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On One Class Eigenvalue Problem with Eigenvalue Parameter in the Boundary Condition at One End-Point

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Abstract. In the present paper we investigate the spectrum of operator corresponding to eigenvalue problem with parameter dependent boundary condition. Trace formula for that operator is also established.

1. Introduction

In this paper we consider the boundary value problem with boundary condition depending on the first degree polynomial of spectral parameter. Namely, we consider Sturm-Liouville equation with unbounded coefficient. Note that in boundary condition some first degree polynomials appear before unknown function, as well as its derivative. Earlier studied problems for Sturm-Liouville operator equation where spectral parameter appears only before function or only before its derivative. For example, in [1, 2, 3] the asymptotics of eigenvalues and self-adjoint extensions of minimal symmetric operators were studied. In [4-10], we have studied asymptotics of spectrum and established trace formulas for operators generated by regular and singular differential operator expressions and spectral parameter dependent boundary conditions.

Here we consider in space $L_2(H,(0,1))$ (where H is separable Hilbert space) the following problem

$$Ly := -y''(t) + Ay(t) + q(t)\,y(t) = \lambda y(t) \tag{1}$$

$$y(0) = 0 \tag{2}$$

$$y(1)(1+\lambda) = y'(1)(h+1+\lambda) \tag{3}$$

where h is real number, $A = A^*$, $A > E$ (E is an identity operator in H) and has compact invers $A^{-1} \in \sigma_\infty$. Clear that under stated conditions A is discrete operator. Denote the eigenvectors of A by $\varphi_1, \varphi_2, \ldots$. Let $\gamma_1 \le \gamma_2 \le \ldots$ be eigenvalues of operator A. Suppose that operator-valued function $q(t)$ is weakly measurable and for each t is defined in H.

Fulton [11] has considered the scalar Sturm-Liouville problem

$$-y''(t) + q(t)\,y(t) = \lambda y(t)$$

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$$\cos \alpha u(a) + \sin \alpha u'(a) = 0, \quad \alpha \in [0, \pi)$$

$$-\beta_1 u(b) - \lambda \beta_1' u(b) = -\beta_2 u'(b) - \lambda \beta_2' u'(b)$$

and given an operator-theoretic formulation of that problem, showing that one can associate a self-adjoint operator with it whenever the relation

$$\beta = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$$

holds. He obtained the expansion theorem and asymptotic formulas for eigenvalues and eigenfunctions.

In our case, obviously

$$\beta = \begin{vmatrix} 1 & 1 \\ 1 & h+1 \end{vmatrix} = h$$

and we take $h > 0$.

Introduce the space $L_2 = L_2(H, (0,1)) \oplus H$. Define in it scalar product of elements $Y = (y(t), y_1)$, $Z = (z(t), z_1) \in L_2$, $(y(t), z(t) \in L_2(H, (0,1)),\ y_1, z_1 \in H)$ by

$$(Y, Z) = \int_0^1 (y(t), z(t))\, dt + \frac{1}{h} (y(1), z(1))$$

where $(\cdot, \cdot)$ is scalar product in H.

About $q(t)$ we assume the next:

1) it is a bounded operator valued function $\|q(t)\| \le const$, $t \in [0,1]$, $q^*(t) = q(t)$;

2) $\sum\limits_{k=1}^{\infty} \left|(q(t)\varphi_k, \varphi_k)\right| < const, \quad \forall t \in [0,1]$;

3) $\int\limits_0^1 (q(t) f, f)\, dt = 0$, for $f = \varphi_k, \quad k = \overline{1, \infty}$.

Formulate (1)-(3) in case $q(t) \equiv 0$ in operator form. Thus define in L_2 operator L_0 as

$$D(L_0) = \{Y = (y(t), y(1) - y'(1)) \in L_2 / y(0) = 0, y_1 = y(1) - y'(1),\ ly \in L_2(H, (0,1))\}.$$

$$L_0 Y = (-y''(t) + Ay(t),\ -(y(1) - (h+1)y'(1))).$$

Denote by L operator defined in L_2 by $L = L_0 + Q$, where $Q\,Y = (q(t)y(t), 0)$.

Our aim is to obtain asymptotic formulae for eigenvalue distribution and establish trace formula for operator L.

2. Asymptotic formulae for eigenvalues

It can be easily verified that L_0 is self adjoint positive-definite operator.

Since A is self adjoint operator in virtue of spectral expansion of A we get the next eigenvalue problem for scalar function $y_k(t) = (y(t), \varphi_k)$ $(y(t) \in L_2(H, (0,1)))$

$$-y_k''(t) + \gamma_k y_k(t) = \lambda y_k(t) \tag{4}$$

$$y_k(0) = 0 \tag{5}$$

$$-\left(y_k(1) - (h+1) y_k'(1)\right) = \lambda \left(y_k(1) - y_k'(1)\right) \tag{6}$$

Solution of problem (4), (5) is $y_k(t) = \sin \sqrt{\lambda - \gamma_k} t$. Obviously, the eigenvalues of L_0 and the problem (1)-(3) are the same. They are obtained from equation

$$-\left(\sin \sqrt{\lambda - \gamma_k} - (h+1)\sqrt{\lambda - \gamma_k} \cos \sqrt{\lambda - \gamma_k}\right) = \lambda \left(\sin \sqrt{\lambda - \gamma_k} - \sqrt{\lambda - \gamma_k} \cos \sqrt{\lambda - \gamma_k}\right) \tag{7}$$

By taking

$$\sqrt{\lambda - \gamma_k} = z$$

we can put (7) in form

$$\sin z - (h+1)\, z\cos z = \left(z^2 + \gamma_k\right)(z\cos z - \sin z) \tag{8}$$

Eigenvectors of operator L_0 are

$$\Psi_n = C_n \left\{\sin\left(\alpha_{k_n,m_n} t\right)\varphi_{k_n}, \left(-\sin\alpha_{k_n,m_n} + \alpha_{k_n,m_n}\cos\alpha_{k_n,m_n}\right)\varphi_{k_n}\right) \tag{9}$$

where α_{k_n,m_n} are roots of (8). Note that $\alpha_{k_n,0}$ are imaginary numbers. C_n are coefficients. Normalizing that vectors we get the next orthonormal eigen-vectors:

$$\Psi_n = \sqrt{\frac{4\alpha_{k_n,m_n}}{K_{k_n,m_n}}} \left\{\sin(\alpha_{k_n,m_n})\varphi_{k_n}, \left(-\sin\alpha_{k_n,m_n} + \alpha_{k_n,m_n}\cos\alpha_{k_n,m_n}\right)\right\}\varphi_{k_n}$$

where $K_{k_n,m_n} = 2\alpha_{k_n,m_n} - h\sin 2\alpha_{k_n,m_n} + 2\alpha_{k_n,m_n} - 2\alpha_{k_n,m_n}\cos 2\alpha_{k_n,m_n} - 4\alpha^2_{k_n,m_n}\sin 2\alpha_{k_n,m_n} + 2\alpha^3_{k_n,m_n} + 2\alpha^3_{k_n,m_n}\cos 2\alpha_{k_n,m_n}$.

Now find the roots of equation (8). Firstly we will investigate is there any imaginary root of that equation. For this reason taking in (8) $z = iy$, $y > 0$, we have

$$\frac{-sh\, y}{i} - (h+1)\, iy\, chy = \left(\gamma_k - y^2\right)\left(iy\, ch\, y + \frac{sh\, y}{i}\right)$$

or expanding into series

$$-\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)!} + (h+1)\, y\sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!} = \left(\gamma_k - y^2\right)\left(\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)!} - y\sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}\right).$$

Simplifying we have

$$\sum_{n=0}^{\infty}\frac{(h+1)(2n+1) - 1 - \gamma_k + \gamma_k(2n+1)}{(2n+1)!} y^{2n+1} = \sum_{n=0}^{\infty}\frac{y^{2n+3}}{(2n)!} - \sum_{n=0}^{\infty}\frac{y^{2n+3}}{(2n+1)!}$$

$$yh + \sum_{n=0}^{\infty}\frac{y^{2n+3}\left(h(2n+3) + (\gamma_k+1)(2n+2)\right)}{(2n+3)!} - \sum_{n=0}^{\infty}\frac{(2n-1)\, y^{2n+3}}{(2n+1)!} = 0.$$

From the last

$$yh + \sum_{n=0}^{\infty}\frac{y^{2n+3}\left(2nh + 3h + 2n\gamma_k + 2\gamma_k + 2n + 2 - (2n+3)(2n+2)(2n-1)\right)}{(2n+3)!} = 0. \tag{10}$$

Consider the function

$$f(z) = 2zh + 3h + 2z\gamma_k + 2\gamma_k + 2z + 2 - 8z^3 - 16z^2 - 2z + 6.$$

Since $f(0) = 3h + 2\gamma_k + 8 > 0$, $f(z) \to -\infty$ as $z \to +\infty$, and $f(z)$ is continuous on $(0, +\infty)$, then it has roots on positive semiaxis. There is only one sign change of terms of

$$f(z) = -8z^3 - 16z^2 + 2zh + 2z\gamma_k + 8 + 3h + 2\gamma_k.$$

Thus, function $f(z)$ has only one positive root by Descartes principe. Denote it by $z = M$. Note that $f(z) > 0$ when $z < M$ and $f(z) < 0$ when $z > M$. Therefore, coefficients of series (10) are positive when $n < [M]$ and negative when $n > [M]$. Thus there is only one change of sign of coefficients (10). But obviously equation

$$-sh\, y + (h+1)\, y\, ch\, y = \left(\gamma_k - y^2\right)(sh\, y - y\, ch\, y)$$

or

$$cth\, y = \frac{\gamma_k - y^2 + 1}{(h+1)\, y + y\,(\gamma_k - y^2)} \tag{11}$$

has no any root for great y values.Left hand side of (11) goes to 1, while write hand side goes to 0 when $y \to \infty$. Consequently we get that eigenvalues corresponding to that roots are

$$\lambda_k = \gamma_k + \alpha_{k,0}^2, \tag{12}$$

where $\alpha_{k,0} = iy$ and form some bounded set.

Now we shall look for real roots of equation (8). Rewrite it in the form

$$ctgz = \frac{z^2 + \gamma_k + 1}{z^3 + \gamma_k z + (h+1)\, z}. \tag{13}$$

From (13) denoting real roots by α_m we get

$$\alpha_m = \frac{\pi}{2} + \pi m + O\left(\frac{1}{m}\right) \tag{14}$$

for large m values. Eigenvalues of L_0 corresponding to that roots are

$$\lambda_{k,m} = \gamma_k + \left[\pi m + \frac{\pi}{2} + O\left(\frac{1}{m}\right)\right]^2. \tag{15}$$

Thus we have proved the next theorem.

Theorem 1. *Eigenvalues of operator L_0 form the next two sequence*

$$\lambda_k = \gamma_k + \alpha_{k,0}^2, \ \ (|\alpha_{k,0}| < M, \text{ where } M \textit{ is some constant})\ \lambda_{k,m} = \gamma_k + \left[\pi m + \frac{\pi}{2} + O\left(\frac{1}{m}\right)\right]^2,$$

when $m \to \infty$.

Lemma 1. *If eigenvalues of A $\gamma_k \sim a \cdot k^\alpha$ $(\alpha > 0, a > 0)$, then eigenvalues of operator L_0 have the next asymptotics at large n*

$$\lambda_n \sim C n^{\frac{2\alpha}{2+\alpha}}$$

where C is some constant.

Proof. Firstly find asymptotics of distribution function $N(\lambda)$ of operator L_0. We have

$$N(\lambda) \equiv \sum_{\lambda_k < \lambda} 1 + \sum_{\gamma_{k,m} < \lambda} 1 \equiv N_1(\lambda) + N_2(\lambda)$$

By using formulas (12) and (15) we get

$$N_1(\lambda) \sim C_1 \lambda^{\frac{1}{\alpha}}$$

and

$$N_2(\lambda) \sim C_2 \lambda^{\frac{2+\alpha}{2\alpha}}.$$

But if $\alpha > 0$ then $\frac{1}{\alpha} < \frac{2+\alpha}{2\alpha}$, thus $N(\lambda) \sim N_2(\lambda)$. From which by using similar arguments as in [11] we get the statement of Lemma 1.

Now we will calculate the first regularized trace of operator L.

Since Q is bounded operator in $L_2(H,(0,1) \oplus H$, operator L is discrete (spectrum is discrete). So, its eigenvalues can be arranged in ascending order. Denote them by μ_n, $(n = \overline{1,\infty})$, $\mu_{1\leq}\mu_{2\leq\dots}$. From boundednes of Q it follows that

$$\mu_n \sim Cn^{\frac{2\alpha}{2+\alpha}}, n \to \infty. \tag{16}$$

In virtue of asymptotics (16) in similar way as in [13] we get

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} \left[\mu_n - \lambda_n - (Q\Psi_n, \Psi_n)\right] = 0, \tag{17}$$

where n_m is some subsequnce of natural numbers.

Lemma 2. *Series*

$$\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{4\alpha_{k,m} h \int\limits_0^1 \sin^2 \alpha_{k,m} t q_k(t)\, dt}{K_{k,m}}$$

where $K_{k,m} = 2\alpha_{k,m}h - h\sin 2\alpha_{k,m} + 2\alpha_{k,m} - 2\alpha_{k,m}\cos 2\alpha_{k,m} - 4\alpha_{k^2,m}\sin 2\alpha_{k,m} + 2\alpha_{k,m}^3 + 2\alpha_{k,m}^3 \cos 2\alpha_{k,m}$, *and* $q_k(t) = (q(t)\varphi_k, \varphi_k)$, *is absolutely convergent.*

Proof. In virtue of formulas (14) and (12) the summund for big m values is equivalent to

$$O\left(\frac{1}{m^2}\right) \int_0^1 |q_k(t)|dt$$

Now validity of statement follows from assumptions 1) and (2).

In virtue of Lemma 2 and expression (9)

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} (Q\Psi_n, \Psi_n) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{2\alpha_{k,m} h \int_0^1 \cos(2\alpha_{k,m} t)\, q_k(t)\, dt}{K_{k,m}}$$

Consider the finite sum

$$S_N = \sum_{m=0}^{N} 2\alpha_{k,m} h \cos(2\alpha_{k,m} t) \tag{18}$$

For evaluating $\lim\limits_{N\to\infty} S_N$ choose a function of complex variable whose poles are roots of equation (8), and residues of that function at poles are terms of sum (18). Taking the function

$$g(z) = \frac{2zh\cos 2zt}{\left(\frac{\sin z - (h+1)2\cos z}{z\cos z - \sin z} - z^2 - \gamma_k\right)(z\cos z - \sin z)^2}$$

we can see that it has such properties. Hence its poles are $\alpha_{k,m}$.

Evaluate residues of $g(z)$ at poles. Thus

$$\operatorname*{res}_{z=\alpha_{k,m}} g(z) = \left[\frac{(\cos z - (h+1)\cos z + (h+1) z \sin z)(z\cos z - \sin z)}{(z\cos z - \sin z)^2} - \right.$$

$$-\left.\frac{(\cos z - z\sin z - \cos z)(\sin z - (h+1)z\cos z)}{(z\cos z - \sin z)^2} - 2z\right]\Bigg|_{z=\alpha_{k,n}} (\alpha_{k,n}\cos\alpha_{k,n} - \sin\alpha_{k,m})^2$$

$$= \frac{2\alpha_{k,m} h\cos 2\alpha_{k,m} t}{A_{k,m}}$$

$$= \frac{2\alpha_{k,n} h\cos(2\alpha_{k,n} t)}{-h\alpha_{k,n} + h\cos\alpha_{k,n}\sin\alpha_{k,n} - 2\alpha_{k^3,n}\cos^2\alpha_{k,n} + 4\alpha_{k^2,n}\sin\alpha_{k,n}\cos\alpha_{k,n} - 2\alpha_{k,n}\sin^2\alpha_{k,n}}$$

$$= \frac{4\alpha_{k,n} h\cos(2\alpha_{k,n} t)}{-2h\alpha_{k,n} + h\sin 2\alpha_{k,n} - 2\alpha_{k^3,n} - 2\alpha_{k^3,n}\cos^2\alpha_{k,n} + 4\alpha_{k^2,n}\sin 2\alpha_{k,n} + 2\alpha_{k,n}\cos 2\alpha_{k,n}} \tag{19}$$

where $A_{k,m} = (-h\cos\alpha_{k,m} + h\alpha_{k,m}\sin\alpha_{k,m} + \alpha_{k,m}\sin\alpha_{k,m})(\alpha_{k,m}\cos\alpha_{k,m} - \sin\alpha_{k,m}) - \alpha_{k,m}^2 (h+1)\cos\alpha_{k,m}\sin\alpha_{k,m} - 2\alpha_{k,m}^3\cos^2\alpha_{k,m} + 4\alpha_{k,m}^2\sin\alpha_{k,m}\cos\alpha_{k,m} - 2\alpha_{k,m}\sin^2\alpha_{k,m}$. As it is seen from (19) residues of $g(z)$ at poles are terms of sum in (18).

But roots of equation $z\cos z - \sin z = 0$ are also poles of $g(z)$.

Denote them by β_n. They have the asymptotics $\beta_n \sim \frac{\pi}{2} + \pi n$ for large n.Thus

$$\underset{z=\beta_n}{res}\, g(z) = \frac{2\beta_n h\cos(2\beta_n t)}{-\left(\sin\beta_n - \beta_n(h+1)\cos\beta_n - (\beta_n^2 + \gamma_k\right)(\beta_n\cos\beta_n - \sin\beta_n))\beta_n\sin\beta_n}$$

in virtue of relation

$$\beta_n\cos\beta_n - \sin\beta_n = 0,$$

it follows that

$$\underset{z=\beta_n}{res}\, g(x) = \frac{2\beta_n h\cos(2\beta_n t)}{(\sin\beta_n - \beta_n(h+1)\cos\beta_n)(-\beta_n\sin\beta_n)} = \frac{2\beta_n h\cos(2\beta_n t)}{-\beta_n h\cos\beta_n(-\beta_n\sin\beta_n)} = \frac{4\cos(2\beta_n t)}{\beta_n\sin 2\beta_n}.$$

We must select for each k a contour which includes all $\alpha_{k,m}\left(m = \overline{0,N}\right)$ values, so integration of $g(z)$ along that contour will yield the sum in (19) by Cauchy theorem.

For that purpose take a rectangular contour C_N with vertices at points $A_N \pm iB$, $\pm iB$, where $A_N = \pi(N+1)$, and $B > \left|\alpha_{k,0}\right|$.

Let C_N by pass the origin and $-\alpha_{k,0}$ along semicircle from the left, and imaginary numbers $\alpha_{k,0}\left(n = \overline{1,M_0}\right)$ from the right. Since $g(z)$ is odd function of argument z the integral along left-hand side of contour vanishes. Consider integral along semicircle by-pasing zero from the left:

$$i = \int\limits_{\substack{z=re^{i\varphi} \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}} g(z)\,dz$$

$$= \int\limits_{\substack{z=re^{i\varphi} \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}} \frac{2zh\cos 2zt\,dz}{(\sin z - (h+1)z\cos z)(z\cos z - \sin z)(z^2+\gamma_k)(z\cos z - \sin z)^2} = \int\limits_{\substack{z=re^{i\varphi} \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}} \frac{2zh\left[1 - \frac{4z^2t^2}{2!} + \ldots\right]}{F(z)},$$

where $F(z) = \left[z - \frac{z^3}{3!} + \ldots (h+1)z\left(1 - \frac{z^2}{2!} + \ldots\right)\right]\left[z\left(1 - \frac{z^2}{2!} + \ldots\right) - \left(z - \frac{z^3}{3!} + \ldots\right)\right] -$
$-\left(z^2+\gamma_k\right)\left[z^2\left(\frac{1}{2} + \frac{1}{2}\left(1 - 2z^2 + \ldots\right)\right) - z\left(2z - \frac{8z^3}{3!} + \ldots\right) + \frac{1}{2}\left(1 - 1 + \frac{4z^2}{2!} + \ldots\right)\right]$.

When $r \to 0$

$$\int\limits_{\substack{z=re^{i\varphi} \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}} g(z)\,dz \sim \int\limits_{\substack{z=re^{i\varphi} \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}} \frac{2zh\left[1 - 2z^2t^2 + \ldots\right]}{-2\gamma_k z^2}dz$$

so

$$\lim_{r\to 0}\int\limits_{\substack{z=re^{i\varphi}\\ -\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2}}} g(z)\,dz=\lim_{r\to 0}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{h\,ire^{i\varphi}d\varphi}{-\gamma_k re^{i\varphi}}=-\frac{h}{\gamma_k}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=-\frac{\pi hi}{\gamma_k}, \tag{20}$$

Since for big z values

$$g(z)\sim\frac{\cos 2zt}{z^3\cos^2 2},$$

taking $z=u+vi$ it is easily seen that $g(z)$ will be of order $e^{2|v|(t-1)}$. That is why integral along upper and lower parts of contour vanishes as $B\to\infty$.

On the right hand side of contour when $N\to\infty$, we have

$$\lim_{B\to\infty}\frac{1}{2\pi i}\int\limits_{A_N-iB}^{A_N+iB} g(z)\,dz\sim\lim_{B\to\infty}\frac{1}{\pi i}\int\limits_{-iB}^{+iB}\frac{\cos(2\pi(N+1)t+2tiv)}{(A_N+iv)^3(1+cosh\ 2v)}dv\sim\frac{1}{A_N^3\pi}\int\limits_{-\infty}^{+\infty}\frac{cosh\ (2tv)}{(1+cosh\ 2v)}dv\to 0.$$

Therefore,

$$\frac{1}{2\pi i}\lim_{N\to\infty}\int\limits_0^1\left(\int\limits_{C_N} g(z)\,dz\right)q_k(t)dt=\lim_{N\to\infty}\int\limits_0^1\left[S_N(t)+T_N(t)-\frac{h}{\gamma_k}\right]q_k(t)dt=\lim_{N\to\infty}\int\limits_0^1\left[S_N(t)+T_N(t)\right]q_k(t)dt \tag{21}$$

where

$$T_N(t)=\sum_{n=1}^{N}\frac{4\cos(2\beta_n t)}{\beta_n\sin 2\beta_n}$$

For evaluating $\lim\limits_{N\to\infty}T_N(t)$ in accordance with above arguments select function of complex variable

$$G(z)=\frac{\cos 2zt}{(z\cos z-\sin z)\cos z}$$

Obviously poles of that function are zeros of $\cos z$, i.e, $\frac{\pi}{2}+\pi n=\delta_n$ and β_n.

$$\underset{z=\beta_n}{res}\,G(z)=\frac{2\cos 2\beta_n t}{(\cos\beta_n-\beta_n\sin\beta_n-\cos\beta_n)\cos\beta_n}=-\frac{2\cos 2\beta_n t}{\beta_n\sin\beta_n\cos\beta_n}=\frac{-4\cos 2\beta_n t}{\beta_n\sin 2\beta_n} \tag{22}$$

and

$$\underset{z=\delta_n}{res}\,G(z)=\frac{\cos(2\delta_n t)}{\sin^2\left(\frac{\pi}{2}+\pi n\right)}=\cos(\pi+2\pi n)\,t=\cos(2n+1)\,\pi t. \tag{23}$$

Selecting corresponding contour in the way similar to one done above and extending it to infinity we can show that limit of $G(z)$ along it vanishes. Denote by $M_N(t)$ the sum

$$\sum_{n=1}^{N}\cos(2n+1)\,\pi t.$$

Thus,

$$\lim_{N\to\infty}\int_0^1 S_N(t)f_k(t)dt = -\lim_{N\to\infty}\int_0^1 T_N(t)\,q_k(t)\,dt = -\lim_{N\to\infty}\int_0^1 M_N(t)\,q_k(t)\,dt = \frac{1}{4}\left[q_k(1) - q_k(0)\right] \tag{24}$$

Summing the last for k, when $k = \overline{1,\infty}$ we will have from (17)

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}\left[\mu_n - \lambda_n\right] = \lim_{m\to\infty}\sum_{n=1}^{n_m}\left(Q\Psi_n, \Psi_n\right) = \frac{1}{4}\left[trq(1) - trq(0)\right]$$

Thus we have proved the next theorem.

Theorem 2. *Under the conditions 1)-3) and Lemma 1, the next trace formula for operator L is true*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}\left[\mu_n - \lambda_n\right] = \frac{1}{4}\left[trq(1) - trq(0)\right].$$

References

[1] Gorbachuk V.I., Rybak M.A. On self-adjoint extensions of the minimal operator associated with Sturm-Liouville expression with operator potential and nonhomogeneous boundary conditions. Dokl. AN URSR, ser A,4 (1975) 300-304.

[2] Rybak, M.A. On asymptotics of eigenvalue distribution of some boundary value problems for Sturm-Liouville operator equation. Ukr. Math. J. 32(2)(1980) 248-252.

[3] Aliev B.A. Asymptotic behavior of eigenvalues of one boundary value problem for elliptic differential operator equation of second order. Ukr. Math. J. 5(8) 1146-1152.

[4] Aslanova N.M. A trace formula of one boundary value problem for the Sturm-Liouville operator equation. Siberian Math. Journal. 49(6) (2008) 1207-1215.

[5] Bayramoglu M. Aslanova, N.M. Formula for second order regularized trace of a problem with spectral parameter dependent boundary condition. Hacet. J. Math. Stat., 40(5)(2011) 635–647.

[6] Aslanova N.M. The asymptotics of eigenvalues and trace formula of operator associated with one singular problem. Bound. Value Probl. (2012)12 8, doi: 10.1186/1687-2770-2012-8.

[7] Aslanova N.M. Investigation of spectrum and trace formula of Bessel's operator equation. Siberian Journal of Mathematics, 51(4)(2010) 722–737.

[8] Bayramoglu M., Aslanova N.M. Eigenvalue distribution and trace formula of operator Sturm-Liouville equation. Ukrain. Math. J. 62(7)(2010) 867–877.

[9] Aslanova N.M., Bayramoglu M. On generalized trace of fourth order differential operator with operator coefficient. Ukrain. Math. J. 66(1) (2014)128-134.

[10] Aslanova N.M. Studu of the asymptotic eigenvalue distribution and trace formula of a second order operator differential equation. Boundary value problems,(2011) 7, doi: 10.1186—1687-2770-2011-7, 22p.

[11] Gorbachuk W.I., Gorbachuk M.L. On some class of boundary value problems for Sturm-Liouville operator with operator potential. Ukr. mathem. journal, 24(3)(1972) 291-305.

[12] Gelfand I.M., Levitan B.M. About one simple identity for eigenvalue of second order differential operator. DAN SSSR, 88(4) (1953) 593-596.

[13] Maksudov F.G., Bayramogly M., Adigezalov A.A., On regularized trace of Sturm-Liouvilles operator on finite segment with unbounded operator coefficient. DAN SSSR. 277(4) (1984) 795-799.

[14] Sadovnichii V.A., Podolskii V.E. Trace of operators with relativly compact perturbation, Matem. Sbor., 193(2)(2002) 129-152.

[15] Fulton Ch.T. Two- point boundary value problems with eigenvalue parameter contained in the boundary condition. Proceedings of the Royal Society of Edinburgh, 77 A(1977) 293-308.

[16] Walter J. Regular eigenvalue problems with eigenvalue parameter in the boundary conditions. Math. Z. 133(1973) 301-312.