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Cartesian Product Graphs and k-Tuple Total Domination

Adel P. Kazemi[a], Behnaz Pahlavsay[a], Rebecca J. Stones[b]

[a]*Department of Mathematics, University of Mohaghegh Ardabili,*
P.O. Box 5619911367, Ardabil, Iran
[b]*College of Computer and Control Engineering,*
Nankai University, China

Abstract. A k-tuple total dominating set (kTDS) of a graph G is a set S of vertices in which every vertex in G is adjacent to at least k vertices in S; the minimum size of a kTDS is denoted $\gamma_{\times k,t}(G)$. We give a Vizing-like inequality for Cartesian product graphs, namely $\gamma_{\times k,t}(G)\gamma_{\times k,t}(H) \leq 2k\gamma_{\times k,t}(G\square H)$ provided $\gamma_{\times k,t}(G) \leq 2k\rho(G)$ holds, where ρ denotes the packing number. We also give bounds on $\gamma_{\times k,t}(G\square H)$ in terms of (open) packing numbers, and consider the extremal case of $\gamma_{\times k,t}(K_n\square K_m)$, i.e., the rook's graph, giving a constructive proof of a general formula for $\gamma_{\times 2,t}(K_n\square K_m)$.

1. Introduction

Domination is well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [11, 12]. Among the many variations of domination, the ones relevant to this paper is k-tuple domination and k-tuple total domination, which were introduced by Harary and Haynes [10], and by Henning and Kazemi [14], respectively. Throughout this paper, we use standard notation as listed in Table 1. All graphs considered here are finite, undirected, and simple.

For a graph $G = (V, E)$ and $k \geq 1$, a set $S \subseteq V$ is called a *k-tuple total dominating set* (kTDS) if every vertex $v \in V$ has at least k neighbors in S, i.e., $|N_G(v) \cap S| \geq k$. A *k-tuple dominating set* (kDS) instead satisfies $|N_G[v] \cap S| \geq k$. The *k-tuple domination number* and the *k-tuple total domination number*, which we denote $\gamma_{\times k}(G)$ and $\gamma_{\times k,t}(G)$, respectively, is the minimum cardinality of a kDS and a kTDS of G, respectively. The familiar *domination number* is thus $\gamma(G) = \gamma_{\times 1}(G)$. We use min-$k$DS and min-$k$TDS to refer to kDSs and kTDSs of minimum size, respectively.

An immediate necessary condition for a graph to have a k-tuple dominating set (resp. k-tuple total dominating set) is that every vertex must have at least $k-1$ (resp. k) neighbors. For example, for $k \geq 1$, a k-regular graph $G = (V, E)$ has only one k-tuple total dominating set, namely V itself.

The *Cartesian product* $G\square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and

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Email addresses: adelpkazemi@yahoo.com (Adel P. Kazemi), pahlavsayb@yahoo.com (Behnaz Pahlavsay), rebecca.stones82@gmail.com (Rebecca J. Stones)

$G = (V, E)$	A graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$.
$N_G(v) = \{u \in V : uv \in E\}$	The *open neighborhood* of vertex v in G.
$N_G[v] = N_G(v) \cup \{v\}$	The *closed neighborhood* of vertex v in G.
$\deg_G(v) = \|N_G(v)\|$	The *degree* of a vertex v in G.
$\delta(G)$, $\Delta(G)$	The *minimum degree* and *maximum degree* of vertices in G.
C_n, K_n	The n-vertex *cycle* and *complete graph*.
$G\square H$	The Cartesian product of graphs G and H.
$K_n\square K_m$	The $n \times m$ rook's graph.
$\gamma_{\times k}(G)$	The k-tuple domination number of G.
$\gamma(G) = \gamma_{\times 1}(G)$	The domination number of G.
$\gamma_{\times k,t}(G)$	The k-tuple total domination number of G.
$\gamma_t(G) = \gamma_{\times 1,t}(G)$	The total domination number of G.
$\rho(G)$	The maximum cardinality of a packing (packing number).
$\rho^{(\text{open})}(G)$	The maximum cardinality of an open packing (open packing number).

Table 1: Table of notation.

$u_1u_2 \in E(G)$. For more information on product graphs see [20]. We will be particularly interested in the case when $K_n\square K_m$, which is known as the *$n \times m$ rook's graph*, as edges represent possible moves by a rook on an $n \times m$ chess board. The 3×4 rook's graph is drawn in Figure 1, along with a min-2TDS.

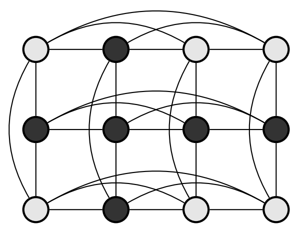


Figure 1: The 3×4 rook's graph, i.e., $K_3\square K_4$. The dark vertices highlight a min-2TDS, so $\gamma_{\times 2,t}(K_3\square K_4) = 6$.

In 1963, and more formally in 1968, Vizing [23] made an elegant conjecture that has subsequently become one the most famous open problems in domination theory.

Conjecture 1.1 (Vizing's Conjecture). *For any graphs G and H,*

$$\gamma(G)\gamma(H) \le \gamma(G\square H).$$

Over more than forty years (see [1] and references therein), Vizing's Conjecture is has been shown to hold for certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have gradually tightened. Additionally, research has explored inequalities (including Vizing-like inequalities) for different variations of domination [12]. A significant breakthrough occurred in 2000, when Clark and Suen [8] proved that

$$\gamma(G)\gamma(H) \le 2\gamma(G\square H)$$

which led to the discovery of a Vizing-like inequality for total domination [15, 17], i.e.,

$$\gamma_t(G)\gamma_t(H) \le 2\gamma_t(G\square H), \tag{1}$$

as well as for paired [3, 6, 18], and fractional domination [9], and the $\{k\}$-domination function (integer domination) [2, 7, 19], and total $\{k\}$-domination function [19].

Burchett, Lane, and Lachniet [5] and Burchett [4] found bounds and exact formulas for the k-tuple domination number and k-domination number of the rook's graph in square cases, i.e., $K_n\square K_n$ (where

k-domination is like holds k-tuple total domination, but only vertices outside of the domination set need to be dominated). The k-tuple total domination number is known for $K_n \times K_m$ [13] and bounds are given for supergeneralized Petersen graphs [21].

In this paper, we investigate inequalities for k-tuple total domination, i.e., we present lower and upper bounds on $\gamma_{\times k,t}(G\Box H)$ in terms of the orders of G and H, the packing numbers and open packing numbers, and in terms of $\gamma_{\times k,t}(G)$ and $\gamma_{\times k,t}(H)$. One new result is Theorem 2.3, which gives a partial generalization of (1). We also find formulas for $\gamma_{\times k,t}(K_n\Box K_m)$, and determine the value of $\gamma_{\times 2,t}(K_n\Box K_m)$ in Proposition 3.8 for all n and m.

2. General graphs

A subset $S \subseteq V(G)$ is a *packing* (resp. *open packing*) if the closed (resp. open) neighborhoods of vertices in S are pairwise disjoint. The *packing number* (resp. *open packing number*) of G, denoted $\rho(G)$ (resp. $\rho^{(\text{open})}(G)$), is the maximum cardinality of a packing (resp. an open packing). Note that vertices in packings S have distance at least 3, i.e., if $u, v \in S$, then $\text{dist}_G(u, v) \geq 3$. Reader can find more information on the packing number in many papers such as [16, 22]. The following two lemmas are from [13].

Lemma 2.1. *If G is an n-vertex graph with $\delta(G) \geq k$, then $\gamma_{\times k,t}(G) \geq \lceil kn/\Delta(G)\rceil$.*

Proof. The sum of the degrees of the vertices in any min-kTDS D is at least kn (since every vertex has at least k neighbors in D) and at most $|D|\Delta(G)$ (by definition of maximum degree). Hence $|D|\Delta(G) \geq kn$ and the lemma follows since $|D| = \gamma_{\times k,t}(G)$, by definition. □

Lemma 2.2. *If G is a graph with $\delta(G) \geq k$, then $\gamma_{\times k,t}(G) \geq k\rho^{(\text{open})}(G) \geq k\rho(G)$.*

Proof. A kTDS must contain k vertices from each of the $\rho^{(\text{open})}(G)$ disjoint open neighborhoods in any maximal open packing. The second inequality is because every packing in a graph is also open packing. □

The following theorem gives an upper bound on the product of the k-tuple total domination numbers of two graphs in terms of the k-tuple total domination number of their Cartesian product.

Theorem 2.3. *Let G and H be two graphs, and suppose $\delta(H) \geq k$. Then*

$$\rho(G)\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G\Box H).$$

Hence, if $\delta(G) \geq k$ and $\gamma_{\times k,t}(G) \leq 2k\rho(G)$, then

$$\gamma_{\times k,t}(G)\gamma_{\times k,t}(H) \leq 2k\gamma_{\times k,t}(G\Box H).$$

Proof. Let S be a min-kTDS of $G\Box H$. Choose a maximal packing $P := \{v_i\}_{i=1}^{\rho(G)}$ of G, and for each vertex v_i in the packing, let H_i be the subgraph of $G\Box H$ induced by $\{v_i\} \times V(H)$. An example is drawn in Figure 2.

We partition S into parts (a) S_i, for $i \in \{1, 2, \ldots, \rho(G)\}$, containing the vertices of S which are in or are adjacent to vertices in H_i, and (b) X, the remaining vertices (if any). (The sets S_i are disjoint, since P is a packing of G.) Hence

$$|S| \geq |S \setminus X| = \left|\bigcup_{i=1}^{\rho(G)} S_i\right| = \sum_{i=1}^{\rho(G)} |S_i| \tag{2}$$

since the sets S_i are disjoint. Moreover, every vertex in H_i has at least k neighbors in S_i.

From S_i, we can construct a kTDS D of H_i of size at most $|S_i|$ as follows:

- Add every vertex in $S_i \cap H_i$ to D.
- For each $x \in S_i \setminus H_i$, by definition of Cartesian product, x has a unique neighbor in H_i; call it x'.

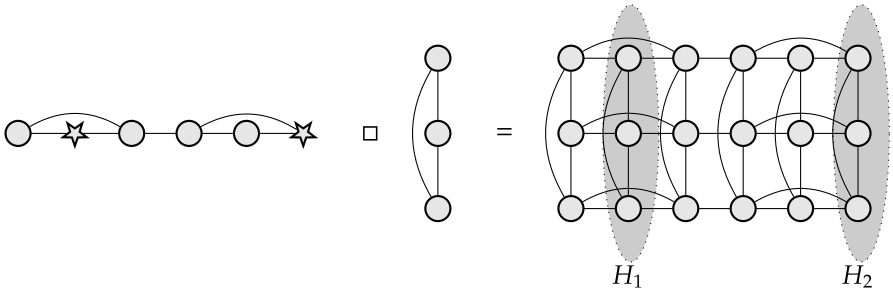


Figure 2: A Cartesian product graph $G \square H$. The vertices of G drawn as stars highlight a maximum packing P of G, and are used to identify H_1 and H_2 in $G \square H$.

- If x' has k or more neighbors in D, do nothing.
- Otherwise, since $\delta(H) \geq k$, we know x' has a neighbor x'' in $H_i \setminus D$. Add x'' to D.

Essentially, any $x \in S_i \setminus H_i$ dominates a unique vertex $x' \in H_i$ so, if necessary, we replace it by some unused $x'' \in H_i \cap N(x')$ which also dominates x'. After performing these operations, $|S_i| \geq |D| \geq \gamma_{\times k,t}(H)$. Thus

$$\begin{aligned}\gamma_{\times k,t}(G \square H) &= |S| \\ &\geq \sum_{i=1}^{\rho(G)} |S_i| && \text{[by (2)]} \\ &\geq \sum_{i=1}^{\rho(G)} \gamma_{\times k,t}(H) && \text{[by the above]} \\ &= \rho(G)\gamma_{\times k,t}(H).\end{aligned}$$

□

The second part of Theorem 2.3 is applicable when $\gamma_{\times k,t}(G) \leq 2k\rho(G)$; in contrast Lemma 2.2 implies that $\gamma_{\times k,t}(G) \geq k\rho(G)$ holds when $\delta(G) \geq k$.

When $k = 1$, i.e., total domination, Theorem 2.3 gives the bound (1) when $\gamma_t(G) \leq 2\rho(G)$. Equality holds in Theorem 2.3 when $k = 1$ in some instances: Modifying a construction in [15], we take a graph $G = (V, E)$ and (a) add at least one pendant vertex to each vertex in V, then (b) subdivide each edge in E twice. Call the result G^*. Then V is both a maximum packing and a minimum dominating set of G^*. So $\rho(G^*) = \gamma(G^*) = |V| = n$ and, in fact, we also find $\gamma_t(G^*) = 2n$. Figure 3 illustrates an example of this construction. Further, since $V(G) \times V(K_2)$ is a $(2n)$-vertex total dominating set of $G^* \square K_2$, we have

$$\gamma_t(G^*)\gamma_t(K_2) = 2\gamma_t(G^* \square K_2).$$

To further illustrate, the Petersen graph $\mathcal{P}$ has packing number $\rho(\mathcal{P}) = 1$ and we compute:

$$\begin{aligned}k = 1 : &\quad \gamma_{\times 1,t}(\mathcal{P}) = 4 > 2k\rho(\mathcal{P}), \\ k = 2 : &\quad \gamma_{\times 2,t}(\mathcal{P}) = 8 > 2k\rho(\mathcal{P}), \\ k = 3 : &\quad \gamma_{\times 3,t}(\mathcal{P}) = 10 > 2k\rho(\mathcal{P}),\end{aligned}$$

so $\gamma_{\times k,t}(\mathcal{P}) \leq 2k\rho(\mathcal{P})$ is not satisfied in all three cases. However, we can still apply the second part of Theorem 2.3 when H is the Petersen graph and G is some other graph which satisfies $\delta(G) \geq k$ and $\gamma_{\times k,t}(G) \leq 2k\rho(G)$.

We now derive lower bounds on $\gamma_{\times k,t}(G \square H)$ (Theorems 2.5 and 2.7) in terms of the packing and open packing numbers of the graphs G and H.

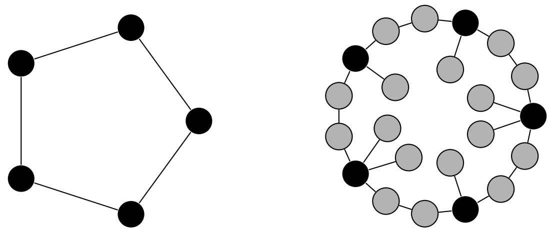


Figure 3: A construction of a graph G^* (right) with $\rho(G^*) = \gamma(G^*) = 5$ and $\gamma_t(G^*) = 10$ from $G = C_5$ (left). The graph $G^*\Box K_2$ has $\gamma_t(G^*\Box K_2) = 10$ and is an instance of equality in Theorem 2.3.

Lemma 2.4. *For graphs G and H, $\rho(G\Box H) \geq \rho(G)\rho(H)$.*

Proof. Let P_G and P_H be maximum packings in G and H, respectively. It is sufficient to show that $P_G \times P_H$, which has size $\rho(G)\rho(H)$, is a packing in $G\Box H$.

If two vertices $(u, v), (x, y) \in P_G \times P_H$ are adjacent, then, by definition of a Cartesian product, either (a) $u = x$, in which case $v, y \in P_H$ are adjacent in H, contradicting the assumption that P_H is a packing of H, or (b) $v = y$, in which case $u, x \in P_G$ are adjacent in G, contradicting the assumption that P_G is a packing of G.

If two distinct vertices $(u, v), (x, y) \in P_G \times P_H$ have a common neighbor, (a, b) say, in $G\Box H$, then by definition

$$(a,b) \in \Big(\overbrace{\{u\} \times N_H[v] \cup N_G[u] \times \{v\}}^{\text{closed neighborhood of (u,v)}}\Big) \bigcap \Big(\overbrace{\{x\} \times N_H[y] \cup N_G[x] \times \{y\}}^{\text{closed neighborhood of (x,y)}}\Big).$$

Four cases arise, and in each case, we contradict the assumption that P_G and P_H are maximum packings, tabulated below:

	$a = u$ and $b \in N_H[v]$	$a \in N_H[u]$ and $b = v$
$a = x$ and $b \in N_H[y]$	$\text{dist}_H(v, y) = 2$	$\text{dist}_G(u, x) = 1$ or $u = x$ and $\text{dist}_H(v, y) = 1$
$a \in N_G[x]$ and $b = y$	$\text{dist}_G(u, x) = 1$ or $u = x$ and $\text{dist}_H(v, y) = 1$	$\text{dist}_G(u, x) = 2$

□

The following theorem follows from Lemmas 2.2 and 2.4.

Theorem 2.5. *If G and H are two graphs with $\delta(G) + \delta(H) \geq k$, then*

$$\gamma_{\times k,t}(G\Box H) \geq k\rho(G)\rho(H).$$

We can also bound the open packing number of Cartesian product graphs, as in the following lemma.

Lemma 2.6. *For graphs G and H, where G and H are not the union of disjoint K_1 or K_2 subgraphs,*

$$\rho^{(\text{open})}(G\Box H) \geq \rho^{(\text{open})}(G) + \rho^{(\text{open})}(H) - 1.$$

Proof. Let O_G and O_H be maximal open packings in G and H. Choose $s \in V(G) \setminus O_G$ and $t \in V(H) \setminus O_H$. (Note that s and t exist because G and H are not the union of disjoint K_2 subgraphs.) An example is drawn in Figure 4.

If s is adjacent to a vertex $s' \in O_G$ in G, then it is adjacent to exactly one vertex in O_G (since O_G is an open packing). If s' exists, we define

$$T = \Big(\{s\} \times O_H\Big) \cup \Big((O_G \setminus \{s'\}) \times \{t\}\Big),$$

otherwise, we define
$$T = \Big(\{s\} \times O_H\Big) \cup \Big(O_G \times \{t\}\Big).$$

Either way, $|T| \geq \rho^{(\mathrm{open})}(G) + \rho^{(\mathrm{open})}(H) - 1$, so it is sufficient to show that T is an open packing of $G\Box H$. Assume, seeking a contradiction, that two distinct vertices $(u, v), (x, y) \in T$ have a common neighbor.

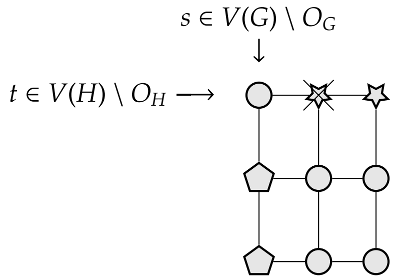


Figure 4: A Cartesian product graph $G\Box H$ as in Figure 2 where both G and H are 3-vertex paths. The stars identify a maximum packing O_G in G (drawn horizontally) and the blue pentagons identify a maximum packing O_H in H (drawn vertically). After deleting (s', t) (crossed out), we obtain an open packing of $G\Box H$.

As elements of T, the vertices (u, v) and (x, y) respectively satisfy (a) either $u = s$ or $v = t$, and (b) either $x = s$ or $y = t$. Since $(s, t) \notin T$, if $u = x = s$, then $v, y \in O_H$ have a common neighbor in H, contradicting that O_H is an open packing. A symmetric contradiction arises if $v = y = t$. Thus, by symmetry, we can assume $u = s$ and $y = t$.

By definition of the Cartesian product, if (s, v) and (x, t) have a common neighbor in $G\Box H$, it is either (s, t) or (x, v) (or both). Either way, we can deduce that x and s are adjacent in G. But, since $(x, t) \in T$, we know that $x \in O_G$ or $x \in O_G \setminus \{s'\}$ (if s' exists). Either way, this is a contradiction. □

The following theorem follows from Lemmas 2.2 and 2.6.

Theorem 2.7. *For graphs G and H with $\delta(G) + \delta(H) \geq k$, where G is not the union of disjoint K_2 subgraphs,*
$$\gamma_{\times k,t}(G\Box H) \geq k(\rho^{(\mathrm{open})}(G) + \rho^{(\mathrm{open})}(H) - 1).$$

We also include the following, simple lower bound on $\gamma_{\times k,t}(G\Box H)$.

Proposition 2.8. *For graphs G and H with $\delta(G) \geq k$,*
$$\gamma_{\times k,t}(G\Box H) \leq \gamma_{\times k,t}(G)\,|V(H)|.$$

Proof. A kTDS D of G gives rise to the k-TDS $\{(d, h) : d \in D \text{ and } h \in V(H)\}$ of $G\Box H$. An example is drawn in Figure 5. □

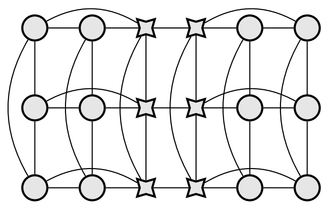


Figure 5: A Cartesian product graph $G\Box H$ as in Figure 2. The stars mark a 1TDS in each copy of G (drawn horizontally), together forming a 1TDS of $G\Box H$.

The following theorem establishes $K_n\Box K_m$, i.e., the rook's graph, as an extremal case, motivating the study of this class of graphs in the next section.

Proposition 2.9. *If G and H are spanning subgraphs of G' and H', respectively, and $\delta(G) + \delta(H) \geq k$, then*

$$\gamma_{\times k,t}(G \Box H) \geq \gamma_{\times k,t}(G' \Box H').$$

In particular,

$$\gamma_{\times k,t}(G \Box H) \geq \gamma_{\times k,t}(K_n \Box K_m)$$

if G has n vertices and H has m vertices.

Thus, any spanning subgraph Q of $K_n \Box K_m$ satisfies $\gamma_{\times k,t}(Q) \geq \gamma_{\times k,t}(K_n \Box K_m)$. The other extreme is achieved by k-regular graphs. If Q is a graph with a k-regular spanning subgraph K, then $\gamma_{\times k,t}(Q) \leq \gamma_{\times k,t}(K) = |V(K)| = |V(Q)|$. Not all graphs Q have a $\delta(Q)$-regular spanning subgraph.

3. The rook's graph

In this section, we find formulas for the k-tuple total domination number of $K_n \Box K_m$, i.e., the $n \times m$ rook's graph. Theorem 2.9 implies that $\gamma_{\times k,t}(K_n \Box K_m)$ is a lower bound on $\gamma_{\times k,t}(G \Box H)$ when G has n vertices and H has m vertices. Assume the vertex set of $V(K_n)$ is $\mathbb{Z}_n$.

For any $n \times m$ $(0,1)$-matrix $M = (m_{ij})$, we define

$$\kappa(i,j) = \overbrace{\left(\sum_{z \in \mathbb{Z}_m} m_{iz}\right)}^{i\text{-th row sum}} + \overbrace{\left(\sum_{z \in \mathbb{Z}_n} m_{zj}\right)}^{j\text{-th column sum}} - 2m_{ij}.$$

A kTDS D of $K_n \Box K_m$ corresponds to an $n \times m$ $(0,1)$-matrix $M = (m_{ij})$ with $m_{ij} = 1$ if and only if $(i,j) \in D$; the matrix M satisfies

$$\kappa(i,j) \geq k$$

for all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$, which we call *the κ bound*. We call an $n \times m$ $(0,1)$-matrix M a *kTDS matrix* if it satisfies the κ bound for all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$. Futher, we call M a *min-kTDS matrix* if it has exactly $\gamma_{\times k,t}(K_n \Box K_m)$ ones. Note that a kTDS matrix (resp. min-kTDS matrix) remains a kTDS matrix (resp. min-kTDS matrix) under permutations of its rows and/or columns, and after taking its matrix transpose.

We can also interpret $(0,1)$-matrices as biadjacency matrices of bipartite graphs (since $K_n \Box K_m$ is isomorphic to the line graph of $K_{n,m}$). Thus, a kTDS D of $K_n \Box K_m$ also corresponds to a bipartite graph with vertex bipartition $\{R_i\}_{i \in \mathbb{Z}_n} \cup \{C_j\}_{j \in \mathbb{Z}_m}$ and an edge R_iC_j whenever $(i,j) \in D$ (or equivalently whenever $m_{ij} = 1$). The bipartite graph has the property that for any pair of vertices (R_i, C_j),

$$k \leq \begin{cases} \deg(R_i) + \deg(C_j) - 2 & \text{if } R_i \text{ is adjacent to } C_j \\ \deg(R_i) + \deg(C_j) & \text{if } R_i \text{ is not adjacent to } C_j. \end{cases} \tag{3}$$

An example of these correspondences is given in Figure 6.

Lemma 3.1. *When $m \geq n \geq 1$ and $n + m \neq 2$,*

$$\gamma_{\times k,t}(K_n \Box K_m) \geq \frac{knm}{n+m-2}.$$

Proof. For any min-kTDS, $\gamma_{\times k,t}(K_n \Box K_m)$ is equal to the number of edges in the corresponding bipartite graph. Let $e = \gamma_{\times k,t}(K_n \Box K_m)$. If we sum (3) over all pairs of vertices (R_i, C_j) for an e-edge bipartite graph determined

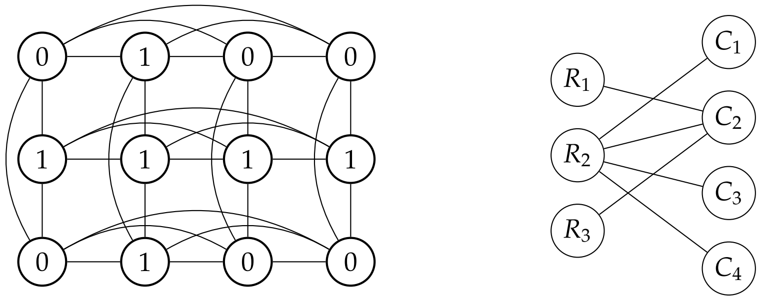


Figure 6: The 3×4 rook's graph $K_3 \Box K_4$ with the vertices in the 2TDS in Figure 1 labeled 1, illustrating the corresponding $(0,1)$-matrix and bipartite graph.

by a min-kTDS, we obtain

$$\begin{aligned}
nmk &\le \sum_{(R_i,C_j)} \text{right-hand side of (3)} \\
&= \sum_{(R_i,C_j)} \deg(R_i) + \sum_{(R_i,C_j)} \deg(C_j) - 2e \\
&= m \sum_{R_i} \deg(R_i) + n \sum_{C_j} \deg(C_j) - 2e \\
&= me + ne - 2e \\
&= (n + m - 2)\gamma_{\times k,t}(K_n \Box K_m).
\end{aligned}$$

□

Lemma 3.1 will be tight when the $\kappa(i, j) = k$ for all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$. This occurs when $n = 2$ and $m \ge k \ge 1$ for an $n \times m$ $(0,1)$-matrix with ones in the first k columns, and zeroes elsewhere. For example:

$k = 1$ $k = 2$ $k = 3$

It is also achieved by any $n \times m$ all-1 matrix when $k = n + m - 2$. If equality does not hold in Lemma 3.1, then equality does not hold in the κ bound for some cell, or equivalently, some vertex of $K_n \Box K_m$ has more than k neighbors in the corresponding kTDS. Of course, to have equality in Lemma 3.1, we must have knm divisible by $n + m - 2$.

Lemma 3.2. *For $n \ge 1$ and $m \ge 1$, an $n \times m$ kTDS matrix with an all-0 column has at least kn ones.*

Proof. If column j^* is an all-0 column, then to achieve $\kappa(i, j^*) \ge k$ for any $i \in \mathbb{Z}_n$, we need k ones in row i. Since this is true for all n rows, we must have kn ones. □

There are instances when kn ones is the least number of ones in any $n \times m$ kTDS matrix; we establish some cases in the following theorem.

Theorem 3.3. *When $m \ge n \ge 2$ and $m \ge k$,*

$$\gamma_{\times k,t}(K_n \Box K_m) \le kn \tag{4}$$

with equality when $m \ge kn - 1$.

When $m \ge k + 1$,

$$\gamma_{\times k,t}(K_1 \Box K_m) = k + 1. \tag{5}$$

Proof. If $m \geq n \geq 2$ and $m \geq k$, the $n \times m$ $(0,1)$-matrix with ones in the first k columns, and zeros elsewhere is a kTDS matrix, and has kn ones, proving (4).

Now also assume $m \geq kn-1$ and let M be an $n \times m$ kTDS matrix. If M has a column of zeros, then M has at least kn ones by Lemma 3.2. If M has no column of zeros but has at least kn columns, then M has at least kn ones. Thus, assume $m = kn-1$ and M has a one in every column. If M has fewer than kn ones, it must have exactly 1 one in each column. Therefore, if $m_{ij} = 1$, then row i must have $k+1$ ones to satisfy $\kappa(i,j) \geq k$. If this is true for every row, then M has at least $(k+1)n \geq kn$ ones. Otherwise, there's a row of zeros, and Lemma 3.2 implies there are at least $km \geq kn$ ones.

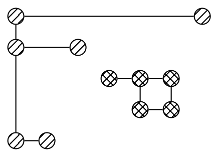
To prove (5), we observe that the $1 \times m$ $(0,1)$-matrix with ones in the first $k+1$ columns, and zeros elsewhere is a kTDS matrix, and has $k+1$ ones. We also observe that if a $1 \times m$ $(0,1)$-matrix has fewer than $k+1$ ones, then $\kappa(i,j) \ngeq k$ for the cells (i,j) containing ones, and thus is not a kTDS. $\square$

In fact, Theorem 3.3 resolves the $k=1$ case since $\gamma_{\times k,t}(K_n \Box K_m)$ is undefined when $n = m = 1$ (as $\delta(K_n \Box K_m) < k$).

3.1. 2-tuple total domination

In this section, we derive a general formula for $\gamma_{\times 2,t}(K_n \Box K_m)$ in Proposition 3.8. Motivated by Burchett, Lane, and Lachniet [5], given a $(0,1)$-matrix M we construct a graph $\Gamma(M)$ with vertices corresponding to the ones in M, and edges between two ones belonging to the same row or column if there are no ones between them. The following gives one such example:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&0&0&0&0&0&1\\\hline 1&0&1&0&0&0&0\\\hline 0&0&0&1&1&1&0\\\hline 0&0&0&0&1&1&0\\\hline 1&1&0&0&0&0&0\\\hline\end{array}$$



In this way, 2TDS matrices M correspond to graphs, which have (connected) components. If the set of vertices in some component of this graph is $\{(R_i, C_i)\}$, then we define the corresponding component of M as the submatrix formed by the intersection of rows $\{R_i\}$ and columns $\{C_i\}$. We shade two components in the example above; in this example, the 5×7 matrix is the union of two components (one 3×4 component, and one 2×3 component). A 2TDS matrix M with a component H, up to permutations of the rows and colums of M, looks like one of the following:

$$\begin{array}{|c|c|}\hline H & \emptyset\\\hline \emptyset & ?\\\hline\end{array}, \quad \begin{array}{|c|}\hline H\\\hline \emptyset\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline H & \emptyset\\\hline\end{array}, \text{ or } \quad \begin{array}{|c|}\hline H\\\hline\end{array},$$

where the question mark (?) denotes some $(0,1)$-submatrix, and $\emptyset$ denotes an all-0 submatrix. Components of 2TDS matrices have the following properties:

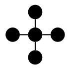
- Components have no all-0 rows and no all-0 columns.
- Components are 2TDS matrices in their own right.
- While $\Gamma(M)$ is a graph, its components correspond to submatrices of M, so we can discuss, say, $x \times y$ components. (Here, if the component has vertex set $\{(R_i, C_i)\}$, then it corresponds to an $|\{R_i\}| \times |\{C_i\}|$ component in M.)

We will now study component switchings in 2TDS matrices. The following two lemmas give conditions on when some kinds of switchings are possible without increasing the number of ones (Lemma 3.4) nor violating the κ bound (Lemma 3.5).

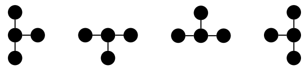
Lemma 3.4. *Let M be a 2TDS matrix and let H be an $x \times y$ component of M. Then the number of ones in H is at least $x+y-1$.*

Proof. Let $\Gamma(H)$ be the subgraph of $\Gamma(M)$ corresponding to H. We choose an arbitrary vertex v of $\Gamma(H)$ which has at least 2 neighbors (since M is a 2TDS matrix) but at most 4 neighbors (by definition of Γ). The closed neighborhood $N_{\Gamma(H)}[v]$ has one of these properties:

- It has cardinality 5 and intersects 3 rows and 3 columns, and looks like the following:



- It has cardinality 4 and either (a) intersects 3 rows and 2 columns, or (b) intersects 2 rows and 3 columns, and looks like one of the following:



- It has cardinality 3 and either (a) intersects 1 row and 3 columns, (b) intersects 2 rows and 2 columns, or (c) intersects 3 row and 1 columns, and looks like one of the following:

We proceed algorithmically. We initialize $S \leftarrow N_{\Gamma(H)}[v]$ and iteratively add vertices to S from $\Gamma(H)$ which (a) do not already belong to S, and (b) have a neighbor in S. As a result of each iteration:

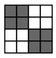
1. the number of vertices in S increases by exactly 1, and
2. one of the following:
   - the number of rows of M that S intersects increases by exactly 1, and the number of columns that S intersects remains unchanged,
   - the number of rows of M that S intersects remains unchanged, and the number of columns that S intersects increases by exactly 1, or
   - the number of rows of M that S intersects remains unchanged, and the number of columns that S intersects remains unchanged.

Since H is a connected component, $\Gamma(H)$ has a spanning tree, and thus every vertex of $\Gamma(H)$ will be added to S at some point. At the end of this process S intersects all x rows and all y columns of H. The number of ones in H is equal to $|S|$, which is (a) at least $5 + (x - 3) + (y - 3)$, (b) at least $4 + (x - 3) + (y - 2)$ or $4 + (x - 2) + (y - 3)$, or (c) at least $3 + (x - 3) + (y - 1)$, or $3 + (x - 2) + (y - 2)$, or $3 + (x - 1) + (y - 3)$. Each of these is equal to $x + y - 1$. □

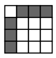
Lemma 3.5. *Let M be a 2TDS matrix with no all-0 rows and no all-0 columns, and with an $x \times y$ union of components K. Let H be an $x \times y$ 2TDS matrix with no all-0 rows and no all-0 columns. Then replacing K with H in M gives a 2TDS matrix.*

Proof. Call the new submatrix $\hat{M} = (\hat{m}_{ij})$. We check the κ bound is satisfied:
- If cell (i, j) is in H, then since H is a 2TDS matrix, the κ bound is satisfied.
- If cell (i, j) neither shares a row nor column with H, then the κ-value for $\hat{M}$ is the same as the κ-value for M, so the κ bound is satisfied.
- If cell (i, j) is in $\hat{M} \setminus H$, and shares a column (resp. row) with H, we know (i) $m_{ij} = 0$, otherwise the submatrix K is not a union of components, and (ii) row i (resp. column j) contains a one in $\hat{M}$, (iii) column j (resp. row i) contains a one in H. Thus the κ bound is satisfied. □

As an example, suppose a 2TDS matrix M with no all-0 rows nor columns contains the union of components

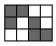


then we can replace it by

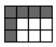


and Lemma 3.5 implies that the matrix obtained after performing this switch is also a 2TDS matrix. Furthermore, since this switch decreases the number of ones, we deduce that the original matrix M is not a min-2TDS matrix.

As another example, if a 2TDS matrix M no all-0 rows nor columns contains the component



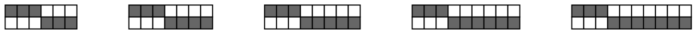
we can replace it by



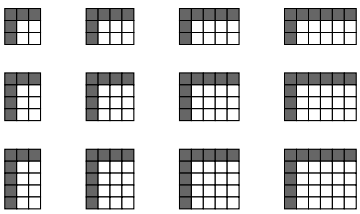
and Lemma 3.5 implies that we obtain a 2TDS matrix. Moreover, since the number of ones is unchanged after performing this switch, if M is a min-2TDS matrix, then we obtain another min-2TDS matrix after switching. Since Lemma 3.4 implies that any 3×4 component of M has at least 6 ones, we can replace every 3×4 component in this way while still preserving the min-2TDS property, thereby reducing the possibilities we need to consider.

In the subsequent material, switchings as per Lemma 3.5 will arise repeatedly, and we will not indicate the use of Lemma 3.5 each time.

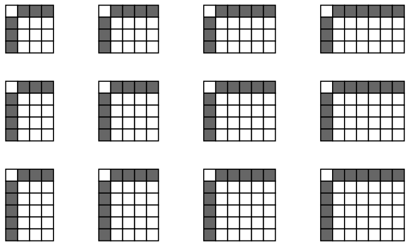
Lemmas 3.4 and 3.5 are the primary motivation for the next theorem (Theorem 3.6). We will repeatedly use the following $(0,1)$-matrices, which we give notation to: For $x \geq 1$ and $y \geq 1$, we define $J(x, y)$ as the $x \times y$ all-1 matrix. For $y \geq 6$, we define the $2 \times y$ matrix $A(2, y)$ to have the first row $(1, 1, 1, 0, 0, \ldots, 0)$ and second row $(0, 0, 0, 1, 1, \ldots, 1)$, depicted below for $y \in \{6, 7, 8, 9, 10\}$:



For $x \geq 3$ and $y \geq 3$, let $B(x, y)$ be the $x \times y$ $(0,1)$-matrix with an all-1 first row, an all-1 first column, and zeroes elsewhere, depicted below for $x \in \{3, 4, 5\}$ and $y \in \{3, 4, 5, 6\}$:



For $x \geq 4$ and $y \geq 4$, let $C(x, y)$ be the $x \times y$ $(0,1)$-matrix first row $(0, 1, 1, \ldots, 1)$, first column $(0, 1, 1, \ldots, 1)^T$, and zeroes elsewhere, depicted below for $x \in \{4, 5, 6\}$ and $y \in \{4, 5, 6, 7\}$:



Theorem 3.6. *For $n \geq 1$ and $m \geq 1$, excluding $(n, m) \in \{(1, 1), (1, 2), (2, 1)\}$, there exists an $n \times m$ min-2TDS matrix M whose components, up to permutations of the rows and columns, are all either $J(x, 1)$ for $x \geq 3$ (or $J(1, y)$ for $y \geq 3$), or $J(x, 2)$ for $x \geq 2$ (or $J(2, y)$ for $y \geq 2$), or $M = B(3, 3)$.*

Proof. We start with a min-2TDS matrix M. (Such a matrix does not exist when $(n, m) \in \{(1, 1), (1, 2), (2, 1)\}$, since $\delta(K_n \Box K_m) \leq 1 < 2$).

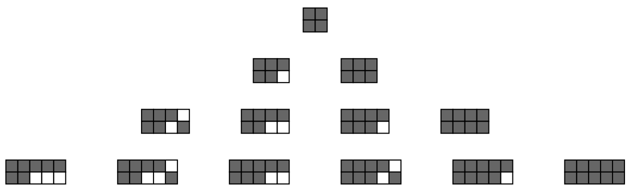
Case I: M has an all-0 column (or, by symmetry, an all-0 row). Then M has at least $2n$ ones by Lemma 3.3, which is the same number of ones as $J(n, 2)$ with $m - 2$ appended columns of zeros, in which case the theorem is true. We henceforth assume M has no all-0 rows nor all-0 columns.

Case II: M has a $2 \times y$ component with $y \geq 6$ (or its transpose). We replace it by $A(2, y)$, but since $A(2, y)$ has y ones, whereas any $2 \times y$ component has at least $y + 1$ ones by Lemma 3.4, we contradict the assumption that M is a min-2TDS matrix.

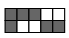
Case III: M has an $x \times y$ component with $x \geq 3$ and $y \geq 3$. We replace it by $B(x, y)$. Lemma 3.4 implies that the number of ones has not increased, so we still have a min-2TDS matrix.

Case IV: M has a $2 \times y$ component H with $2 \leq y \leq 5$ (or its transpose). There must be an all-1 column for the component to be connected. Hence H has at least $y + 1$ ones.

- If there are no other components, then $M = H = J(2, 2)$, as the other possible $2 \times y$ components are not min-2TDSs.
- If there is at least one other component K, then:
  - If H has at least two all-1 columns, then, since $2 \leq y \leq 5$, it is equivalent to one of the following:

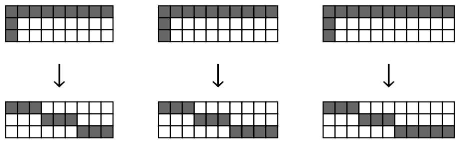
    

    In this case, the union $H \cup K$ thus has dimensions $x' \times y'$ where $x' \geq 3$ and $y' \geq 3$ and at least $x' + y' - 1$ ones. We replace this union of components with $B(x', y')$, which has $x' + y' - 1$ ones.
  - If H has exactly one all-1 column, then, since $y \leq 5$ and H is a 2TDS, it is equivalent to:

    * If the components of M except for H are all $J(1, 3)$ matrices, then M has more than $2n$ ones, whence Theorem 3.3 contradicts the assumption that M is a min-2TDS matrix.
    * Otherwise, we can choose K to have at least 2 rows. The union $H \cup K$ has dimensions $x' \times y'$ where $x' \geq 4$ and $y' \geq 4$ and at least $x' + y' - 2$ ones by Lemma 3.4. We replace $H \cup K$ by $C(x', y')$.

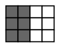
Case V: M has a $3 \times y$ component with $y \geq 9$ (or its transpose). As a result of Case III, components have the form $B(3, y)$, and we perform the switches indicated below:



and so on. By Lemma 3.5 we obtain a 2TDS matrix with fewer ones than M, giving a contradiction.

Case VI: M has a $3 \times y$ component with $3 \leq y \leq 8$ (or its transpose). As a result of Case III, components have the form $B(3, y)$:

If there are no other components, then $M = B(3, y)$. However, $B(3, y)$ is not a min-2TDS for $5 \leq y \leq 8$, so $M \in \{B(3, 3), B(3, 4)\}$ (or $M = B(3, 4)^T = B(4, 3)$). However, If $M = B(3, 4)$, we can replace it with



which has the same number of ones as $B(3, 4)$ (and likewise if $M = B(4, 3)$). If there is another component K, then $B(3, y) \cup K$ has dimensions $x' \times y'$ where $x' \geq 4$ and $y' \geq 4$ and at least $x' + y' - 2$ ones. We replace $B(3, y) \cup K$ with $C(x', y')$, which has exactly $x' + y' - 2$ ones.

Case VII: M has an $x \times y$ component with $x \geq 4$ and $y \geq 4$. As a result of Case III, components have the form $B(x, y)$, which we can replace by $C(x, y)$ to obtain a 2TDS matrix with fewer ones than M, giving a contradiction. □

The proof of Theorem 3.6 implies that, for all $n \geq 1$ and $m \geq 1$ except when $(n, m) \in \{(1, 1), (1, 2), (2, 1)\}$, there is some min-2TDS matrix whose components belong only to a strongly restricted family of components. In the next theorem, we restrict this family of component further when considering matrices with no all-0 rows and no all-0 columns.

Theorem 3.7. *For $n \geq 1$ and $m \geq 1$ except $(n, m) \in \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3)\}$, if there exists an $n \times m$ min-2TDS matrix with no all-0 rows and no all-0 columns, then there exists an $n \times m$ min-2TDS matrix M whose components, up to permutations of the rows and columns, are all $J(1, 3)$ or $J(3, 1)$, except possibly for*

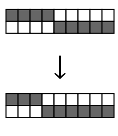
- *exactly one $J(x, 1)$ component with $4 \leq x \leq 7$;*
- *exactly one $J(1, y)$ component with $4 \leq y \leq 7$; or*
- *exactly one $J(x, 1)$ component with $4 \leq x \leq 5$ and exactly one $J(1, y)$ component with $4 \leq y \leq 5$.*

Further, the number of ones in M is

$$\begin{cases} \lceil 3(n+m)/4 \rceil + 1 & \text{if } m \equiv 3n + 4 \pmod 8 \\ \lceil 3(n+m)/4 \rceil & \text{otherwise.} \end{cases}$$

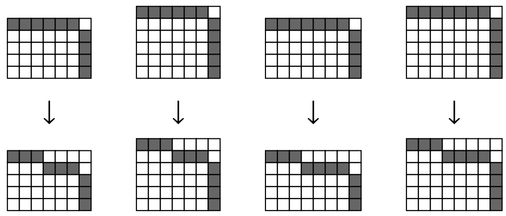
Proof. The proof of Theorem 3.6 implies we can assume that each component of M has one of the forms: $J(x, 1)$ for $x \geq 3$, or $J(1, y)$ for $y \geq 3$.

Case I: M has two components $J(1, y)$ and $J(1, y')$ with $y \geq 4$ and $y' \geq 4$ (and possibly more components). (Or, by symmetry, M has two components $J(x, 1)$ and $J(x', 1)$ with $x \geq 4$ and $x' \geq 4$.) We replace them by the two components $J(1, 3)$ and $J(1, y + y' - 3)$. An example is drawn below when $y = 4$ and $y' = 5$:



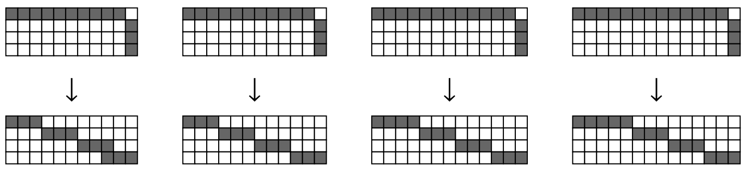
We repeat this process until there is at most one component of the form $J(1, y)$ with $y \geq 4$, and likewise at most one component of the form $J(x, 1)$ with $x \geq 4$.

Case II: M has a $J(1, y)$ component with $y \geq 6$ and a $J(x, 1)$ component with $x \geq 4$. (Or, by symmetry, M has a $J(x, 1)$ component with $x \geq 6$ and a $J(1, y)$ component with $y \geq 4$.) We apply the switch indicated below:



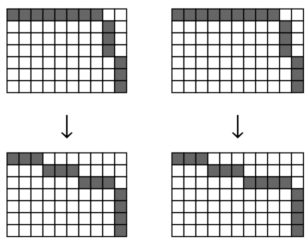
and so on in other cases. This reduces the number of ones, contradicting that M is a min-2TDS matrix.

Case III: M has a $J(1, y)$ component with $y \geq 10$. (Or, by symmetry, M has a $J(x, 1)$ component with $x \geq 10$.) There must also be a $J(3, 1)$ component in M, otherwise every row contains at least 3 ones, contradicting that M is a min-2TDS matrix. We apply the switch indicated below:



and so on in other cases. These switches all reduce the number of ones, contradicting that M is a min-2TDS matrix.

Case IV: M has a $J(1, y)$ component with $8 \leq y \leq 9$. (Or, by symmetry, M has a $J(x, 1)$ component with $8 \leq x \leq 9$.) There must also be at least two $J(3, 1)$ components in M, otherwise the average number of ones per row is at least $(8 + 3)/4 > 2$, contradicting that M is a min-2TDS matrix. We apply the switch indicated below:



These switches all reduce the number of ones, contradicting that M is a min-2TDS matrix.

Cases I–IV prove the first half of the theorem statement. Now let a be the number of $J(3, 1)$ components and let b be the number of $J(1, 3)$ components.

Case V: M only has $J(1, 3)$ and $J(3, 1)$ components. Then

$$\begin{aligned} n &= 3a + b, \\ m &= a + 3b, \end{aligned}$$

and the number of ones in M is $3(a + b) = 3(n + m)/4 = \lceil 3(n + m)/4 \rceil$. In this case, we have $m \equiv 3n \pmod 8$, since by adding in a $J(1, 3)$ or $J(3, 1)$ component, we either increase m by 1 and n by 3, or we increase n by 1 and m by 3, and either way $m \equiv 3n \pmod 8$ remains true.

Case VI: M has a $J(1,4)$ component and a $J(4,1)$ component. Then

$$n = 5 + 3a + b,$$
$$m = 5 + a + 3b,$$

and the number of ones in M is $3(a+b)+8 = 3(n+m-10)/4+8 = \lceil 3(n+m)/4 \rceil$. In this case, we have $m \equiv 3n+6 \pmod 8$.

Case VII: M has a $J(1,5)$ component and a $J(4,1)$ component. (Or, by symmetry, M has a $J(1,4)$ component and a $J(5,1)$ component.) Then

$$n = 5 + 3a + b,$$
$$m = 6 + a + 3b,$$

and the number of ones in M is $3(a+b)+9 = 3(n+m-11)/4+9 = \lceil 3(n+m)/4 \rceil$. In this case, we have $m \equiv 3n+7 \pmod 8$ (or $m \equiv 3n+3 \pmod 8$ in the transposed case).

Case VIII: M has a $J(1,5)$ component and a $J(5,1)$ component. Then

$$n = 6 + 3a + b,$$
$$m = 6 + a + 3b,$$

and the number of ones in M is $3(a+b)+10 = 3(n+m-12)/4+10 = \lceil 3(n+m)/4 \rceil + 1$. In this case, we have $m \equiv 3n+4 \pmod 8$.

Case IX: M has a $J(1,y)$ component with $4 \le y \le 7$ and no $J(x,1)$ component with $x \ge 4$. (Or, by symmetry, M has a $J(x,1)$ component with $4 \le x \le 7$ and no $J(1,y)$ component with $y \ge 4$.) We have

$$n = 1 + 3a + b$$
$$m = y + a + 3b$$

and the number of ones in M is

$$\begin{aligned} 3(a+b)+y &= 3(n+m-1-y)/4+y \\ &= 3(n+m)/4+(y-3)/4 \\ &= \begin{cases} 3(n+m)/4+1/4 & \text{if } y=4 \\ 3(n+m)/4+1/2 & \text{if } y=5 \\ 3(n+m)/4+3/4 & \text{if } y=6 \\ 3(n+m)/4+1 & \text{if } y=7. \end{cases} \end{aligned}$$

Since the number of ones is an integer quantity, this is equal to $\lceil 3(n+m)/4 \rceil$ except when $y=7$ when it is equal to $\lceil 3(n+m)/4 \rceil + 1$. In this case we have $m \equiv 3n+y-3 \pmod 8$, i.e., $m \equiv 3n+1, 3n+2, 3n+3, 3n+4 \pmod 8$ when $y = 4,5,6,7$, respectively (and $m \equiv 3n-3x+1 \pmod 8$ in the transposed case, i.e., $m \equiv 3n+5, 3n+2, 3n+7, 3n+4 \pmod 8$ when $x = 4,5,6,7$, respectively). □

With our highly restricted families of $n \times m$ min-2TDS matrices, we find a general formula for $\gamma_{\times 2,t}(K_n \square K_m)$ by simply counting the ones in all possible cases. This gives the following proposition. The subsequent Proposition 3.9 summarizes the matrices that need to be considered to find an example min-2TDS matrix for arbitrary $n \ge 1$ and $m \ge 1$, when possible.

Proposition 3.8. *For $n \ge 1$ and $m \ge n$, excluding $(n,m) \in \{(1,1),(1,2)\}$,*

$$\gamma_{\times 2,t}(K_n \square K_m) = \begin{cases} 3 & \text{if } n = 1 \text{ and } m \ge 3, \\ 2n & \text{if } n \ge 2 \text{ and } m \ge \lfloor (5n-4)/3 \rfloor + 1, \\ \lceil 3(n+m)/4 \rceil + 1 & \text{if } m \le \lfloor (5n-4)/3 \rfloor \text{ and } m \equiv 3n+4 \pmod 8, \\ \lceil 3(n+m)/4 \rceil & \text{otherwise.} \end{cases}$$

Hence, in the square case, for $n \geq 2$,

$$\gamma_{\times 2,t}(K_n \Box K_n) = \begin{cases} \lceil 3n/2 \rceil + 1 & \textit{if } n \equiv 2 \pmod 4, \\ \lceil 3n/2 \rceil & \textit{otherwise.} \end{cases}$$

Proof. If $(n, m) \in \{(1, 1), (1, 2)\}$, then there are no $n \times m$ min-2TDS matrices. If $n = 1$ and $m \geq 3$, then any $1 \times m$ $(0, 1)$-matrix with exactly 3 ones is a min-2TDS matrix. If $(n, m) \in \{(2, 2), (3, 3)\}$, then the following are $n \times m$ min-2TDS matrices:

$\lceil 3(2 + 2)/4 \rceil + 1 = 4$ ones

$\lceil 3(3 + 3)/4 \rceil = 5$ ones

Now assume $n \geq 2$ and $(n, m) \notin \{(2, 2), (3, 3)\}$.

Let M be an $n \times m$ min-2TDS matrix. If M has an all-0 row, then M has at least $2n$ ones, as in the proof of Theorem 3.3. Likewise, if M has an all-0 column, then M has at least $2m \geq 2n$ ones. If M has no all-0 rows and no all-0 columns, then the number of ones in M is

$$\begin{cases} \lceil 3(n + m)/4 \rceil + 1 & \text{if } m \equiv 3n + 4 \pmod 8 \\ \lceil 3(n + m)/4 \rceil & \text{otherwise} \end{cases}$$

by Theorem 3.7. Hence

$$\gamma_{\times 2,t}(K_n \Box K_m) = \begin{cases} 3 & \text{if } n = 1 \text{ and } m \geq 3, \\ \min(2n, \lceil 3(n + m)/4 \rceil + 1) & \text{if } n \geq 2 \text{ and } m \equiv 3n + 4 \pmod 8, \\ \min(2n, \lceil 3(n + m)/4 \rceil) & \text{otherwise.} \end{cases}$$

If $m > (5n - 4)/3$ (which occurs when $m \geq \lfloor (5n - 4)/3 \rfloor + 1$), then $3(n + m)/4 > 2n - 1$ implying $3(n + m)/4 \geq 2n$, in which case

$$\min(2n, \lceil 3(n + m)/4 \rceil) = \min(2n, \lceil 3(n + m)/4 \rceil + 1) = 2n.$$

If $m \leq (5n - 4)/3$, then $3(n + m)/4 \leq 2n - 1 < 2n$, in which case

$$\min(2n, \lceil 3(n + m)/4 \rceil) = \lceil 3(n + m)/4 \rceil$$

and

$$\min(2n, \lceil 3(n + m)/4 \rceil + 1) = \lceil 3(n + m)/4 \rceil + 1.$$

□

Theorem 2.3 implies that if $n \geq 3$ and $m \geq 3$ then

$$\gamma_{\times 2,t}(K_n \Box K_m) \geq \frac{1}{4} \gamma_{\times 2,t}(K_n) \gamma_{\times 2,t}(K_m) = \frac{9}{4} = 2.25.$$

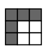
In contrast, Proposition 3.8 implies

$$\gamma_{\times 2,t}(K_n \Box K_m) = \Theta(n)$$

when $3 \leq n \leq m$ and $n \to \infty$.

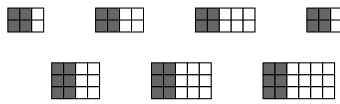
Proposition 3.9. *Equality is realized in Proposition 3.8 by the following min-2TDS matrices:*

- *Two sporadic cases:*



- *When $n = 1$ and $m \geq 3$, any $1 \times m$ $(0,1)$-matrix with exactly 3 ones, e.g.*

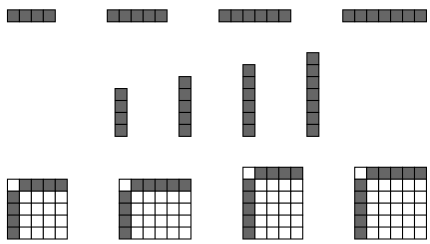
- *When $n \geq 2$ and $m \geq \lfloor (5n-4)/3 \rfloor + 1$, any $n \times m$ $(0,1)$-matrix with two columns of ones and zeroes elsewhere, e.g.*



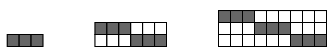
- *When $n \geq 3$ and $m \leq \lfloor (5n-4)/3 \rfloor$, except $(n,m) = (3,3)$, the $(0,1)$-matrices with block structure*

A	$\emptyset$	$\emptyset$
$\emptyset$	B	$\emptyset$
$\emptyset$	$\emptyset$	C

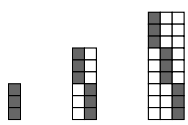
where $A = \emptyset$ or A is one of the following:



and $B = \emptyset$ or one of the following:



and so on, and $C = \emptyset$ or one of the following:



and so on.

4. Concluding remarks

Tables 2, 3, and 4 give examples of min-kTDS matrices for small n, m, and $k \in \{2, 3, 4\}$, found by computer search. Where possible, we include a representative with no all-0 rows and no all-0 columns. In some cases, the construction given in Proposition 3.9 (when $k = 2$) is the only possible construction (up to permutations of the rows and columns, and matrix transposition).

A natural way to extend this work is to find a general formula for $\gamma_{\times k,t}(K_n \square K_m)$ in the $k = 3$ or $k = 4$ cases (as in Tables 3 and 4). It seems reasonable to suspect that the component switching method used here for $k = 2$ will continue to be useful for larger k values. Other possible ways to extend this work are: (a) consider higher dimensions, e.g., $K_n \square K_m \square K_\ell$, and (b) consider graphs which have $K_n \square K_m$ as a spanning subgraph, such as Latin rectangle graphs and the $n \times m$ queen's graph.

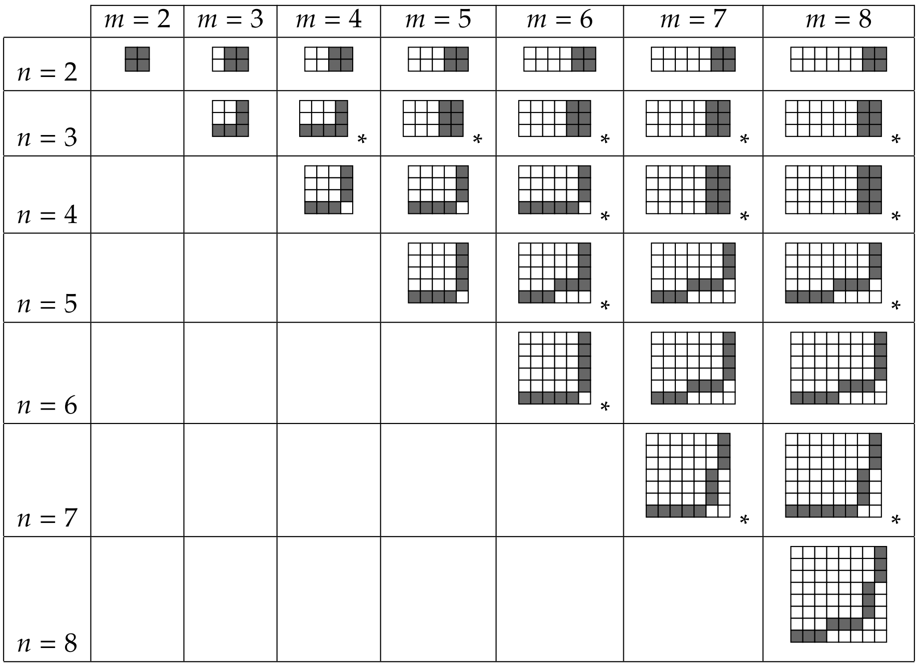


Table 2: Small min-2TDS matrices. Asterisks indicate that there are unlisted min-2TDS matrices with the same dimensions that are inequivalent under row and column permutations, and transposition (when $m = n$).

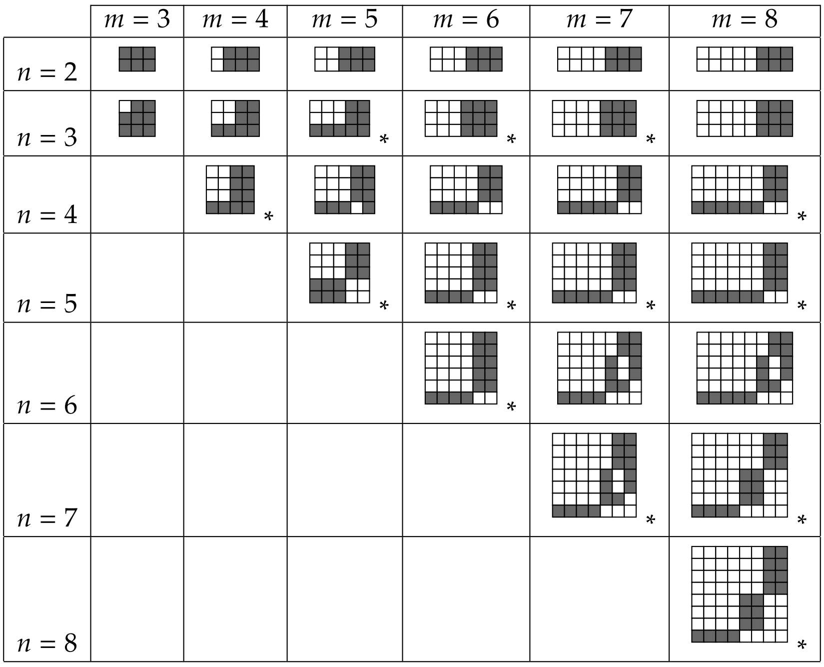


Table 3: Small min-3TDS matrices.

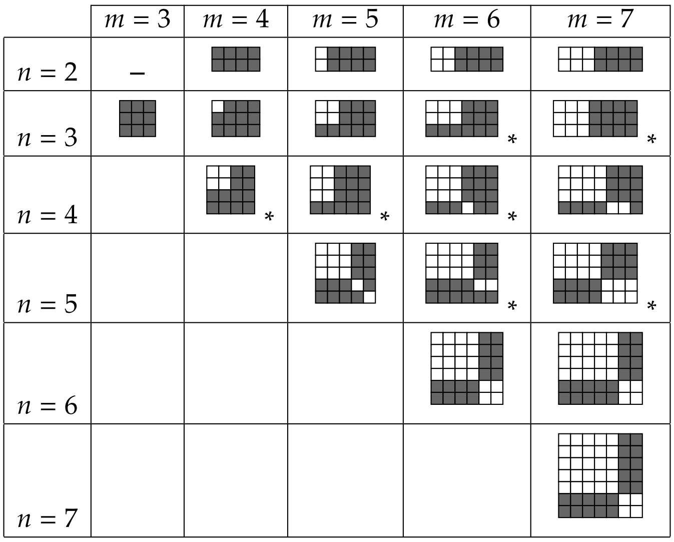


Table 4: Small min-4TDS matrices.

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