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z_c-Ideals and Prime Ideals in the Ring $\mathcal{R}_cL$

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Abstract. The ring $\mathcal{R}_cL$ is introduced as a sub-f-ring of $\mathcal{R}L$ as a pointfree analogue to the subring $C_c(X)$ of $C(X)$ consisting of elements with the countable image. We introduce z_c-ideals in $\mathcal{R}_cL$ and study their properties. We prove that for any frame L, there exists a space X such that $\beta L \cong \mathfrak{O}X$ with $C_c(X) \cong \mathcal{R}_c(\mathfrak{O}X) \cong \mathcal{R}_c\beta L \cong \mathcal{R}_c^*L$, and from this, we conclude that if $\alpha, \beta \in \mathcal{R}_cL$, $|\alpha| \le |\beta|^q$ for some $q > 1$, then α is a multiple of β in $\mathcal{R}_cL$. Also, we show that $IJ = I \cap J$ whenever I and J are z_c-ideals. In particular, we prove that an ideal of $\mathcal{R}_cL$ is a z_c-ideal if and only if it is a z-ideals. In addition, we study the relation between z_c-ideals and prime ideals in $\mathcal{R}_cL$. Finally, we prove that $\mathcal{R}_cL$ is a Gelfand ring.

1. Introduction

An ideal I of a ring A is a *z-ideal* if whenever two elements of A are in the same set of maximal ideals and I contains one of the elements, then it also contains the other (the term "ring" means a commutative ring with identity). A study of z-ideals in rings generally has been carried by Mason in the article [25]. We refer to z-ideals as defined in [25] as "z-ideals á la Mason". This algebraic definition of z-ideal was coined in the context of rings of continuous functions by Kohls in [22] and is also in the text *Rings of continuous functions* by Gillman and Jerison [16]. In pointfree topology, z-ideals were introduced by Dube in [8] in terms of the cozero map. z-Ideals have been studied in the theory of abelian lattice-ordered groups [4, 27] and in the context of Riesz space in [17] and [18].

This paper is mainly about the study of prime ideals and z_c-ideals in the ring $\mathcal{R}_cL$, where $\mathcal{R}_cL$ is the sub-f-ring of $\mathcal{R}L$ consisting of all elements which have the pointfree countable image. The ring $\mathcal{R}_cL$ is introduced in [21] as the pointfree version of $C_c(X)$, the subalgebra of $C(X)$ of all continuous functions with a countable image on a topological space X.

This paper is organized as follows. Section 2 is introductory. It is where we present relevant definitions pertaining to frames and give relevant background for the other sections. In Section 3, we introduce z_c-ideals in $\mathcal{R}_cL$ (see Definition 3.2) and study some properties of z_c-ideals. In addition, we show that $IJ = I \cap J$ whenever I and J are z_c-ideals, just as in $C(X)$ and in $\mathcal{R}L$ (Proposition 3.14). We prove that for any frame L, there exists a space X such that $\beta L \cong \mathfrak{O}X$ with $C_c(X) \cong \mathcal{R}_c(\mathfrak{O}X) \cong \mathcal{R}_c\beta L \cong \mathcal{R}_c^*L$, (see Lemmas 3.16 and 3.18).

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By this result, we show that if $\alpha, \beta \in \mathcal{R}_c L$, $|\alpha| \leq |\beta|^q$ for some $q > 1$, then α is a multiple of β in $\mathcal{R}_c L$ (Lemma 3.19). Also, we prove that a z-ideal in $\mathcal{R}_c L$ is a z_c-ideal if and only if it is a z-ideal á la Mason (see Proposition 3.27). In Section 4, we study the relation between prime ideals and z_c-ideals in the ring $\mathcal{R}_c L$ (see Proposition 4.4). In addition, we show that $\mathcal{R}_c L$ is a Gelfand ring (see Corollary 4.7).

2. Preliminaries

Here, we recall some concepts and terminologies with frames, frame maps and the pointfree version of the ring of continuous real-valued functions. Our references for frames are [20, 26] and our references for the ring $\mathcal{R}L$ are [1, 2].

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top_L$ and $\bot_L$ respectively, dropping the decorations if L is clear from the context. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m \in L$ is said to be *maximal (or dual atom)* if $m < \top$ and $m \leq x \leq \top$ implies $m = x$ or $x = \top$. As it is well known, every maximal element is prime. A lattice-ordered ring A is called *f- ring*, if $(f \wedge g)h = fh \wedge gh$ for every $f, g \in A$ and every $0 \leq h \in A$.

Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L : a \nleq p\}$ $(a \in L)$ as its open sets.

Let L be a frame. We say that a is *rather below* b, and write $a \prec b$, if there exists a *separating element* s of L with $a \wedge s = \bot$ and $s \vee b = \top$. A frame L is called *regular* if each of its elements is a join of elements rather below it. An element a of a frame L is said to be *completely below* b, written $a << b$, if there exists a sequence (c_q), $q \in \mathbb{Q} \cap [0, 1]$, where $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ whenever $p < q$. A frame L is called *completely regular* if each $a \in L$ is a join of elements completely below it.

A *frame homomorphism (or frame map)* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An ideal J of L is said to be completely regular if for each $x \in J$ there exists $y \in J$ such that $x << y$. The set βL of all completely regular ideals of a frame L under set inclusion is a compact completely regular frame, and $j_L : \beta L \to L$, defined by $j_L(I) = \bigvee I$, is a dense onto frame homomorphism, so that βL is a compactification of L. The compactification βL is known as the *Stone-Čech compactification* of the frame L. It is clear that βL is finite if and only if L is finite. The right adjoint $j_* : L \to \beta L$ of the surjective frame homomorphism j_L is denoted by r_L, and $r_L(a) = \{x \in L : x << a\}$ for all $a \in L$ (see [2, 3, 10, 26]).

Recall from [2] (see also [1]) that the frame of reals $\mathcal{L}(\mathbb{R})$ is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee \{(r, s) : p < r < s < q\}$.

(R4) $\top = \bigvee \{(p, q) : p, q \in \mathbb{Q}\}$.

For the pairs $(p, q) \in \mathbb{Q}^2$, we let:

$$\langle p, q \rangle := \{x \in \mathbb{Q} : p < x < q\} \qquad \text{and} \qquad]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}.$$

The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [2].

Corresponding to every operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p, q) = \bigvee \{\alpha(r, s) \wedge \beta(u, w) :< r, s > \diamond < u, w > \leq < p, q >\},$$

where $< r, s > \diamond < u, w > \leq < p, q >$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For any $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$, $(-\alpha)(p, q) = \alpha(-q, -p)$ and for every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$. An element α of $\mathcal{R}L$ is said to be bounded if there exists $n \in \mathbb{N}$ such that $\alpha(-n, n) = \top$. The set of all bounded elements of $\mathcal{R}L$ is denoted by $\mathcal{R}^*L$ which is a sub-f-ring of $\mathcal{R}L$. In connection with the Stone-Čech compactification of a frame L, it is also well known $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$.

The *cozero map* is the map $coz : \mathcal{R}L \to L$, defined by

$$\operatorname{coz}(\alpha) = \bigvee \{\alpha(p, 0) \vee \alpha(0, q) : p, q \in \mathbb{Q}\}.$$

A *cozero element* of L is an element of the form $\operatorname{coz}(\alpha)$ for some $\alpha \in \mathcal{R}L$ (see [2]). The cozero part of L, is denoted by $CozL$. It is well known that L is completely regular if and only if $\operatorname{coz}(L)$ generates L. For $A \subseteq \mathcal{R}L$, let $Coz[A] = \{\operatorname{coz}(\alpha) : \alpha \in A\}$ and for $A \subseteq CozL$, we write $Coz^{\leftarrow}[A]$ to designate the family maps $\{\alpha \in \mathcal{R}L : \operatorname{coz}(\alpha) \in A\}$. An ideal I of $\mathcal{R}L$ is a z-ideal if, for any $\alpha \in \mathcal{R}L$ and $\beta \in I$, $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ implies $\alpha \in I$ (for more details, see [6–8, 11, 13]).

Here we recall some notations from [12]. Let $a \in L$ and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$ are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively. For $a \neq \top$ it is obvious that for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$, $r \leq s$. In fact, we have that if $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$ (see [12]). Throughout this paper, for every $\alpha \in \mathcal{R}L$, we define $\alpha[p] = \widetilde{p}(\alpha)$ where p is a prime element of L (see [13]).

It is well known that the homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ taking (p, q) to $]\!]p, q[\![$ is an isomorphism (see [2, Proposition 2]). Now, we recall some concepts and results from [21] that we need to establish the principal results of our paper.

Definition 2.1. [21] *For any $\alpha \in \mathcal{R}L$, we say that α has the pointfree countable image if there exists $\mathbb{S} \subseteq \mathbb{R}$ such that*

(1) $|\mathbb{S}| \leq \aleph_0$
(2) $\tau(u) \cap \mathbb{S} \subseteq \tau(v) \cap \mathbb{S}$ *implies* $\alpha(u) \leq \alpha(v)$*, for every* $u, v \in \mathcal{L}(\mathbb{R})$ *(denoted by* $\alpha \blacktriangleleft \mathbb{S}$ *and say* α *is an overlap of* $\mathbb{S}$*).*

Lemma 2.2. [21] *For any $\alpha \in \mathcal{R}L$ and any $\mathbb{S} \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) $\alpha \blacktriangleleft \mathbb{S}$.
(2) $\tau(u) \cap \mathbb{S} = \tau(v) \cap \mathbb{S}$ *implies* $\alpha(u) = \alpha(v)$*, for any* $u, v \in \mathcal{L}(\mathbb{R})$.
(3) $\tau(p, q) \cap \mathbb{S} = \tau(v) \cap \mathbb{S}$ *implies* $\alpha(p, q) = \alpha(v)$*, for any* $v \in \mathcal{L}(\mathbb{R})$ *and any* $p, q \in \mathbb{Q}$.
(4) $\tau(p, q) \cap \mathbb{S} \subseteq \tau(v) \cap \mathbb{S}$ *implies* $\alpha(p, q) \leq \alpha(v)$*, for any* $v \in \mathcal{L}(\mathbb{R})$ *and any* $p, q \in \mathbb{Q}$.

Definition 2.3. [21] *For every frame L, we put*

$$\mathcal{R}_cL := \{\alpha \in \mathcal{R}L : \alpha \text{ has the pointfree countable image}\}.$$

Corollary 2.4. [21] *For any completely regular frame L, the set $\mathcal{R}_cL$ is a sub-f-ring of $\mathcal{R}L$.*

3. z_c-Ideals in $\mathcal{R}_cL$

Throughout this paper, all frames are assumed to be completely regular. We recall the notation z-ideal of a ring A as was introduced by Mason in [25]. We refer to z-ideals as defined in [25] as "z-ideals á la Mason".

Denoted by Max(A) the set of all maximal ideals of a ring A. For $a \in A$ and $S \subseteq A$, let

$$\mathfrak{M}(a) = \{M \in Max(A) : a \in M\} \quad \text{and} \quad \mathfrak{M}(S) = \{M \in Max(A) : M \supseteq S\}.$$

Definition 3.1. *An ideal I of a ring A is a z-ideal á la Mason if whenever $\mathfrak{M}(a) \supseteq \mathfrak{M}(b)$ and $b \in I$, then $a \in I$.*

In [8, Corollary 3.8], Dube shows that an ideal of $\mathcal{R}L$ is a z-ideal if and only if it is a z-ideal á la Mason. Here we introduce and study z_c-ideals in $\mathcal{R}_cL$. We begin by below definition.

Definition 3.2. *An ideal I in $\mathcal{R}_cL$ is called a z_c-ideal if, for every $\alpha \in \mathcal{R}_cL$ and $\beta \in I$,* $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ *implies $\alpha \in I$.*

Remark 3.3. It is evident that for a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of z_c-ideals of $\mathcal{R}_cL$, $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a z_c-ideal.

Recall from [9] that for each $a \in L$ with $a < \top$, the subset M_a of $\mathcal{R}L$ is defined by

$$\mathbf{M}_a = \{\, \alpha \in \mathcal{R}L : \mathrm{coz}(\alpha) \leq a\}.$$

They are distinct for distinct points. By [14, Lemma 4.2], if p is a prime element of L, then

$$\mathbf{M}_p = \{\, \alpha \in \mathcal{R}L : \alpha[p] = 0\,\}.$$

Definition 3.4. *For every $a \in L$, we let $\mathbf{M}^c_a := \{\alpha \in \mathcal{R}_cL \,:\, \mathrm{coz}(\alpha) \leq a\}$.*

Proposition 3.5. *The following statements are equivalent for an ideal I of $\mathcal{R}_cL$.*

(1) *I is a z_c-ideal.*
(2) *For any $\alpha, \beta \in \mathcal{R}_cL$, $\alpha \in I$ and* $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$ *imply $\beta \in I$.*
(3) *$I = \bigcup\{\mathbf{M}^c_{\mathrm{coz}(\alpha)} : \alpha \in I\}$.*

Proof. (1) ⇒ (2). Assume $\alpha \in I$ and $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$. Then

$$\mathrm{coz}(\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta).$$

Since $\alpha\beta \in I$, by statement (1), we infer that $\beta \in I$.

(2) ⇒ (3). Clearly $I \subseteq \bigcup\{\mathbf{M}^c_{\mathrm{coz}(\alpha)} : \alpha \in I\}$, because for every $\gamma \in \mathcal{R}_cL$, $\gamma \in \mathbf{M}^c_{\mathrm{coz}(\gamma)}$. To see the inverse inclusion, let $\alpha \in I$ and consider $\beta \in \mathbf{M}^c_{\mathrm{coz}(\alpha)}$. This means $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$, so that, by (2), $\beta \in I$. Therefore $\mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq I$, and hence the desired inclusion.

(3) ⇒ (1). Let $\alpha \in I$ and $\beta \in \mathcal{R}_cL$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$. Then $\beta \in \mathbf{M}^c_{\mathrm{coz}(\beta)} = \mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq I$, and hence (1) follows. □

Remark 3.6. Recall from [1] that if $\alpha \in \mathcal{R}L$ be a unit element of $\mathcal{R}L$ and we define $\beta \in \mathcal{R}L$ by $\beta(p,q) = \alpha(\tau^{-1}(\{\frac{1}{x} : x \in \tau(p,q)\,,\, x \neq 0\}))$, then $\beta = \alpha^{-1}$.

Lemma 3.7. *Let α be a unit element of $\mathcal{R}L$. If $\alpha \in \mathcal{R}_cL$, then $\alpha^{-1} \in \mathcal{R}_cL$.*

Proof. Since $\alpha \in \mathcal{R}_cL$, we infer from Definitions 2.1 and 2.3 that there is a countable subset $\mathbb{S} \subseteq \mathbb{R}$ such that $\alpha \blacktriangleleft \mathbb{S}$. Put $\mathbb{S}^1 := \{\frac{1}{s} : s \in \mathbb{S}\,,\, s \neq 0\}$. We claim that $\alpha^{-1} \blacktriangleleft \mathbb{S}^1$. To do this, suppose that $(p,q), u \in \mathcal{L}(\mathbb{R})$ and $\tau(p,q) \cap \mathbb{S}^1 = \tau(u) \cap \mathbb{S}^1$. Since

$$\{\frac{1}{s} : s \in \tau(p,q)\,,\, s \neq 0\} \cap \mathbb{S} = \{\frac{1}{s} : s \in \tau(u)\,,\, s \neq 0\} \cap \mathbb{S},$$

we conclude from Remark 3.6 and Lemma 2.2 that

$$\begin{aligned}\alpha^{-1}(p,q) &= \alpha(\tau^{-1}(\{\tfrac{1}{s} : s \in \tau(p,q)\,,\, s \neq 0\}))\\ &= \alpha(\tau^{-1}(\{\tfrac{1}{s} : s \in \tau(u)\,,\, s \neq 0\}))\\ &= \alpha^{-1}(u).\end{aligned}$$

Hence, by Lemma 2.2 and Definition 2.3, $\alpha^{-1} \blacktriangleleft \mathbb{S}^1$, which shows that $\alpha^{-1} \in \mathcal{R}_cL$. □

Lemma 3.8. *Every maximal ideal of $\mathcal{R}_cL$, is a z_c-ideal.*

Proof. Let I be a maximal ideal of $\mathcal{R}_cL$ and $\gamma \in \mathcal{R}_cL$ be an element with $\mathrm{coz}(\beta) = \mathrm{coz}(\gamma)$, where $\beta \in I$. It suffices to show that $\gamma \in I$. Suppose that $\gamma \notin I$. Since I is maximal, we infer that there exist $\alpha \in \mathcal{R}_cL$ and $\psi \in I$ such that $\mathbf{1} = \psi + \alpha\gamma$. So

$$\begin{aligned} \top &= \mathrm{coz}(\psi + \alpha\gamma) \\ &\leq \mathrm{coz}(\psi) \vee (\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma)) \\ &\leq (\mathrm{coz}(\psi) \vee \mathrm{coz}(\alpha)) \wedge (\mathrm{coz}(\psi) \vee \mathrm{coz}(\gamma)) \\ &\leq \mathrm{coz}(\psi) \vee \mathrm{coz}(\gamma) \\ &= \mathrm{coz}(\psi) \vee \mathrm{coz}(\beta). \end{aligned}$$

Therefore $\mathrm{coz}(\psi) \vee \mathrm{coz}(\beta) = \top$, thus $\mathrm{coz}(\psi^2 + \beta^2) = \top$. So, by Lemma 3.7, $\psi^2 + \beta^2$ is invertible in $\mathcal{R}_cL$ which is a contradiction. Hence $\gamma \in I$ and the proof is complete. □

Proposition 3.9. *For any ideal I in $\mathcal{R}_cL$, $Ann_{\mathcal{R}_cL}(I)$ is a z_c-ideal.*

Proof. Let $\alpha \in \mathcal{R}_cL$, $\beta \in Ann_{\mathcal{R}_cL}(I)$ and $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$. Thus

$$\begin{aligned} \beta\gamma = 0 \quad &\Rightarrow \mathrm{coz}(\beta) \wedge \mathrm{coz}(\gamma) = \bot \\ &\Rightarrow \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma) = \bot \\ &\Rightarrow \mathrm{coz}(\alpha\gamma) = \bot \\ &\Rightarrow \alpha\gamma = 0, \end{aligned}$$

for every $\gamma \in I$. Therefore $\alpha \in Ann_{\mathcal{R}_cL}(I)$. □

Remark 3.10. Let I be a z_c-ideal and $\alpha, \beta \in \mathcal{R}_cL$. If $\alpha^2 + \beta^2 \in I$, then $\alpha, \beta \in I$. For we have

$$\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2).$$

Since I is a z_c-ideal, we conclude that $\alpha, \beta \in I$.

Definition 3.11. *Let L be a frame. We define:*

$$\mathrm{Coz}_c[L] := \{\mathrm{coz}(\alpha) : \alpha \in \mathcal{R}_cL\}.$$

Proposition 3.12. *The following statements hold for any frame L.*

(1) *If I is a proper ideal of $\mathcal{R}_cL$, then $\mathrm{Coz}_c[I]$ is a proper ideal of $\mathrm{Coz}_c[L]$.*
(2) *If I is a proper ideal of $\mathrm{Coz}_c[L]$, then $\mathrm{Coz}_c^{\leftarrow}[I]$ is a proper ideal of $\mathcal{R}_cL$.*
(3) *If M is a maximal ideal of $\mathcal{R}_cL$, then $\mathrm{Coz}_c[M]$ is a maximal ideal of $\mathrm{Coz}_c[L]$.*
(4) *If M is a maximal ideal of $\mathrm{Coz}_c[L]$, then $\mathrm{Coz}_c^{\leftarrow}[M]$ is a maximal ideal of $\mathcal{R}_cL$.*

Proof. (1). Let I be a proper ideal of $\mathcal{R}_cL$ and $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}_c[I]$. Then

$$\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2) \in \mathrm{Coz}_c[I].$$

Thus $\mathrm{Coz}_c[I]$ is directed. Now, assume $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathrm{Coz}_c[I]$ and $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$. Then

$$\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta) \in \mathrm{Coz}_c[I].$$

Therefore $\mathrm{Coz}_c[I]$ is a downset and so $\mathrm{Coz}_c[I]$ is an ideal of $\mathrm{Coz}_c[L]$. If $\mathrm{Coz}_c[I]$ is not proper, there is $\gamma \in I$ such that $\mathrm{coz}(\gamma) = \top$. Thus $\gamma \in I$ is invertible, that is a contradiction.

(2). Consider $\alpha, \beta \in \mathrm{Coz}_c^{\leftarrow}[I]$, then $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in I$. Since I is an ideal of $\mathrm{Coz}_c[L]$, we have $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in I$. Therefore $\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) \in I$ implies that $\mathrm{coz}(\alpha + \beta) \in I$. So $\alpha + \beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Now, assume $\alpha \in \mathrm{Coz}_c^{\leftarrow}[I]$ and $\gamma \in \mathcal{R}_cL$. Then, $\mathrm{coz}(\alpha) \in I$ and $\mathrm{coz}(\gamma) \in \mathrm{Coz}_c\, L$. Also

$$\mathrm{coz}(\alpha) \geq \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha\gamma).$$

Thus, $\mathrm{coz}(\alpha\gamma) \in I$ and so, $\alpha\gamma \in \mathrm{Coz}_c^{\leftarrow}[I]$. If $\mathrm{Coz}_c^{\leftarrow}[I]$ is not proper, there is an invertible element $\beta \in \mathcal{R}_c L$ such that $\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Therefore $\top = \mathrm{coz}(\beta) \in I$, which is a contradiction.

(3). Let M be a maximal ideal of $\mathcal{R}_c L$ and J be a proper ideal of $\mathrm{Coz}_c[L]$ such that $\mathrm{Coz}_c[M] \subseteq J$. Since M is maximal, we conclude from Lemma 3.8 that $M = \mathrm{Coz}_c^{\leftarrow}[Coz[M]]$. Now

$$M = \mathrm{Coz}_c^{\leftarrow}[\mathrm{Coz}_c[M]] \subseteq \mathrm{Coz}_c^{\leftarrow}[J] \subseteq \mathrm{Coz}_c[L].$$

Since M is maximal, we infer that $M = \mathrm{Coz}_c^{\leftarrow}[J]$, so $\mathrm{Coz}_c[M] = J$.

(4). Assume $\alpha \notin \mathrm{Coz}_c^{\leftarrow}[M]$. Then $\mathrm{coz}(\alpha) \notin M$, and so there is $b \in M$ such that $\mathrm{coz}(\alpha) \vee b = \top$. Since M is an ideal of $\mathrm{Coz}_c[L]$, we can choose $\gamma \in \mathcal{R}_c L$ such that $\mathrm{coz}(\gamma) = b$. Then

$$\top = \mathrm{coz}(\alpha) \vee b = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha^2) \vee \mathrm{coz}(\gamma^2) = \mathrm{coz}(\alpha^2 + \gamma^2),$$

which implies that $\alpha^2 + \gamma^2$ is invertible in $\mathcal{R}_c L$, by Lemma 3.7. Therefore for every $\alpha \in \mathcal{R}_c L \setminus \mathrm{Coz}_c^{\leftarrow}[M]$, the ideal $< \alpha, \mathrm{Coz}_c^{\leftarrow}[M] >$ is not a proper ideal of $\mathcal{R}_c L$. Hence $\mathrm{Coz}_c^{\leftarrow}[M]$ is a maximal ideal of $\mathcal{R}_c L$. □

In [24], Mason shows that if I and J are z-ideals, then IJ is a z-ideal precisely when $IJ = I \cap J$. In $\mathcal{R}L$, just as in $C(X)$, the product of two z-ideals is always a z-ideal. We study this result in $\mathcal{R}_c L$ as we show next. To do this, we utilize the following lemma.

Lemma 3.13. *Let $\alpha \in \mathcal{R}L$ and $\rho_3 : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ by $\rho_3(p,q) = (p^3, q^3)$. Then*

(1) $\rho_3 \in \mathcal{R}(\mathcal{L}(\mathbb{R}))$.
(2) $\rho_3^3 = id_{\mathcal{L}(\mathbb{R})}$.
(3) $(\alpha \circ \rho_3)^3 = \alpha$.
(4) $\mathrm{coz}(\alpha \circ \rho_3) = \mathrm{coz}(\alpha)$.
(5) *If $\alpha \in \mathcal{R}_c L$, then $\alpha \circ \rho_3 \in \mathcal{R}_c L$.*

Proof. (1). We check the conditions (R1)-(R4).

(R1). Let $(p,q),(r,s) \in \mathcal{L}(\mathbb{R})$. Then

$$\begin{aligned}\rho_3(p,q) \wedge \rho_3(r,s) &= (p^3,q^3) \wedge (r^3,s^3)\\ &= (\max\{p^3,r^3\}, \min\{q^3,s^3\})\\ &= ((\max\{p,r\})^3, (\min\{q,s\})^3)\\ &= \rho_3(p \vee r, q \wedge s).\end{aligned}$$

(R2). Assume $p \leq r < q \leq s \in \mathbb{Q}$. Then

$$\rho_3(p,q) \vee \rho_3(r,s) = (p^3,q^3) \vee (r^3,s^3) = (p^3,s^3) = \rho_3(p,s),$$

because $p^3 \leq r^3 < q^3 \leq s^3$.

(R3). We trivially have

$$\begin{aligned}\bigvee\{\rho_3(r,s) : p < r < s < q\} &= \bigvee\{(r^3,s^3) : p < r < s < q\}\\ &= \bigvee\{(r^3,s^3) : p^3 < r^3 < s^3 < q^3\}\\ &= (p^3,q^3)\\ &= \rho_3(p,q).\end{aligned}$$

(R4). We have

$$\bigvee\{\rho_3(p,q) : p,q \in \mathbb{Q}\} = \bigvee\{(p^3,q^3) : p,q \in \mathbb{Q}\} = \top.$$

Thus ρ_3 is a frame map, so $\rho_3 \in \mathcal{R}(\mathcal{L}(\mathbb{R}))$.

(2). Consider $(p,q) \in \mathcal{L}(\mathbb{R})$, then

$$\begin{aligned} \rho_3^3(p,q) &= \bigvee\{\rho_3(r_1,s_1) \wedge \rho_3(r_2,s_2) \wedge \rho_3(r_3,s_3) :< r_1,s_1 > . < r_2,s_2 > . < r_3,s_3 > \subseteq < p,q >\} \\ &\geq (p,q). \end{aligned}$$

Thus $\rho_3^3 = id_{\mathcal{L}(\mathbb{R})}$ by regularity of L.

(3). Let $(p,q) \in \mathcal{L}(\mathbb{R})$. Then, we conclude from (2) that

$$(\alpha \circ \rho_3)^3(p,q) = \alpha \circ \rho_3^3(p,q) = \alpha \circ id(p,q) = \alpha(p,q).$$

Hence, $(\alpha \circ \rho_3)^3 = \alpha$.

(4). First, we note that

$$\mathrm{coz}(\rho_3) = \rho_3(-,0) \vee \rho_3(0,-) = (-,0) \vee (0,-).$$

Also, we infer from (3) that $\alpha^{1/3} = \alpha \circ \rho_3$. Therefore

$$\mathrm{coz}(\alpha^{1/3}) = \mathrm{coz}(\alpha \circ \rho_3) = \alpha(\mathrm{coz}(\rho_3)) = \alpha((-,0) \vee (0,-)) = \mathrm{coz}(\alpha).$$

(5). Let $\alpha \in \mathcal{R}_c L$. Then, by Definitions 2.1 and 2.3, there is a countable subset $\mathbb{S} \subseteq \mathbb{R}$ such that $\alpha \blacktriangleleft \mathbb{S}$. Put $\mathbb{S}_0 = \{\sqrt[3]{s} : s \in \mathbb{S}\}$. We show that $\alpha \circ \rho_3 \blacktriangleleft \mathbb{S}_0$. Assume $(p,q), u \in \mathcal{L}(\mathbb{R})$ with $u = \bigvee_{i\in I}(a_i,b_i)$ and $\tau(p,q) \cap \mathbb{S}_0 = \tau(u) \cap \mathbb{S}_0$. Since $\tau(p^3,q^3) \cap \mathbb{S} = \tau(\bigvee(a_i^3,b_i^3)) \cap \mathbb{S}$, we conclude from Lemma 2.2 that $\alpha(p^3,q^3) = \alpha(\bigvee(a_i^3,b_i^3))$, which follows that $\alpha \circ \rho_3(p,q) = \alpha \circ \rho_3(u)$. Thus, by Lemma 2.2, $\alpha \circ \rho_3 \blacktriangleleft \mathbb{S}_0$. Hence $\alpha \circ \rho_3 \in \mathcal{R}_c L$ and the proof is complete. □

Proposition 3.14. *If P and Q are z_c-ideals in $\mathcal{R}_c L$, then $PQ = P \cap Q$.*

Proof. Since $PQ \subseteq P \cap Q$ always holds, we show the reverse inclusion. Let $\alpha \in P \cap Q$. Suppose that ρ_3 be the same in Lemma 3.13. Then, by Lemma 3.13(3,5), we have $\alpha^{1/3} \in \mathcal{R}_c L$ and $\alpha^{1/3}\alpha^{1/3} \in \mathcal{R}_c L$. Also, $\alpha = (\alpha^{1/3})^3 = \alpha^{1/3}\alpha^{2/3}$ and $\mathrm{coz}(\alpha) = \mathrm{coz}(\alpha^{1/3})$. Now, since $\alpha \in P \cap Q$ and P, Q are z_c-ideals, we infer that $\alpha^{1/3} \in P$ and $\alpha^{1/3} \in Q$ Hence, $(\alpha^{1/3})^2 \in Q$. Therefore $\alpha = \alpha^{1/3}(\alpha^{1/3})^2 \in PQ$ and proof is complete. □

Remark 3.15. By [2, Proposition 4], we know that the map

$$\begin{aligned} \theta : \mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathfrak{O}X) &\longrightarrow \mathbf{Top}(X, \mathbb{R}) \\ \varphi &\longmapsto \widetilde{\varphi} \end{aligned}$$

such that $p < \widetilde{\varphi}(x) < q$ if and only $x \in \varphi(p,q)$ is an isomorphism (also, see [5]).

Lemma 3.16. *For any space X, $\mathcal{R}_c(\mathfrak{O}X) \cong C_c(X)$.*

Proof. Define

$$\begin{aligned} \theta|_{\mathcal{R}_c(\mathfrak{O}X)} : \mathcal{R}_c(\mathfrak{O}X) &\longrightarrow C_c(X) \\ \varphi &\longmapsto \widetilde{\varphi} \end{aligned}$$

such that $p < \widetilde{\varphi}(x) < q$ if and only $x \in \varphi(p,q)$.

Consider $\varphi \in \mathcal{R}_c(\mathfrak{O}X)$. Then, by Definitions 2.1 and 2.3, there is a countable subset $\mathbb{S} \subseteq \mathbb{R}$ such that $\varphi \blacktriangleleft \mathbb{S}$. We claim that $Im\widetilde{\varphi} \subseteq \mathbb{S}$. Suppose that $Im\widetilde{\varphi} \nsubseteq \mathbb{S}$ and $y \in Im\widetilde{\varphi} \setminus \mathbb{S}$. So there is an element $x \in X$ such that $y = \widetilde{\varphi}(x)$. Since τ is an isomorphism, there is an element $v \in \mathcal{L}(\mathbb{R})$ such that $\tau(v) = \mathbb{R} \setminus \{y\}$ and also $\tau(\top_{\mathcal{L}(\mathbb{R})}) = \mathbb{R}$. Now, by Definition 2.1, $\tau(v) \cap \mathbb{S} = \tau(\top_{\mathcal{L}(\mathbb{R})}) \cap \mathbb{S}$, it follows that

$$\varphi(v) = \varphi(\top_{\mathcal{L}(\mathbb{R})}) = \varphi(\mathbb{R}) = \top_{\mathfrak{O}X} = X.$$

Thus $x \in X = \varphi(v)$. Therefore $\widetilde{\varphi}(x) \in \mathbb{R} \setminus \{y\}$, which is a contradiction with $\widetilde{\varphi}(x) = y$. Thus $Im\widetilde{\varphi} \subseteq \mathbb{S}$, which follows that $\theta(\varphi) \in C_c(X)$.

Now, we show that $\theta|_{\mathcal{R}_c(\mathfrak{O}X)}$ is onto. Suppose that $f \in C_c(X)$. Then $Imf := \mathbb{S}$ is a countable subset of $\mathbb{R}$. By Remark 3.15, θ is onto implies that there is $\varphi \in \mathcal{R}(\mathfrak{O}X)$ such that $\theta(\varphi) = f$. We claim that $\varphi \in \mathcal{R}_c(\mathfrak{O}X)$. Assume $(a, b), v \in \mathcal{L}(\mathbb{R})$ with $v = \bigvee_{\lambda\in\Lambda}(a_\lambda, b_\lambda)$ and $\tau(a, b) \cap \mathbb{S} \subseteq \tau(v) \cap \mathbb{S}$. Therefore,

$$\begin{aligned} x \in \varphi(a, b) &\Rightarrow a < f(x) < b \\ &\Rightarrow f(x) \in \tau(a, b) \cap \mathbb{S} \\ &\Rightarrow f(x) \in \tau(v) \cap \mathbb{S}. \end{aligned}$$

Since $\tau(v)$ is an open subset of $\mathbb{R}$, there is $p, q \in \mathbb{Q}$ such that

$$f(x) \in \tau(p, q) \cap \mathbb{S} \subseteq \tau(v) \cap \mathbb{S}$$

and hence $x \in \varphi(p, q) \leq \varphi(v)$. Thus $x \in \varphi(v)$, so $\varphi(a, b) \subseteq \varphi(v)$. Now, by Lemma 2.2 and Definition 2.3, $\varphi \in \mathcal{R}_c(\mathfrak{O}X)$. Therefore, by Remark 3.15, $\theta|_{\mathcal{R}_c(\mathfrak{O}X)}$ is an isomorphism and hence $\mathcal{R}_c(\mathfrak{O}X) \cong C_c(X)$. □

Remark 3.17. Recall from [9] that we denote by t_L the ring isomorphism

$$\mathrm{t}_L : \mathcal{R}\beta L \to \mathcal{R}^*L \qquad \text{given by} \qquad \mathrm{t}_L(\alpha) = j_L(\alpha),$$

the inverse of which we will denote by $\varphi \mapsto \varphi^\beta$. It is also important to note that $\bigvee \alpha^\beta(p, q) = \alpha(p, q)$, for all $p, q \in \mathbb{Q}$.

Lemma 3.18. *For any frame L, $\mathcal{R}^*_cL \cong \mathcal{R}_c\beta L$, where $\mathcal{R}^*_cL = \mathcal{R}_cL \cap \mathcal{R}^*L$*

Proof. We define

$$\begin{aligned} \mathrm{t}_L|_{\mathcal{R}_c\beta L} : \mathcal{R}_c\beta L &\longrightarrow \mathcal{R}^*_cL \\ \alpha &\longmapsto j_L \circ \alpha \end{aligned}$$

Consider $\alpha \in \mathcal{R}_c\beta L$. So, by Definitions 2.1 and 2.3, there is a countable subset $\mathbb{S} \subseteq \mathbb{R}$ such that $\alpha \blacktriangleleft \mathbb{S}$. Assume $(p, q), v \in \mathcal{L}(\mathbb{R})$, and $\tau(p, q) \cap \mathbb{S} = \tau(v) \cap \mathbb{S}$. Then we conclude from Lemma 2.2 that

$$\begin{aligned} \alpha(p, q) = \alpha(v) &\Rightarrow j_L \circ \alpha(p, q) = j_L \circ \alpha(v) \\ &\Rightarrow \mathrm{t}_L|_{\mathcal{R}_c\beta L}(\alpha)(p, q) = \mathrm{t}_L|_{\mathcal{R}_c\beta L}(\alpha)(v) \end{aligned}$$

Thus, by Lemma 2.2, $\mathrm{t}_L(\alpha) \blacktriangleleft \mathbb{S}$.

Now, suppose that $\alpha \in \mathcal{R}^*_cL$. Then there is a countable subset $\mathbb{S} \subseteq \mathbb{R}$ such that $\alpha \blacktriangleleft \mathbb{S}$. Let $(p, q), v \in \mathcal{L}(\mathbb{R})$ and $\tau(p, q) \cap \mathbb{S} = \tau(v) \cap \mathbb{S}$. Then we conclude from Lemma 2.2 that

$$\begin{aligned} \alpha(p, q) = \alpha(v) &\Rightarrow \bigvee \alpha^\beta(p, q) = \bigvee \alpha^\beta(v) \\ &\Rightarrow \alpha^\beta(p, q) = \alpha^\beta(v). \qquad \text{(since } \beta L \text{ is compact)} \end{aligned}$$

Therefore $\alpha^\beta = \mathrm{t}_L^{-1}|_{\mathcal{R}_c\beta L}(\alpha) \in \mathcal{R}_c\beta L$. Hence $\mathrm{t}_L(\alpha^\beta) = \bigvee \alpha^\beta = \alpha$, which shows that $\mathrm{t}_L|_{\mathcal{R}_c\beta L}$ is onto. Consequently, by Remark 3.17, $\mathrm{t}_L|_{\mathcal{R}_c\beta L}$ is an isomorphism. □

We shall study the relation between z_c-ideal and prime ideal minimal over an ideal. For this, we recall that in [16, 1D] the following results play a useful role in the context of $C(X)$. It is shown that the pointfree version of this results is also true (see [19]). The following results are the counterpart for $\mathcal{R}_cL$.

Lemma 3.19. *Let $\alpha, \beta \in \mathcal{R}_cL$. If $|\alpha| \leq |\beta|^q$ for some $q > 1$, then α is a multiple of β. In particular, if $|\alpha| \leq |\beta|$, then whenever α^q is defined for every $q > 1$, α^q is a multiple of β.*

Proof. Multiply by $\frac{1}{1+|\alpha|}.(\frac{1}{1+|\beta|})^q$ both sides of the stated inequality to obtain

$$\frac{\alpha}{1+|\alpha|}.(\frac{1}{1+|\beta|})^q \leq \frac{1}{1+|\alpha|}.(\frac{|\beta|}{1+|\beta|})^q.$$

Since of each of the factors in this inequality is in $\mathcal{R}^*_cL$, and by Corollaries 3.16 and 3.18, $\mathcal{R}^*_cL$ is isomorphic to a $C_c(X)$ via an f-ring isomorphism, we deduce from [15, Corollary 2.5], that $\frac{\alpha}{1+|\alpha|}$ is a multiple of $\frac{|\beta|}{1+|\beta|}$. This implies α is a multiple of β, as desired. □

Proposition 3.20. *Let Q be an ideal of $\mathcal{R}_cL$, and $\alpha \in \mathcal{R}_cL$. If $\mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq \sqrt{Q}$, then $\mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq Q$.*

Proof. Let $\beta \in \mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq \sqrt{Q}$. Without loss of generality, we assume that $|\beta| \leq 1$. We define $\gamma = \Sigma^{\infty}_{n=1} 2^{-n}.\beta^{\frac{1}{n}}$. Hence

$$\begin{aligned} \mathrm{coz}(\gamma) &= \bigvee_n \mathrm{coz}(2^{-n}.\beta^{\frac{1}{n}}) \\ &= \bigvee_n (\mathrm{coz}(2^{-n}) \wedge \mathrm{coz}(\beta^{\frac{1}{n}})) \\ &= \bigvee_n \mathrm{coz}(\beta^{\frac{1}{n}}) \\ &= \mathrm{coz}(\beta). \end{aligned}$$

Since $\mathrm{coz}(\gamma) = \mathrm{coz}(\beta)$ and $\mathbf{M}^c_{\mathrm{coz}(\alpha)}$ is a z_c-ideal, then $\gamma \in \mathbf{M}^c_{\mathrm{coz}(\alpha)}$. Hence $\gamma \in \sqrt{Q}$ and hence there is $m \in \mathbb{N}$ such that $\gamma^m \in Q$. Furthermore, since $2^{-n}.\beta^{\frac{1}{n}} \leq \gamma$, for every $n \in \mathbb{N}$, we have $2^{-2m}.\beta^{\frac{1}{2m}} \leq \gamma$ which implies that $(2^{-2m}.\beta^{\frac{1}{2m}})^m \leq \gamma^m$ and hence $2^{-2m^2}.\beta^{\frac{1}{2}} \leq \gamma^m$. Therefore, by Lemma 3.19, there exists $\tau \in \mathcal{R}_cL$ such that $\beta = \tau.\gamma^m$. This shows that $\beta \in Q$, and hence $\mathbf{M}^c_{\mathrm{coz}(\alpha)} \subseteq Q$. □

Corollary 3.21. *An ideal of $\mathcal{R}_cL$ is a z_c-ideal if and only if its radical is a z_c-ideal.*

Proof. $(\Rightarrow)$: It is evident.

$(\Leftarrow)$: Let Q be an ideal of $\mathcal{R}_cL$. Suppose that for $\alpha, \beta \in \mathcal{R}_cL$, $\alpha \in Q$ and $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$. Since $\sqrt{Q}$ is a z_c-ideal, $\beta \in \sqrt{Q}$. By Proposition 3.20, $\mathbf{M}^c_{\mathrm{coz}(\beta)} \subseteq \sqrt{Q}$ and hence $\mathbf{M}^c_{\mathrm{coz}(\beta)} \subseteq Q$. Since $\beta \in \mathbf{M}^c_{\mathrm{coz}(\beta)} \subseteq Q$, it implies that $\beta \in Q$. Therefore Q is a z_c-ideal. □

Corollary 3.22. *Let Q be an ideal of $\mathcal{R}_cL$. Then Q is a z_c-ideal if and only if every prime ideal minimal over it is a z_c-ideal.*

Proof. Suppose every prime ideal minimal over Q is a z_c-ideal. Then, by Corollary 3.21, it is sufficient to show that $\sqrt{Q}$ is a z_c-ideal. We know that $\sqrt{Q}$ is the intersection of prime ideals minimal over Q . Hence $\sqrt{Q}$ is an intersection of z_c-ideals, thus it is a z_c-ideal.

Conversely, let Q be a z_c-ideal and $P \in Min(Q)$. Consider $\alpha, \beta \in \mathcal{R}_cL$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$, $\alpha \in P$ and $\beta \notin P$. We put

$$S = (\mathcal{R}_cL \setminus P) \bigcup \{\gamma\alpha^n : \gamma \in \mathcal{R}_cL \setminus P \ , \ n \in \mathbb{N}\}.$$

It is clear that S is a multiplicatively closed set of $\mathcal{R}_cL$. If $\varphi \in S \cap Q$, then there are $n \in \mathbb{N}$ and $\gamma \in \mathcal{R}_cL \setminus P$ such that $\varphi = \gamma\alpha^n \in Q \subseteq P$. We have

$$\mathrm{coz}(\varphi) = \mathrm{coz}(\gamma\alpha^n) = \mathrm{coz}(\gamma) \wedge \mathrm{coz}(\alpha) = \mathrm{coz}(\gamma) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\gamma\beta).$$

From Q is a z_c-ideal and $\varphi \in Q$, we conclude that $\gamma\beta \in Q \subseteq P$, which follows that $\gamma \in P$ or $\beta \in P$. That is a contradiction. Therefore $S \cap Q = \emptyset$. By [28, Theorem 3.44], there exists a prime ideal $P' \in \mathcal{R}_cL$ such that $S \cap P' = \emptyset$ and $Q \subseteq P'$. Now, if $\varphi \in P'$, then $\varphi \notin S$, it implies that $\varphi \in P$. Thus $Q \subseteq P' \subseteq P$ and since $P \in Min(Q)$, we infer that $P' = P$. We have $\alpha \in P = P'$ and $\alpha \in S$, and so $\alpha \in P'$ and $\alpha \notin P'$, which is a contradiction. □

Now, we discuss on the z_c-ideals of $\mathcal{R}_cL$ and contraction of z-ideals of $\mathcal{R}L$.

Proposition 3.23. *An ideal J in $\mathcal{R}_cL$ is a z_c-ideal if and only if it is a contraction of a z-ideal in $\mathcal{R}L$.*

Proof. Suppose that J is a z_c-ideal of $\mathcal{R}_cL$. Put

$$I = \{\alpha \in \mathcal{R}L : \mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta), \quad \text{for some } \beta \in J\}.$$

Clearly, I is a z-ideal in $\mathcal{R}L$ and $J \subseteq I^c$. On the other hand, if $\alpha \in I^c$, there exists $\beta \in J$ with $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$. Since J is z_c-ideal, we conclude that $\alpha \in J$, as desired.

Conversely, let $J = I^c$, where I is a z-ideal in $\mathcal{R}L$. Then J is clearly a z_c-ideal in $\mathcal{R}_cL$. □

Corollary 3.24. *An ideal P in $\mathcal{R}_cL$ is a prime z_c-ideal if and only if it is a contraction of a prime z-ideal in $\mathcal{R}L$.*

Proof. Let P be a prime z_c-ideal in $\mathcal{R}_cL$. Consider $S = \mathcal{R}_cL \setminus P$ as a multiplicatively closed set in $\mathcal{R}L$. By Proposition 3.23, P is a contraction of a z-ideal in $\mathcal{R}L$, I say. Clearly, $I \cap S = \emptyset$, so there is a prime ideal $Q \in \mathcal{R}L$ minimal over I with $Q \cap S = \emptyset$. Now, from [25] we have that Q is a z-ideal in $\mathcal{R}L$. It is evident that $P = I^c \subseteq Q^c \subseteq P$. Therefore $P = Q^c$, as desired. The converse is evident. □

Corollary 3.25. *Every maximal ideal N of $\mathcal{R}_cL$ is a contraction of a maximal ideal in $\mathcal{R}L$.*

Proof. Let N be a maximal ideal in $\mathcal{R}_cL$. By Lemma 3.8, N is a z_c-ideal. Hence, from Proposition 3.23, we infer that $N = I^c$, where I is a z-ideal in $\mathcal{R}L$. But there is a maximal ideal M in $\mathcal{R}L$ containing I. Therefore $N = I^c \subseteq M^c$ implies that $N = M^c$ and we are done. □

We shall see the relation between z_c-ideals in $\mathcal{R}_cL$ and z-ideal á la Mason.

For $\alpha \in \mathcal{R}_cL$, we put $\mathfrak{M}_c(\alpha) := \{M \in Max(\mathcal{R}_cL) : \alpha \in M\}$.

Lemma 3.26. *For $\alpha, \beta \in \mathcal{R}_cL$, the following statements are equivalent:*

(1) $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$.
(2) $\mathbf{M}^c_{\mathrm{coz}(\beta)} \subseteq \mathbf{M}^c_{\mathrm{coz}(\alpha)}$.
(3) $\mathfrak{M}_c(\alpha) \subseteq \mathfrak{M}_c(\beta)$.

Proof. (1) ⇒ (2). It is evident.

(2) ⇒ (3). Suppose that $M \in \mathfrak{M}_c(\alpha)$. Then, by Proposition 3.12, $\mathrm{Coz}_c[M]$ is a maximal ideal of $\mathrm{Coz}_c[L]$ such that $\mathrm{coz}(\alpha) \in \mathrm{Coz}_c[M]$. By hypothesis, $\mathrm{coz}(\beta) \in Coz[M]$. So, by Proposition 3.12, $\beta \in \mathrm{Coz}_c^{\leftarrow}[\mathrm{Coz}_c[M]] = M$. Thus $M \in \mathfrak{M}_c(\beta)$. Hence $\mathfrak{M}_c(\alpha) \subseteq \mathfrak{M}_c(\beta)$.

(3) ⇒ (1). By Corollary 3.25, we have

$$\mathfrak{M}_c(\alpha) = \{M^c : M \in \mathfrak{M}(\alpha)\} \text{ and } \mathfrak{M}_c(\beta) = \{M^c : M \in \mathfrak{M}(\beta)\}.$$

Suppose that $M \in \mathfrak{M}(\alpha)$. Then, by (3), we have $M^c \in \mathfrak{M}_c(\alpha) \subseteq \mathfrak{M}_c(\beta)$, which follows that $M \in \mathfrak{M}(\beta)$. Thus $\mathfrak{M}(\alpha) \subseteq \mathfrak{M}(\beta)$, and so $\beta \in \bigcap \mathfrak{M}(\beta) \subseteq \bigcap \mathfrak{M}(\alpha)$. Now, from [8, Lemma 3.7] and [23, Lemma 3.1], we have $\beta \in \bigcap \mathfrak{M}(\alpha) = \{\varphi \in \mathcal{R}L : \mathrm{coz}(\varphi) \leq \mathrm{coz}(\alpha)\}$. Therefore $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$. □

Proposition 3.27. *An ideal I in $\mathcal{R}_cL$ is a z_c-ideal if and only if it is a z-ideal á la Mason.*

Proof. Let I be a z_c-ideal and suppose that $\alpha, \beta \in \mathcal{R}_cL$ such that $\mathfrak{M}(\alpha) \subseteq \mathfrak{M}(\beta)$ and $\alpha \in I$. Since $\mathfrak{M}_c(\alpha) \subseteq \mathfrak{M}_c(\beta)$, we conclude by Lemma 3.26 that $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$, which follows that $\beta \in I$, because I is a z_c-ideal. Therefore I is a z-ideal á la Mason.

Conversely, let I be a z-ideal á la Mason. Suppose that $\mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha)$ and $\alpha \in I$. Then, by Lemma 3.26, $\mathfrak{M}_c(\alpha) \subseteq \mathfrak{M}_c(\beta)$, which follows that $\mathfrak{M}(\alpha) \subseteq \mathfrak{M}(\beta)$. Therefore, we have $\beta \in I$ because I is a z-ideal á la Mason. □

4. The relation between z_c-ideals and prime ideals

In this section, we study the relation between prime ideals and z_c-ideals in the ring $\mathcal{R}_cL$. We begin by some evident instances.

Lemma 4.1. *Let I be a proper ideal and P be a prime ideal in $\mathcal{R}_cL$. If $I \cap P$ is a z_c-ideal and $I \nsubseteq P$, then P is a z_c-ideal.*

Proof. Let $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ where $\alpha \in P$ and $\beta \in \mathcal{R}_cL$. Since $I \nsubseteq P$, there is $\gamma \in I \setminus P$. But $\mathrm{coz}(\alpha\gamma) = \mathrm{coz}(\beta\gamma)$ and $\alpha\gamma \in P \cap I$. Since $P \cap I$ is a z_c-ideal, it follows that $\beta\gamma \in P \cap I$. So $\beta\gamma \in P$, we infer that $\beta \in P$ (since P is a prime ideal). Hence P is a z_c-ideal. □

Corollary 4.2. *Let I be an ideal and P be a prime ideal in $\mathcal{R}_cL$ such that $P \cap I$ is a z_c-ideal. Then I or P is a z_c-ideal.*

Proof. If $I \nsubseteq P$, then we conclude from Lemma 4.1 that P is a z_c-ideal. If $I \subseteq P$, then we have $I \cap P = I$. Hence, by assumptions, I is a z_c-ideal. □

Corollary 4.3. *Let P and Q be two prime ideals in $\mathcal{R}_cL$ that are not in a chain. If $P \cap Q$ is a z_c-ideal, then either P or Q are z_c-ideals.*

Proof. Let $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ where $\alpha \in P$ and $\beta \in \mathcal{R}_cL$. As P and Q are not the chain, so $Q \nsubseteq P$ and $P \nsubseteq Q$. Since $Q \nsubseteq P$, there is $\gamma \in Q \setminus P$. But $\mathrm{coz}(\alpha\gamma) = \mathrm{coz}(\beta\gamma)$, $\alpha\gamma \in P \cap Q$. Since $P \cap Q$ is a z_c-ideal, it follows that $\beta\gamma \in P \cap Q$. So $\beta\gamma \in P$, we infer that $\beta \in P$ (since P is prime). Hence P is a z_c-ideal. Similarly to prove that Q is a z_c-ideal. □

It is well known in the classical situation that a z-ideal of $C(X)$ is prime if and only if it contains a prime ideal (see [16, Theorem 2.9]). It is shown that the pointfree version of this result is also true (see [6]). If we apply the proof of [23, Lemma 4.8] word-for-word, we obtain the following for $\mathcal{R}_cL$.

Proposition 4.4. *Let I be a proper z_c-ideal in $\mathcal{R}_cL$. The following statements are equivalent:*

(1) *I is a prime ideal in $\mathcal{R}_cL$.*
(2) *I contains a prime ideal in $\mathcal{R}_cL$.*
(3) *For all $\alpha, \beta \in \mathcal{R}_cL$, if $\alpha\beta = \mathbf{0}$, then $\alpha \in I$ or $\beta \in I$.*
(4) *Given $\alpha \in \mathcal{R}_cL$, there exists a cozero element $a \in \mathrm{Coz}_c[I]$ such that*

$$\alpha(0, -) \leq a \text{ or } \alpha(-, 0) \leq a.$$

Corollary 4.5. *Let I be a proper ideal of $\mathrm{Coz}_c[L]$ such that for every $\alpha, \beta \in \mathcal{R}_cL$, $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ implies that $\mathrm{coz}(\alpha) \in I$ or $\mathrm{coz}(\beta) \in I$. Then the following statements hold:*

(1) *$\mathrm{Coz}_c^{\leftarrow}[I]$ is a prime z_c-ideal of $\mathcal{R}_cL$.*
(2) *I is a prime ideal of $\mathrm{Coz}_c[L]$.*

Proof. (1). Let $\alpha, \beta \in \mathcal{R}_cL$ and $\alpha\beta = \mathbf{0}$. Then $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and, by assumption, $\mathrm{coz}(\alpha) \in I$ or $\mathrm{coz}(\beta) \in I$. This means that $\alpha \in \mathrm{Coz}_c^{\leftarrow}[I]$ or $\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Since $\mathrm{Coz}_c^{\leftarrow}[I]$ is a z_c-ideal of $\mathcal{R}_cL$, by Proposition 4.4, $\mathrm{Coz}_c^{\leftarrow}[I]$ is a prime z_c-ideal of $\mathcal{R}_cL$.

(2). Let $\alpha, \beta \in \mathcal{R}_cL$ and $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \in I$. Then $\alpha\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$ and, by (1), $\alpha \in \mathrm{Coz}_c^{\leftarrow}[I]$ or $\beta \in \mathrm{Coz}_c^{\leftarrow}[I]$. Hence $\mathrm{coz}(\alpha) \in I$ or $\mathrm{coz}(\beta) \in I$. Thus I is a prime ideal of $\mathrm{Coz}_c[L]$. □

In proof of Proposition 4.6, we use this fact: Let J, J' be two ideals. If $J \cap J'$ is prime then either $J \subseteq J'$ or $J' \subseteq J$. About the following proposition, we must say that it was established by Dube in [7] in the context of $\mathcal{R}L$.

Proposition 4.6. *Every prime ideal of $\mathcal{R}_cL$ is included in a unique maximal ideal.*

Proof. We know that every prime ideal is included in at least one maximal ideal. Let M and M' be two distinct maximal ideals. Then, by Lemma 3.8 and Remark 3.3, $M \cap M'$ is a z_c-ideal. But it is not prime, by Proposition 4.4, $M \cap M'$ contains no prime ideal. □

A commutative ring with identity is called *Gelfand ring* [20] if every prime ideal is contained in a unique maximal ideal. In [7], Dube shows that $\mathcal{R}L$ is a Gelfand ring. As a result of Proposition 4.6, we have the following.

Corollary 4.7. *$\mathcal{R}_cL$ is a Gelfand ring.*

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