ e = e \otimes e$ and $\varepsilon_C \circ e = id_K$, the triple (A, C, ψ) is a quasi-entwining structure and, if we define the coaction $\rho_A = \psi \circ (e \otimes A)$, we can prove that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Also, under these conditions, $\rho_A \circ \eta_A = \eta_A \otimes e$. Moreover, if for all $(M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)$, we denote by M^{coC} the equalizer of ρ_M and $\zeta_M = M \otimes e$ and by i_M the injection of M^{coC} in M, it is easy to show that A^{coC} is a unital magma where $\eta_{A^{coC}}$ and $\mu_{A^{coC}}$ are the unique morphisms such that $i_A \circ \eta_{A^{coC}} = \eta_A$, $i_A \circ \mu_{A^{coC}} = \mu_A \circ (i_A \otimes i_A)$. Then, by Definition 1.7 of [7], we will say that $A^{coC} \hookrightarrow A$ is a cleft extension if there exist a morphism of right C-comodules, $h : C \to A$ and a morphism $h^{-1} : C \to A$ such that:

(f1) $h \circ e = \eta_A$,

(f2) $\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h \otimes h^{-1}) \circ \delta_C)) = \mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) = A \otimes \varepsilon_C$,

(f3) $\psi \circ (C \otimes h^{-1}) \circ \delta_C = h^{-1} \otimes e$,

hold.

For example, if H is a Hopf quasigroup and (A, ρ_A) is a right H-comodule magma, note that $\rho_A = \psi \circ (e \otimes A)$ for ψ the morphism defined in (11) and $e = \eta_H$. Moreover, (A, μ_A, ρ_A) is a entwining quasi-module and the equality

$$\rho_A \circ h^{-1} = (h^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \tag{22}$$

holds for all morphisms $h, h^{-1} : H \to A$ such that h is a morphism of right H-comodules and satisfying (f2). Also, if h is a morphism of right H-comodules, (f3) is a consequence of (f2). Therefore, in this particular case, the definition of cleft extension is the one introduced in [4] with the name of cleft comodule algebra. This last notion is the "quasi-Hopf" version of the notion of cleft extension for Hopf algebras. In [4] the reader can find interesting examples of these kind of extensions.

If $A^{coC} \hookrightarrow A$ is a cleft extension for a quasi-entwining structure (A, C, ψ), $A^{coC} \hookrightarrow A$ is a weak cleft extension. Indeed, trivially $u_\psi * h^{-1} = h^{-1}$ because in this setting $u_\psi = \varepsilon_C \otimes \eta_A$. Also, by (f2) we have that

$h * h^{-1} = h^{-1} * h = \varepsilon_C \otimes \eta_A$ and then (d2) and (d3) of Definition 3.2 hold. Finally, by (f3), $\rho_A \circ \eta_A = \eta_A \otimes e$, and by the comodule condition for A, we obtain

$$\rho_A \circ q_A = q_A \otimes e = \zeta_A \circ \rho_A$$

and, therefore, (d4) of Definition 3.2 holds.

Example 3.4. In the associative setting there exists a theory of weak cleft extensions associated to weak entwining structures and they are examples of weak cleft extensions as the ones introduced in Definition 3.2. A weak entwining structure in $\mathcal{C}$ consists of a triple (A, C, ψ), where A is a monoid, C a comonoid, and $\psi : C \otimes A \to A \otimes C$ is a morphism satisfying the identities (a1), (a3), (a4) of Definition 2.1 and (1). If we define A^{coC} by the equalizer diagram (17), we obtain that A^{coC} is a monoid. Moreover, if there exists a coaction $\rho_A : A \to A \otimes C$ such that (A, μ_A, ρ_A) is in $\mathcal{M}_A^C(\psi)$ (in this case the objects of $\mathcal{M}_A^C(\psi)$ are also right A-modules and the morphisms are also A-linear), we say that $A^{coC} \hookrightarrow A$ is a weak cleft extension, or a weak C-cleft extension (see Definition 2.3 of [1]), if there exist a morphism of right C-comodules, $h : C \to A$ and a morphism $h^{-1} : C \to A$ such that

(g1) $h^{-1} * h = u_\psi$,

(g2) $\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ (u_\psi * h^{-1})$.

Note that under these assumptions $h * u_\psi = h$. Moreover, if we put $g = h$ and $g^{-1} = u_\psi * h^{-1}$ we have that $g^{-1} * g = u_\psi * h^{-1} * h = u_\psi * u_\psi = u_\psi$ and $u_\psi * g^{-1} = u_\psi * u_\psi * h^{-1} = u_\psi * h^{-1} = g^{-1}$. Then we can assume that $u_\psi * h^{-1} = h^{-1}$, i.e., h^{-1} satisfies (d1) of Definition 3.2, and we can change (g2) by

(g3) $\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ h^{-1}$.

In this associative setting, if for $A^{coC} \hookrightarrow A$ there exist a morphism of right C-comodules, $h : C \to A$ and a morphism $h^{-1} : C \to A$ satisfiying (d1)-(d4) of Definition 3.2, we have that (g1), (g2) and (g3) hold. Indeed, by (d2) of Definition 3.2 we get (g1). Also, the equality

$$\psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \tag{23}$$

holds because

$$\begin{aligned}
&(\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \\
&= (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ (\rho_A \otimes A))) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \text{ (by (14))} \\
&= (\mu_A \otimes C) \circ ((h^{-1} * h) \otimes \psi) \circ (\delta_C \otimes A) \text{ (by the coassociativity of } \delta_C \text{ and the condition of comodule morphism for } h) \\
&= (\mu_A \otimes C) \circ (u_\psi \otimes \psi) \circ (\delta_C \otimes A) \text{ (by (g1))} \\
&= (\psi \otimes \varepsilon_C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \text{ (by (a4) of Definition 2.1)} \\
&= \psi \text{ (by (a3) of Definition 2.1 and counit properties).}
\end{aligned}$$

Therefore, we obtain (g2) and (g3):

$$\begin{aligned}
&\psi \circ (C \otimes h^{-1}) \circ \delta_C \\
&= (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \circ (C \otimes h^{-1}) \circ \delta_C \text{ (by (23))} \\
&= (\mu_A \otimes C) \circ (h^{-1} \otimes (\rho_A \circ (h * h^{-1}))) \circ \delta_C \text{ (by the coassociativity of } \delta_C) \\
&= (\mu_A \otimes C) \circ (h^{-1} \otimes (\rho_A \circ q_A \circ h)) \circ \delta_C \text{ (by the condition of comodule morphism for } h) \\
&= (\mu_A \otimes C) \circ (h^{-1} \otimes (\zeta_A \circ q_A \circ h)) \circ \delta_C \text{ (by (d4) of Definition 3.2)}
\end{aligned}$$

$= \zeta_A \circ (u_\psi * h^{-1})$ (by (d2) of Definition 3.2, the coassociativity of δ_C and the associativity of μ_A)

$= \zeta_A \circ h^{-1}$ (by (d1) of Definition 3.2).

Conversely, let $A^{coC} \hookrightarrow A$ be an extension and assume that there exist a morphism of right C-comodules, $h : C \to A$ and a morphism $h^{-1} : C \to A$ satisfiying (g1), (g2) ((g3)). Then (d1) of Definition 3.2 follows by the properties of h^{-1}, (d3) of Definition 3.2 holds trivially by the associativity of μ_A, (d2) of Definition 3.2 follows by (g1) and the associativity of μ_A, and by (g2) we obtain (d4) of Definition 3.2 because

$\rho_A \circ q_A$

$= (\mu_A \otimes C) \circ (A \otimes (\psi \circ (C \otimes h^{-1}) \circ \delta_C)) \circ \rho_A$ (by the comodule condition for A)

$= (\mu_A \otimes C) \circ (A \otimes (\zeta_A \circ h^{-1})) \circ \rho_A$ (by (g3))

$= \zeta_A \circ q_A$ (by the associativity of μ_A).

Therefore, in an associative context Definition 3.2 is the definition of weak C-cleft extension introduced in [1].

Example 3.5. Let H be a weak Hopf quasigroup and let (A, ρ_A) be a right H-comodule magma. Following Definition 2.7 of [8], we will say that $h : H \to A$ is an anchor morphism if it is a multiplicative total integral (i.e., a right H-comodule morphism and a morphism of unital magmas) and the following equalities hold:

$$\mu_A \circ ((\mu_A \circ (A \otimes h)) \otimes (h \circ \lambda_H)) \circ (A \otimes \delta_H) = \mu_A \circ (A \otimes (h \circ \Pi_H^L)), \tag{24}$$

$$\mu_A \circ ((\mu_A \circ (A \otimes (h \circ \lambda_H))) \otimes h) \circ (A \otimes \delta_H) = \mu_A \circ (A \otimes (h \circ \Pi_H^R)). \tag{25}$$

If there exists an anchor morphism $h : H \to A$, the extension $A^{coC} \hookrightarrow A$ associated to the weak quasi-entwining structure defined in (11) is a weak cleft extension with cleaving morphism h and $h^{-1} = h \circ \lambda_H$. Indeed, first note that, using that h is a comodule morphism and $h \circ \eta_A = \eta_H$, we have that

$$u_\psi = h \circ \Pi_H^R. \tag{26}$$

Then, as a consequence of (26), the multiplicative condition for h and (b4-3) of the definition of weak Hopf quasigroup, we have

$$u_\psi * h^{-1} = (h \circ \Pi_H^R) * (h \circ \lambda_H) = h \circ (\Pi_H^R * \lambda_H) = h^{-1}.$$

Therefore, (d1) of Definition 3.2 holds. The equality (d2) holds by (26) and (25). Similarly, we obtain (d3) by (24) and by $h \circ \Pi_H^L = h * h^{-1}$ (this last equality follows by the multiplicative condition for h). Finally, by Proposition 2.6 of [8] we know that

$$\rho_A \circ q_A = (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ q_A, \tag{27}$$

and then, using (c3) of the definition of right H-comodule magma, we obtain (d4). As a particular instance of this case, we have that $H_L \hookrightarrow H$ is a weak cleft extension associated to the weak quasi-entwining structure $(H, H, \psi = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H))$ where $h = id_H$ (anchor morphism), $h^{-1} = \lambda_H$, $q_H = \Pi_H^L$ and H_L is the image of Π_H^L.

For example, if H is a cocommutative weak Hopf quasigroup and C is symmetric, $(H^{op}, \rho_{H^{op}} = (H \otimes \lambda_H) \circ \delta_H)$ is an example of right H-comodule magma. In this case, it is easy to show that λ_H is an anchor morphism for $(H^{op}, \rho_{H^{op}})$. Of course, the same result holds for cocommutative Hopf quasigroups (in this case $\Pi_H^R = \Pi_H^L = \varepsilon_H \otimes \eta_H$).

On the other hand, let H and A be Hopf quasigrous in C. Let $g : A \to H$, $h : H \to A$ be morphisms of Hopf quasigroups such that $g \circ h = id_H$. Consider the right H-comodule structure on A defined by $\rho_A = (A \otimes g) \circ \delta_A$. Then, h is an anchor morphism and, as a consequence, the examples of strong projections that we can find in [2] give examples of anchor morphisms.

Finally, let H be a Hopf quasigroup and let D be a unital magma in C. If there exists a morphism $\varphi_D : H \otimes D \to D$ such that

$$\varphi_D \circ (\eta_H \otimes D) = id_D, \tag{28}$$

$$\varphi_D \circ (H \otimes \eta_D) = \varepsilon_H \otimes \eta_D, \tag{29}$$

hold, the smash product $D\sharp H = (D \otimes H, \eta_{D\sharp H}, \mu_{D\sharp H})$ defined by

$$\eta_{D\sharp H} = \eta_D \otimes \eta_H, \quad \mu_{D\sharp H} = (\mu_D \otimes \mu_H) \circ (D \otimes \psi_H^D \otimes H),$$

where $\psi_H^D = (\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ (\delta_H \otimes D)$, is a right H-comodule magma with comodule structure given by $\varrho_{D\sharp H} = D \otimes \delta_H$. For this H-comodule magma, $h = \eta_D \otimes H$ is an anchor morphism.

Example 3.6. Let H be a weak Hopf quasigroup and let (A, ρ_A) be a right H-comodule magma. Following Definition 4.1 of [6], we will say that the extension $A^{coC} \hookrightarrow A$, associated to the weak quasi-entwining structure defined in (11), is a weak H-cleft extension if there exists a right H-comodule morphism $h : H \to A$ (called the cleaving morphism) and a morphism $h^{-1} : H \to A$ such that

(h1) $h^{-1} * h = (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A))$.

(h2) $(A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ h^{-1})) \circ \delta_H = (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ h^{-1}$.

(h3) $\mu_A \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_H) = \mu_A \circ (A \otimes (h^{-1} * h))$.

(h4) $\mu_A \circ (\mu_A \otimes A) \circ (A \otimes h \otimes h^{-1}) \circ (A \otimes \delta_H) = \mu_A \circ (A \otimes (h * h^{-1}))$.

Then, $A^{coC} \hookrightarrow A$ is a weak cleft extension, in the sense of Definition 3.2, for the weak quasi-entwining structure introduced in (11), with cleaving morphism h. Indeed, first note that, by (c3) of Example 2.4, (h2) is equivalent to say that

$$\psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}. \tag{30}$$

Then, if (h2) holds, so hold (d1) and (d4) of Definition 3.2. Indeed,

$$u_\psi * h^{-1}$$
$$= (A \otimes \varepsilon_H) \circ \psi \circ (H \otimes h^{-1}) \circ \delta_H \text{ (by (a4) of Definition 2.1)}$$
$$= (A \otimes \varepsilon_H) \circ \zeta_A \circ h^{-1} \text{ (by (30))}$$
$$= h^{-1} \text{ (by the right } H\text{-comodule condition for } A \text{ and the unit properties)},$$

and then (d1) of Definition 3.2 holds. On the other hand,

$$\rho_A \circ q_A$$
$$= (\mu_A \otimes H) \circ (A \otimes \psi) \circ (\rho_A \otimes h^{-1}) \circ \rho_A \text{ (by (14))}$$
$$= (\mu_A \otimes H) \circ (A \otimes (\psi \circ (H \circ h^{-1}) \circ \delta_H)) \circ \rho_A \text{ (by the right } H\text{-comodule condition for } A)$$
$$= (\mu_A \otimes H) \circ (A \otimes (\zeta_A \circ h^{-1})) \circ \rho_A \text{ (by (30))}$$

$= (\mu_A \otimes \overline{\Pi}_H^R) \circ (A \otimes (\zeta_A \circ h^{-1})) \circ \rho_A$ (by (c5) of Example 2.4).

Therefore, $\rho_A \circ q_A = (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ q_A$ and, applying (c3) of Example 2.4, we get (d4) of Definition 3.2. Moreover, (h1) is just the same that $h^{-1} * h = u_\psi$, and then (h3) is equal to (d2) of Definition 3.2 and (h4) is the same that (d3).

Lemma 3.7. *Let (A, C, ψ) be a weak quasi-entwining structure satisfying (19) and let $A^{coC} \hookrightarrow A$ be a weak cleft extension with cleaving morphism h. Then, the following equalities hold:*

$$\mu_A \circ (A \otimes (\mu_A \circ (q_A \otimes A))) = \mu_A \circ ((\mu_A \circ (A \otimes q_A)) \otimes A), \tag{31}$$

$$\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((q_A \otimes h) \circ \rho_A)) = \mu_A, \tag{32}$$

$$\mu_A \circ (q_A \otimes h) \circ \rho_A = id_A, \tag{33}$$

$$\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \rho_A) = \mu_A \circ (A \otimes q_A). \tag{34}$$

Proof. If $A^{coC} \hookrightarrow A$ is a weak cleft extension with cleaving morphism h, by (d4) of Definition 3.2 we have that $\rho_A \circ q_A = \zeta_A \circ q_A$. Then, there exists a unique morphism $p_A : A \to A^{coC}$ such that $q_A = i_A \circ p_A$. Therefore, if (19) holds we prove (31) composing in (19) with $A \otimes p_A \otimes A$. On the other hand,

$\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((q_A \otimes h) \circ \rho_A))$

$= \mu_A \circ (A \otimes (\mu_A \circ (q_A \otimes h) \circ \rho_A))$ (by (31))

$= \mu_A \circ (A \otimes (\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ \rho_A))$ (by the comodule condition for A)

$= \mu_A \circ (A \otimes (\mu_A \circ (A \otimes u_\psi) \circ \rho_A))$ (by (d2) of Definition 3.2)

$= \mu_A \circ (A \otimes ((\mu_A \otimes \varepsilon_C) \circ (A \otimes \psi) \circ (\rho_A \otimes \eta_A)))$ (by the definition of u_ψ)

$= \mu_A \circ (A \otimes ((A \otimes \varepsilon_C) \circ \rho_A \circ \mu_A \circ (A \otimes \eta_A)))$ (by (14))

$= \mu_A$ (by unit and counit properties).

Therefore, we obtain (32). Composing in this identity with $\eta_A \otimes A$ we prove (33). Finally, (34) follows from

$\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \rho_A)$

$= \mu_A \circ ((\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((q_A \otimes h) \circ \rho_A))) \otimes h^{-1}) \circ (A \otimes \rho_A)$ (by (32))

$= \mu_A \circ (\mu_A \otimes A) \circ ((\mu_A \circ (A \otimes q_A)) \otimes ((h \otimes h^{-1}) \circ \delta_C)) \circ (A \otimes \rho_A)$ (by the comodule condition for A)

$= \mu_A \circ ((\mu_A \circ (A \otimes q_A)) \otimes (h * h^{-1})) \circ (A \otimes \rho_A)$ (by (d3) of Definition 3.2)

$= \mu_A \circ (A \otimes (\mu_A \circ (q_A \otimes (h * h^{-1})) \circ \rho_A))$ (by (31))

$= \mu_A \circ (A \otimes (\mu_A \circ (\mu_A \otimes A) \circ (q_A \otimes ((h \otimes h^{-1}) \circ \delta_C)) \circ \rho_A))$ (by (d3) of Definition 3.2)

$= \mu_A \circ (A \otimes (\mu_A \circ ((\mu_A \circ (q_A \otimes h) \circ \rho_A) \otimes h^{-1}) \circ \rho_A))$ (by the comodule condition for A)

$= \mu_A \circ (A \otimes q_A)$ (by (33)).

□

4. Galois extensions for weak quasi-entwining structures

A classical result in Hopf algebra theory proved by Doi and Takeuchi in [18] gives a characterization of Galois extensions with normal basis in terms of cleft extensions. A generalization of this theorem to weak entwining structures, and therefore to weak Hopf algebras, can be found in [1]. The aim of this section is to prove a similar theorem for weak quasi-entwining structures containing, as a particular instance, the characterization obtained in [6] for Galois extensions with normal basis for weak Hopf quasigroups.

In this section we will assume that (A, C, ψ) is a weak quasi-entwining structure in $\mathcal{C}$ such that there exists a coaction $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) is in $\mathcal{M}_A^C(\psi)$. Also, unless otherwise stated we assume that the identities (19) and (20) holds.

Then, if A^{coC} is the equalizer object and $i_A : A^{coC} \to A$ the equalizer morphism of the morphisms ρ_A and $\zeta_A = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \eta_A))$, we have that $(A^{coC}, \eta_{A^{coC}}, \mu_{A^{coC}})$ is a monoid, where $\eta_{A^{coC}} : K \to A^{coC}$, $\mu_{A^{coC}} : A^{coC} \otimes A^{coC} \to A^{coC}$ are the factorizations of the morphisms η_A and $\mu_A \circ (i_A \otimes i_A)$ through i_A, respectively. That is, $\eta_{A^{coC}}$ is the unique morphism satisfying (18), and $\mu_{A^{coC}}$ the unique morphism such that (21) holds.

Under these assumptions, let $A\square C$ be the image of the idempotent morphism $\nabla_{A\otimes C}$, and let $i_{A\otimes C} : A\square C \to A \otimes C$ and $p_{A\otimes C} : A \otimes C \to A\square C$ be the associated injection and projection respectively, i.e., $i_{A\otimes C}$ and $p_{A\otimes C}$ are the unique morphisms such that $\nabla_{A\otimes C} = i_{A\otimes C} \circ p_{A\otimes C}$ and $p_{A\otimes C} \circ i_{A\otimes C} = id_{A\square C}$. If we define the morphism $t_A : A \otimes A \to A\square C$ by

$$t_A = p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A), \tag{35}$$

using (19) and (20), we have the following:

$$t_A \circ ((\mu_A \circ (A \otimes i_A)) \otimes A) = p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (i_A \otimes \rho_A))) = t_A \circ (A \otimes (\mu_A \circ (i_A \otimes A))).$$

Then, if the object $A \otimes_{A^{coC}} A$ is defined by the following coequalizer diagram

$$\begin{tikzcd}[column sep=large] A \otimes A^{coC} \otimes A \arrow[r, shift left, "(\mu_A \circ (A \otimes i_A)) \otimes A"] \arrow[r, shift right, "A \otimes (\mu_A \circ (i_A \otimes A))"'] & A \otimes A \arrow[r, "n_A^A"] & A \otimes_{A^{coC}} A, \end{tikzcd} \tag{36}$$

there exists a unique morphism $\gamma_A : A \otimes_{A^{coC}} A \to A\square C$, called the canonical morphism, such that

$$\gamma_A \circ n_A^A = t_A. \tag{37}$$

Definition 4.1. *Let (A, C, ψ) be a weak quasi-entwining structure in $\mathcal{C}$. We will say that $A^{coC} \hookrightarrow A$ is a weak Galois extension if γ_A is an isomorphism.*

By the properties of the coequalizer (36) and (20), we have

$$(n_A^A \otimes C) \circ ((\mu_A \circ (A \otimes i_A)) \otimes \rho_A) = (n_A^A \otimes C) \circ (A \otimes (\rho_A \circ (\mu_A \circ (i_A \otimes A)))).$$

As a consequence, there exists a unique coaction $\rho_{A\otimes_{A^{coC}}A} : A \otimes_{A^{coC}} A \to (A \otimes_{A^{coC}} A) \otimes C$ satisfying

$$\rho_{A\otimes_{A^{coC}}A} \circ n_A^A = (n_A^A \otimes C) \circ (A \otimes \rho_A). \tag{38}$$

Using the comodule structure of A, it is easy to show that $(A \otimes_{A^{coC}} A, \rho_{A\otimes_{A^{coC}}A})$ is a right C-comodule.

On the other hand, $(A\square C, \rho_{A\square C} = (p_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ i_{A\otimes C})$ is a right C-comodule because, by the counit properties,

$$(A\square C \otimes \varepsilon_C) \circ \rho_{A\square C} = p_{A\otimes C} \circ i_{A\otimes C} = id_{A\square C},$$

and, by (3) and the coassociativity of δ_C,

$$(\rho_{A\square C} \otimes C) \circ \rho_{A\square C} = (((p_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \nabla_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C) \circ i_{A\otimes C} = (A\square C \otimes \delta_C) \circ \rho_{A\square C}.$$

Moreover, we have that

$$
\begin{aligned}
&(\gamma_A \otimes C) \circ \rho_{A\otimes_{A^{coC}}A} \circ n_A^A \\
&= ((p_{A\otimes C} \circ (\mu_A \otimes A) \circ (A \otimes \rho_A)) \otimes C) \circ (A \otimes \rho_A) \text{ (by (38), (37))} \\
&= (p_{A\otimes C} \otimes C) \circ (\mu_A \otimes \delta_C) \circ (A \otimes \rho_A) \text{ (by coassociativity of } \delta_C) \\
&= (p_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) \text{ (by (3))} \\
&= \rho_{A\square C} \circ \gamma_A \circ n_A^A \text{ (by definition).}
\end{aligned}
$$

Thus, the canonical morphism is a morphism of right C-comodules, i.e.,

$$(\gamma_A \otimes C) \circ \rho_{A\otimes_{A^{coC}}A} = \rho_{A\square C} \circ \gamma_A. \tag{39}$$

As a consequence, by (3),

$$\rho_{A\otimes_{A^{coC}}A} \circ \gamma_A^{-1} \circ p_{A\otimes C} = ((\gamma_A^{-1} \circ p_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C), \tag{40}$$

i.e., $\gamma_A^{-1} \circ p_{A\otimes C} : A \otimes C \to A \otimes_{A^{coC}} A$ is a morphism of right C-comodules.

If the endofunctor $- \otimes A$ preserves coequalizers (for example if $\mathcal{C}$ is closed) and the equality

$$\mu_A \circ (\mu_A \otimes i_A) = \mu_A \circ (A \otimes (\mu_A \circ (A \otimes i_A))) \tag{41}$$

holds, we have

$$n_A^A \circ ((\mu_A \circ (A \otimes (\mu_A \circ (A \otimes i_A)))) \otimes A) = n_A^A \circ (\mu_A \otimes (\mu_A \circ (i_A \otimes A))).$$

Thus, there exists a unique morphism $\varphi_{A\otimes_{A^{coC}}A} : A \otimes (A \otimes_{A^{coC}} A) \to A \otimes_{A^{coC}} A$ satisfying the identitity

$$\varphi_{A\otimes_{A^{coC}}A} \circ (A \otimes n_A^A) = n_A^A \circ (\mu_A \otimes A). \tag{42}$$

Definition 4.2. *Let (A, C, ψ) be a weak quasi-entwining structure in $\mathcal{C}$. Assume that $A^{coC} \hookrightarrow A$ is a weak Galois extension and that (41) holds. We will say that γ_A^{-1} is almost linear if it satisfies the following equality:*

$$\gamma_A^{-1} \circ p_{A\otimes C} = \varphi_{A\otimes_{A^{coC}}A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C}) \circ (\eta_A \otimes C)). \tag{43}$$

Definition 4.3. *Let (A, C, ψ) be a weak quasi-entwining structure in $\mathcal{C}$ satisfying the assumptions of this section. We will say that $A^{coC} \hookrightarrow A$ is a weak Galois extension with normal basis if:*

- *(i1) $A^{coC} \hookrightarrow A$ is a weak Galois extension.*
- *(i2) There exists an idempotent morphism $\Omega_{A^{coC}\otimes C} : A^{coC} \otimes C \to A^{coC} \otimes C$ of left A^{coC}-modules, where the action and the coaction are $\varphi_{A^{coC}\otimes C} = \mu_{A^{coC}} \otimes C$, and $\rho_{A^{coC}\otimes C} = A^{coC} \otimes \delta_C$.*
- *(i3) If we denote by $A^{coC} \times C$ the image of $\Omega_{A^{coC}\otimes C}$, there is a left A^{coC}-module and right C-comodule isomorphism $b_A : A \to A^{coC} \times C$ where the actions and the coactions are*

$$\varphi_A = \mu_A \circ (i_A \otimes A), \quad \varphi_{A^{coC}\times C} = r_{A^{coC}\otimes C} \circ (\mu_{A^{coC}} \otimes C) \circ (A^{coC} \otimes s_{A^{coC}\otimes C}),$$

$$\rho_A, \quad \rho_{A^{coC}\otimes C} = (r_{A^{coC}\otimes C} \otimes C) \circ (A^{coC} \otimes \delta_C) \circ s_{A^{coC}\otimes C},$$

and $r_{A^{coC}\otimes C} : A^{coC} \otimes C \to A^{coC} \times C$, $s_{A^{coC}\otimes C} : A^{coC} \times C \to A^{coC} \otimes C$ are the morphisms such that $s_{A^{coC}\otimes C} \circ r_{A^{coC}\otimes C} = \Omega_{A^{coC}\otimes C}$ and $r_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} = id_{A^{coC}\times C}$.

Remark 4.4. Let H be a weak Hopf quasigroup and let (A, ρ_A) be a right H-comodule magma. Let $A^{coH} \hookrightarrow A$ be the extension associated to the weak quasi-entwining structure defined in (11). Then, in this particular case, the notions of weak Galois extension and weak Galois extension with normal basis introduced in Definitions 4.1 and 4.3 are the notions of weak H-Galois extension and weak H-Galois extension with normal basis defined in [6] for weak Hopf quasigrous in a symmetric setting (see Definitions 3.10 and 3.11, respectively).

Lemma 4.5. *Let (A, C, ψ) be a weak quasi-entwining structure in C. If $A^{coC} \hookrightarrow A$ is a weak Galois extension with normal basis, there exists a unique morphism $m_A : A \otimes_{A^{coC}} A \to A$ such that*

$$m_A \circ n_A^A = \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ s_{A^{coC} \otimes C} \circ b_A)). \tag{44}$$

Also, m_A satisfies the following identity:

$$m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A = (i_A \otimes \varepsilon_H) \circ s_{A^{coC}\otimes C} \circ b_A. \tag{45}$$

Moreover, if (41) holds and $A \otimes -$ preserves coequalizers, the equality

$$m_A \circ \varphi_{A\otimes_{A^{coC}} A} = \mu_A \circ (A \otimes m_A) \tag{46}$$

holds.

Proof. We have that

$$\mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ s_{A^{coC}\otimes C} \circ b_A)) \circ (A \otimes (\mu_A \circ (i_A \otimes A))$$

$= \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ \Omega_{A^{coC}\otimes C} \circ (\mu_{A^{coC}} \otimes C) \circ (A^{coC} \otimes (s_{A^{coC}\otimes C} \circ b_A)))$ (by the condition of left A^{coC}-module morphism for b_A)

$= \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_H) \circ (\mu_{A^{coC}} \otimes C) \circ (A^{coC} \otimes (s_{A^{coC}\otimes C} \circ b_A)))$ (by the condition of left A^{coC}-module morphism for $\Omega_{A^{coC}\otimes C}$ and $s_{A^{coC}\otimes C} \circ \Omega_{A^{coC}\otimes C} = s_{A^{coC}\otimes C}$)

$= \mu_A \circ (A \otimes (\mu_A \circ (i_A \otimes i_A \otimes \varepsilon_C) \circ (A^{coC} \otimes (s_{A^{coC}\otimes C} \circ b_A))))$ (by (21))

$= \mu_A \circ ((\mu_A \circ (A \otimes i_A)) \otimes ((i_A \otimes \varepsilon_H) \circ s_{A^{coC}\otimes C} \circ b_A))$ (by (19)).

Then, by the properties of the coequalizer (36) we can assure that there exists a unique morphism $m_A : A \otimes_{A^{coC}} A \to A$ satisfying (44). The equality (45) holds because

$$m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A$$

$= m_A \circ \gamma_A^{-1} \circ t_A \circ (\eta_A \otimes A)$ (by unit properties)

$= m_A \circ n_A^A \circ (\eta_A \otimes A)$ (by (37))

$= (i_A \otimes \varepsilon_H) \circ s_{A^{coC}\otimes C} \circ b_A$ (by (44) and the unit properties).

Finally, we get (46):

$$m_A \circ \varphi_{A\otimes_{A^{coC}} A} \circ (A \otimes n_A^A)$$

$= \mu_A \circ (\mu_A \otimes ((i_A \otimes \varepsilon_C) \circ s_{A^{coC}\otimes C} \circ b_A))$ (by (42) and (44))

$= \mu_A \circ (A \otimes (\mu_A \circ (A \otimes ((i_A \otimes \varepsilon_C) \circ s_{A^{coC}\otimes C} \circ b_A))))$ (by (41))

$= \mu_A \circ (A \otimes m_A) \circ (A \otimes n_A^A)$ (by (44)).

□

Lemma 4.6. *Let (A, C, ψ) be a weak quasi-entwining structure in C satisfying the assumptions of this section. If $A^{coC} \hookrightarrow A$ is a weak Galois extension with normal basis and $\overline{\mu}_A$ is the factorization of μ_A through the coequalizer morphism n_A^A, the equality*

$$\overline{\mu}_A = \mu_A \circ (m_A \otimes h) \circ \rho_{A\otimes_{A^{coC}} A} \tag{47}$$

holds for $h = b_A^{-1} \circ r_{A^{coC}\otimes C} \circ (\eta_{A^{coC}} \otimes C)$. Moreover, $\overline{\mu}_A$ satisfies that

$$\overline{\mu}_A = (A \otimes \varepsilon_C) \circ i_{A\otimes C} \circ \gamma_A. \tag{48}$$

Proof. First note that by (19) we can assure that μ_A factorizes through the coequalizer morphism n_A^A. Then, there exists a unique morphism $\overline{\mu}_A$ such that

$$\overline{\mu}_A \circ n_A^A = \mu_A. \tag{49}$$

On the other hand, the equality

$$(m_A \otimes C) \circ \rho_{A\otimes_{A^{coC}}A} \circ n_A^A = ((\mu_A \circ (A \otimes i_A)) \otimes C) \circ (A \otimes (s_{A^{coC}\otimes C} \circ b_A)) \tag{50}$$

holds because

$$\begin{aligned}
&(m_A \otimes C) \circ \rho_{A\otimes_{A^{coC}}A} \circ n_A^A \\
&= ((m_A \circ n_A^A) \otimes C) \circ \rho_A \text{ (by (38))} \\
&= (\mu_A \otimes C) \circ (A \otimes ((i_A \otimes \varepsilon_C) \circ s_{A^{coC}\otimes C} \circ b_A) \otimes C) \circ (A \otimes \rho_A) \text{ (by (44))} \\
&= (\mu_A \otimes C) \circ (A \otimes ((i_A \otimes \varepsilon_C) \circ \Omega_{A^{coC}\otimes C}) \otimes C) \circ (A \otimes ((A^{coC} \otimes \delta_C) \circ s_{A^{coC}\otimes C} \circ b_A)) \text{ (by the condition of left } A^{coC}\text{-module} \\
&\quad \text{morphism for } b_A). \\
&= ((\mu_A \circ (i_A \otimes A)) \otimes C) \circ (A \otimes (s_{A^{coC}\otimes C} \circ b_A)) \text{ (by the condition of left } A^{coC}\text{-module morphism for } \Omega_{A^{coC}\otimes C} \text{ and the counit properties).}
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\mu_A \circ (m_A \otimes h) \circ \rho_{A\otimes_{A^{coC}}A} \circ n_A^A \\
&= \mu_A \circ ((\mu_A \circ (A \otimes i_A)) \otimes (b_A^{-1} \circ r_{A^{coC}\otimes C} \circ (\eta_{A^{coC}} \otimes C))) \circ (A \otimes (s_{A^{coC}\otimes C} \circ b_A)) \text{ (by (50))} \\
&= \mu_A \circ (A \otimes (\mu_A \circ (i_A \otimes (b_A^{-1} \circ r_{A^{coC}\otimes C} \circ (\eta_{A^{coC}} \otimes C))) \circ s_{A^{coC}\otimes C} \circ b_A)) \text{ (by (19))} \\
&= \mu_A \circ (A \otimes (b_A^{-1} \circ r_{A^{coC}\otimes C} \circ (\mu_{A^{coC}} \otimes C) \circ (A^{coC} \otimes (\Omega_{A^{coC}\otimes C} \circ (\eta_{A^{coC}} \otimes C))) \circ s_{A^{coC}\otimes C} \circ b_A)) \text{ (by the condition of left} \\
&\quad A^{coC}\text{-module morphism for } b_A^{-1}) \\
&= \mu_A \text{ (by the condition of left } A^{coC}\text{-module morphism for } \Omega_{A^{coC}\otimes C}, r_{A^{coC}\otimes C} \circ \Omega_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} = id_{A^{coC}\times C} \text{ and unit properties)} \\
&= \overline{\mu}_A \circ n_A^A \text{ (by (49)),}
\end{aligned}$$

and then (47) holds.

Finally,

$$\begin{aligned}
&(A \otimes \varepsilon_C) \circ i_{A\otimes C} \circ \gamma_A \circ n_A^A \\
&= (A \otimes \varepsilon_C) \circ \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) \text{ (by (37))} \\
&= \mu_A \circ (A \otimes ((A \otimes \varepsilon_C) \circ \nabla_{A\otimes C} \circ \rho_A)) \text{ (by (a2) of Definition 2.1)} \\
&= \mu_A \text{ (by (16)),}
\end{aligned}$$

and, applying (49), we obtain (48). □

Theorem 4.7. *Let (A, C, ψ) be a weak quasi-entwining structure in C. If (41) holds and $A \otimes -$ preserves coequalizers, the following assertions are equivalent:*

(i) $A^{coC} \hookrightarrow A$ *is a weak Galois extension with normal basis and* γ_A^{-1} *is almost lineal.*

(ii) $A^{coC} \hookrightarrow A$ *is a weak cleft extension.*

Proof. First we will prove that (ii)$\Longrightarrow$(i). If $A^{coC} \hookrightarrow A$ is a weak cleft extension with cleaving morphism h, define $\gamma'_A : A\square C \to A\otimes_{A^{coC}} A$ by

$$\gamma'_A = n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ i_{A\otimes C}.$$

Then,

$$\gamma_A \circ \gamma'_A$$
$$= p_{A\otimes C} \circ (\mu_A \otimes C) \circ ((\mu_A \circ (A \otimes h^{-1})) \otimes (\rho_A \circ h))(A \otimes \delta_C) \circ i_{A\otimes C} \text{ (by (37))}$$
$$= p_{A\otimes C} \circ ((\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C))) \otimes C) \circ (A \otimes \delta_C) \circ i_{A\otimes C} \text{ (by the condition of morphism of right } C\text{-comodules for } h \text{ and the coassociativity of } \delta_C)$$
$$= p_{A\otimes C} \circ ((\mu_A \circ (A \otimes u_\psi)) \otimes C) \circ (A \otimes \delta_C) \circ i_{A\otimes C} \text{ (by (d2) of Definition 3.2)}$$
$$= p_{A\otimes C} \circ \nabla_{A\otimes C} \circ i_{A\otimes C} \text{ (by (6))}$$
$$= id_{A\square C} \text{ (by properties of } \nabla_{A\otimes C}),$$

and

$$\gamma'_A \circ \gamma_A \circ n_A^A$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) \text{ (by (37))}$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ (\mu_A \otimes C) \circ (A \otimes (\nabla_{A\otimes C} \circ \rho_A)) \text{ (by (a2) of Definition 2.1)}$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) \text{ (by (15))}$$
$$= n_A^A \circ ((\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \rho_A)) \otimes h) \circ (A \otimes \rho_A) \text{ (by the right } C\text{-comodule condition for } A)$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((q_A \otimes h) \circ \rho_A)) \text{ (by (34))}$$
$$= n_A^A \circ (A \otimes (\mu_A \circ (q_A \otimes h) \circ \rho_A)) \text{ (by (33))}$$
$$= n_A^A \text{ (by (33))}.$$

Therefore, $\gamma'_A \circ \gamma_A = id_{A\otimes_{A^{coC}} A}$ and, as a consequence, the canonical morphism is an isomorphism with inverse $\gamma_A^{-1} = \gamma'_A$. The next step is to prove that γ_A^{-1} is almost lineal. This property holds because, in one hand

$$\varphi_{A\otimes_{A^{coC}} A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C}) \circ (\eta_A \otimes C))$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes (((u_\psi * h^{-1}) \otimes h) \circ \delta_C)) \text{ (by (7) and the coassociativity of } \delta_C)$$
$$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C)) \text{ (by (d1) of Definition 3.2)},$$

and, on the other hand,

$$\gamma_A^{-1} \circ p_{A\otimes C}$$
$$= n_A^A \circ (\mu_A \otimes A) \circ ((\mu_A \circ (A \otimes u_\psi)) \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C \otimes C) \circ (A \otimes \delta_C) \text{ (by (6) and (3))}$$
$$= n_A^A \circ (\mu_A \otimes A) \circ ((\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C))) \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C \otimes C) \circ (A \otimes \delta_C)$$

(by (d2) of Definition 3.2)

$= n_A^A \circ ((\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \rho_A)) \otimes A) \circ ((\mu_A \circ (A \otimes h^{-1})) \otimes h \otimes h) \circ (A \otimes \delta_C \otimes C) \circ (A \otimes \delta_C)$

(by the condition of morphism of right C-comodules for h and coassociativity of δ_C)

$= n_A^A \circ ((\mu_A \circ (A \otimes q_A)) \otimes A) \circ ((\mu_A \circ (A \otimes h^{-1})) \otimes h \otimes h) \circ (A \otimes \delta_C \otimes C) \circ (A \otimes \delta_C)$ (by (34))

$= n_A^A \circ ((\mu_A \circ (A \otimes h^{-1})) \otimes (\mu_A \circ ((q_A \circ h) \otimes h))) \circ (A \otimes \delta_C \otimes C) \circ (A \otimes \delta_C)$ (by (31))

$= n_A^A \circ ((\mu_A \circ (A \otimes h^{-1})) \otimes (\mu_A \circ (q_A \otimes h) \circ \rho_A \circ h)) \circ (A \otimes \delta_C)$ (by the condition of morphism of right C-comodules for h and coassociativity of δ_C)

$= n_A^A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C))$ (by (33)).

Finally we will prove that $A^{coC} \hookrightarrow A$ satisfies the normal basis condition. Let $\omega_A = \mu_A \circ (i_A \otimes h) : A^{coC} \otimes C \to A$ and $\omega'_A = (p_A \otimes C) \circ \rho_A : A \to A^{coC} \otimes C$ where, by (d4) of Definition 3.2, $p_A : A \to A^{coC}$ is the unique morphism such that

$$i_A \circ p_A = q_A. \tag{51}$$

Then, by (33), we have that $\omega_A \circ \omega'_A = id_A$, and $\Omega_{A^{coC} \otimes C} = \omega'_A \circ \omega_A$ is an idempotent morphism. For $\Omega_{A^{coC} \otimes C}$ we have that the following identities

$$\Omega_{A^{coC} \otimes C} = ((p_A \circ \mu_A \circ (i_A \otimes h)) \otimes C) \circ (A^{coC} \otimes \delta_C), \tag{52}$$

$$\Omega_{A^{coC} \otimes C} = ((\mu_{A^{coC}} \circ (A^{coC} \otimes (p_A \circ h))) \otimes C) \circ (A^{coC} \otimes \delta_C), \tag{53}$$

hold. Indeed, the first one follows by (20) and the condition of morphism of right C-comodules for h. To prove (53), firstly we will show that

$$i_A \circ \mu_{A^{coC}} \circ (A^{coC} \otimes p_A) = q_A \circ \mu_A \circ (i_A \otimes A) \tag{54}$$

holds. The proof is the following:

$i_A \circ \mu_{A^{coC}} \circ (A^{coC} \otimes p_A)$

$= \mu_A \circ (i_A \otimes q_A)$ (by (21) and (51))

$= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (i_A \otimes \rho_A)$ (by (34))

$= q_A \circ \mu_A \circ (i_A \otimes A)$ (by (20)).

Thus, using that i_A is a monomorphism, we have

$$\mu_{A^{coC}} \circ (A^{coC} \otimes p_A) = p_A \circ \mu_A \circ (i_A \otimes A), \tag{55}$$

and then (53) holds.

Therefore, by (52), we obtain that $\Omega_{A^{coC} \otimes C}$ is a morphism of right C-comodules for the coaction $\rho_{A^{coC} \otimes C} = A^{coC} \otimes \delta_C$ and, by (53), $\Omega_{A^{coC} \otimes C}$ is a morphism of left A^{coC}-modules for the action $\varphi_{A^{coC} \otimes C} = \mu_{A^{coC}} \otimes C$.

Let $r_{A^{coC} \otimes C} : A^{coC} \otimes C \to A^{coC} \times C$, $s_{A^{coC} \otimes C} : A^{coC} \times C \to A^{coC} \otimes C$ be the morphisms such that $s_{A^{coC} \otimes C} \circ r_{A^{coC} \otimes C} = \Omega_{A^{coC} \otimes C}$ and $r_{A^{coC} \otimes C} \circ s_{A^{coC} \otimes C} = id_{A^{coC} \times C}$, where $A^{coC} \times C$ is the image of $\Omega_{A^{coC} \otimes C}$. Define

$$b_A = r_{A^{coC} \otimes C} \circ \omega'_A : A \to A^{coC} \times C$$

and

$$b'_A = \omega_A \circ s_{A^{coC} \otimes C} : A^{coC} \times C \to A.$$

Then,

$$b'_A \circ b_A = \omega_A \circ s_{A^{coC}\otimes C} \circ r_{A^{coC}\otimes C} \circ \omega'_A = \omega_A \circ \omega'_A \circ \omega_A \circ \omega'_A = id_A$$

and

$$b_A \circ b'_A = r_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} \circ r_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} = id_{A^{coC}\times C}.$$

Therefore, b_A is an isomorphism with inverse b'_A and a morphism of right C-comodules because, by the equality $\Omega_{A^{coC}\otimes C} \circ \omega'_A = \omega'_A$ and the condition of right C-comodule morphism for ω'_A (consequence of the right C-comodule structure of A), we have

$$\rho_{A^{coC}\times C} \circ b_A = (r_{A^{coC}\otimes C} \otimes C) \circ (A^{coC} \otimes \delta_C) \circ \Omega_{A^{coC}\otimes C} \circ \omega'_A = (b_A \otimes C) \circ \rho_A.$$

Moreover, b_A is a morphism of left A^{coC}-modules. Indeed:

$b_A^{-1} \circ \varphi_{A^{coC}\times C}$

$= \mu_A \circ ((i_A \circ \mu_{A^{coC}}) \otimes h) \circ (A^{coC} \otimes s_{A^{coC}\otimes C})$ (by the condition of morphism of left A^{coC}-modules for $\Omega_{A^{coC}\otimes C}$ and $\Omega_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} = s_{A^{coC}\otimes C}$)

$= \mu_A \circ ((\mu_A \circ (i_A \otimes i_A)) \otimes h) \circ (A^{coC} \otimes s_{A^{coC}\otimes C})$ (by (21))

$= \mu_A \circ (i_A \otimes b_A^{-1})$ (by (19))

$= \varphi_A \circ (A^{coC} \otimes b_A^{-1})$ (by definition of φ_A).

Now we get (i)$\Longrightarrow$(ii): Assume that $A^{coC} \hookrightarrow A$ is a weak Galois extension with normal basis and that γ_A^{-1} is almost lineal. Let $\Omega_{A^{coC}\otimes C}$ be the associated idempotent morphism of right C-comodules and left A^{coC}-modules. Denote by $r_{A^{coC}\otimes C} : A^{coC} \otimes C \to A^{coC} \times C$ and $s_{A^{coC}\otimes C} : A^{coC} \times C \to A^{coC} \otimes C$ the morphisms such that $s_{A^{coC}\otimes C} \circ r_{A^{coC}\otimes C} = \Omega_{A^{coC}\otimes C}$ and $r_{A^{coC}\otimes C} \circ s_{A^{coC}\otimes C} = id_{A^{coC}\times C}$, where $A^{coC} \times C$ is the image of $\Omega_{A^{coC}\otimes C}$. Let $b_A : A \to A^{coC} \times A$ be the isomorphism of right C-comodules and left A^{coC}-modules. Then, define

$$\omega_A = b_A^{-1} \circ r_{A^{coC}\otimes C} : A^{coC} \otimes C \to A$$

and

$$\omega'_A = s_{A^{coC}\otimes C} \circ b_A : A \to A^{coC} \otimes C.$$

Trivially, $\omega'_A \circ \omega_A = \Omega_{A^{coC}\otimes C}$ and $\omega_A \circ \omega'_A = id_A$. Moreover, ω_A and ω'_A are morphisms of right C-comodules and left A^{coC}-modules because $\Omega_{A^{coC}\otimes C}$ and b_A are morphisms of right C-comodules and left A^{coC}-modules.

Consider

$$h = \omega_A \circ (\eta_{A^{coC}} \otimes C), \quad h^{-1} = m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C),$$

where m_A is the morphism introduced in Lemma 4.5. Using that ω_A is a morphism of right C-comodules, we obtain that h is a morphism of right C-comodules. The proof for (d1) of Definition 3.2 is the following:

$u_\psi * h^{-1}$

$= m_A \circ \varphi_{A\otimes_{A^{coC}}A} \circ (u_\psi \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \delta_C$ (by (46))

$= m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (u_\psi \otimes C) \circ \delta_C$ (by (43))

$= h^{-1}$ (by (5)).

By

$\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h^{-1} \otimes h) \circ \delta_C))$

$= \mu_A \circ ((m_A \circ \varphi_{A\otimes_{A^{coC}}A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C)))) \otimes h) \circ \delta_C$ (by (46))

$= \mu_A \circ ((m_A \circ \gamma_A^{-1} \circ p_{A\otimes C}) \otimes h) \circ (A \otimes \delta_C)$ (by (43))

$= \mu_A \circ (m_A \otimes h) \circ \rho_{A\otimes_{A^{coC}}A} \circ \gamma_A^{-1} \circ p_{A\otimes C}$ (by (40))

$= \overline{\mu}_A \circ \gamma_A^{-1} \circ p_{A\otimes C}$ (by (47))

$= (A \otimes \varepsilon_C) \circ i_{A\otimes C} \circ \gamma_A \circ \gamma_A^{-1} \circ p_{A\otimes C}$ (by (48))

$= (A \otimes \varepsilon_C) \circ \nabla_{A\otimes C}$ (by the factorization of $\nabla_{A\otimes C}$)

$= \mu_A \circ (A \otimes u_\psi)$ (by (6)),

we obtain (d2) of Definition 3.2. To prove (d3), we will obtain previously that

$$q_A = (i_A \otimes \varepsilon_C) \circ \omega'_A \tag{56}$$

and (34) hold. Indeed, first note that (56) follows by

q_A

$= m_A \circ \varphi_{A\otimes_{A^{coC}}A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \rho_A$ (by (46))

$= m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A$ (by (43))

$= (i_A \otimes \varepsilon_C) \circ \omega'_A$ (by (45)).

Thus, we get (34) because

$\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \rho_A)$

$= m_A \circ \varphi_{A\otimes_{A^{coC}}A} \circ (\mu_A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \circ (A \otimes \rho_A)$ (by (46))

$= m_A \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A)$ (by (43))

$= m_A \circ n_A^A$ (by (37))

$= \mu_A \circ (A \otimes ((i_A \otimes \varepsilon_C) \circ \omega'_A))$ (by (44))

$= mu_A \circ (A \otimes q_A)$ (by (56)).

Then

$\mu_A \circ (\mu_A \otimes A) \circ (A \otimes ((h \otimes h^{-1}) \circ \delta_C))$

$= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes (\rho_A \circ h))$ (by the condition of morphism of right C-comodules for h)

$= \mu_A \circ (A \otimes (q_A \circ h))$ (by (34))

$= \mu_A \circ (A \otimes (h * h^{-1}))$ (by the condition of morphism of right C-comodules for h),

i.e., (d3) of Definition 3.2 holds.

Finally, (d4) of Definition 3.2 follows by (56) and (17).

□

As a consequence we have the following corollary:

Corollary 4.8. *Let H be a weak Hopf quasigroup in a strict symmetric monoidal category $\mathcal{C}$. Let (A, ρ_A) be a right H-comodule magma. Let $A^{coH} \hookrightarrow A$ be the extension associated to the weak quasi-entwining structure $(A, H, \psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A))$ defined in (11). If $A \otimes -$ preserves coequalizers and (19) and (41) hold, the following assertions are equivalent:*

(i) $A^{coH} \hookrightarrow A$ is a weak Galois extension with normal basis and γ_A^{-1} is almost lineal.

(ii) $A^{coH} \hookrightarrow A$ is a weak cleft extension.

(iii) $A^{coH} \hookrightarrow A$ is a weak H-cleft extension.

Proof. The proof follows by Theorem 4.7 and Theorem 5.1 of [6] because, as was pointed in Remark 3.1, under the conditions of this corollary, the equality (20) holds.

References

[1] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez and A. B. Rodríguez Raposo, Weak C-cleft extensions and weak Galois extensions, *J. Algebra* 299 (2006), 276-293.

[2] Alonso Álvarez J.N., Fernández Vilaboa J.M. and González Rodríguez R., Soneira Calvo, C. Projections and Yetter-Drinfel'd modules over Hopf (co)quasigroups, *J. Algebra* 443 (2015), 153-199.

[3] Alonso Álvarez J.N., Fernández Vilaboa J.M. and González Rodíguez R., Soneira Calvo, C. Lax entwining structures, groupoid algebras and cleft extensions, *B. Braz. Math. Soc.* 45 (2014), 133-178.

[4] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, C. Soneira Calvo, Cleft comodules over Hopf quasi-groups, *Commun. Contemp. Math.* 17 (2015), 1550007.

[5] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Weak Hopf quasigroups, *Asian J. of Math.* 20 (2016), 665-694.

[6] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Cleft and Galois extensions associated to a weak Hopf quasigroup, *J. Pure Appl. Algebra* 220 (2016), 1002-1034.

[7] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Cleft extensions for quasi-entwining structures, *Math. Slovaca* 68 (2018), 339-352.

[8] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Fundamental theorems of Doi-Hopf modules in a non-associative setting, *Bull. Malays. Math. Sci. Soc.* (2018). https://doi.org/10.1007/s40840-018-0624-y

[9] G. Böhm, G., F. Nill and K. Szlachányi, Weak Hopf algebras, I. Integral theory and C^*-structure, *J. Algebra* 221 (1999), 385-438.

[10] T. Brzeziński, On modules associated to coalgebra Galois extensions, *J. Algebra* 215 (1999), 290-317.

[11] T. Brzeziński and P.M. Hajac, Coalgebra extensions and algebra coextensions of Galois type, *Commun. Algebra* 2 (1999), 1347-1367.

[12] T. Brzeziński and S. Majid, Coalgebra bundles, *Comm. Math. Phys.* 191 (1998), 467-492.

[13] T. Brzeziński and R. Wisbauer, *Corings and comodules*, London Mathematical Society Lecture Note Series, 309. Cambridge University Press, Cambridge, 2003.

[14] S. Caenepeel, E. De Groot, Modules over weak entwining structures, *Contemp. Math.* 267 (2000), 31-54.

[15] S. Caenepeel and K. Janssen, Partial (co)cctions of Hopf algebras and partial Hopf-Galois theory, *Comm. Algebra* 36 (2008), 2923-2946.

[16] S. U. Chase and M. E. Sweedler, *Hopf algebras and Galois theory*, Springer-Verlag, Berlin-Heilderberg-New York, 1969.

[17] Y. Doi, On the structure of relative Hopf modules, *Comm. Algebra* 11 (1983), 243-255.

[18] Y. Doi and M. Takeuchi, Cleft comodule algebras for a bialgebra, *Comm. Algebra* 14 (1986), 801-817.

[19] Y. Doi and M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action, and Azumaya algebras, *J. Algebra* 121 (1989), 488-516.

[20] Y. Doi Unifying Hopf modules, *J. Algebra* 153 (1992), 373-385.

[21] M. Koppinen Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* 104 (1994), 61-80.

[22] J. Klim and S. Majid, Hopf quasigroups and the algebraic 7-sphere, *J. Algebra* 323 (2010), 3067-3110.

[23] J. Klim and S. Majid, Bicrossproduct Hopf quasigroups, *Comment. Math. Univ. Carolin.* 51 (2010), 287-304.

[24] H. F. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* 30 (1981), 675-691.

[25] J.M. Pérez-Izquierdo and I.P. Shestakov, An envelope for Malcev algebras, *J. Algebra* 272 (2004), 379-393.

[26] J.M. Pérez-Izquierdo, Algebras, hyperalgebras, non-associative bialgebras and loops, *Adv. Math.* 208 (2007), 834-876.

[27] M.E., Sweedler, *Hopf algebras*, New York: Benjamin, New York 1969.

[28] R. Wisbauer, Weak Corings, *J. Algebra* 245 (2001), 123-160.

[29] D.N. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Phil. Soc.* 108 (1990), 261-290.