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Local K-Convoluted C-Semigroups and Complete Second Order Abstract Cauchy Problems

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Abstract. Let $C : X \to X$ be a bounded linear operator on a Banach space X over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$, and $K : [0, T_0) \to \mathbb{F}$ a locally integrable function for some $0 < T_0 \leq \infty$. Under some suitable assumptions, we deduce some relationship between the generation of a local (or an exponentially bounded) K-convoluted $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and one of the following cases: (i) the well-posedness of a complete second-order abstract Cauchy problem ACP(A, B, f, x, y): $w''(t) = Aw'(t) + Bw(t) + f(t)$ for a.e. $t \in (0, T_0)$ with $w(0) = x$ and $w'(0) = y$; (ii) a Miyadera-Feller-Phillips-Hille-Yosida type condition; (iii) B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X for which A may not be bounded; (iv) A is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on X for which B may not be bounded.

1. Introduction

Let X be a non-trivial Banach space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with norm $\|\cdot\|$, and let L(X) denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following two abstract Cauchy problems:

$$\text{ACP}(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ u(0) = x \end{cases}$$

and

$$\text{ACP}(A, B, f, x, y) \qquad \begin{cases} w''(t) = Aw'(t) + Bw(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ w(0) = x, w'(0) = y, \end{cases}$$

where $x, y \in X$, $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ are closed linear operators, and $f \in L^1_{loc}([0, T_0), X)$ (the family of all locally integrable functions from $[0, T_0)$ into X). A function u is called a (strong) solution

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of ACP(A, f, x) if $u \in C([0, T_0), X)$ satisfies ACP(A, f, x) (that is $u(0) = x$ and for a.e. $t \in (0, T_0)$, $u(t)$ is differentiable and $u(t) \in D(A)$, and $u'(t)=Au(t)+f(t)$ for a.e. $t \in (0, T_0)$). For each $C \in L(X)$ and $K \in L^1_{loc}([0, T_0), \mathbb{F})$, a subfamily $S(\cdot)(= \{S(t) \,|\, 0 \le t < T_0\})$ of $L(X)$ is called a local K-convoluted C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x = [\int_0^{t+s} - \int_0^t - \int_0^s]K(t + s - r)S(r)Cxdr \tag{1}$$

for all $0 \le t, s, t+s < T_0$ and $x \in X$ (see [10,11,15]). In particular, $S(\cdot)$ is called a local (0-times integrated) C-semigroup on X if $K = j_{-1}$ (the Dirac measure at 0) or equivalently, $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x = S(t + s)Cx \quad \text{for all } 0 \le t, s, t + s < T_0 \text{ and } x \in X \tag{2}$$

(see [2,5,23,33,36-37]). Moreover, we say that $S(\cdot)$ is nondegenerate if $x = 0$ whenever $S(t)x = 0$ for all $0 \le t < T_0$ or exponentially bounded if $T_0 = \infty$ and there exist M, $\omega > 0$ such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$. The nondegeneracy of a local K-convoluted C-semigroup $S(\cdot)$ on X implies that $S(0) = C$ if $K = j_{-1}$, and $S(0) = 0$ (zero operator on X) otherwise, and the (integral) generator $A : D(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in X defined by $D(A) =\{x \in X \,|\, S(\cdot)x - K_0(\cdot)Cx=\widetilde{S}(\cdot)y_x$ on $[0, T_0)$ for some $y_x \in X\}$ and $Ax = y_x$ for all $x \in D(A)$. Here $K_\beta(t) = K * j_\beta(t) = \int_0^t K(t - s)j_\beta(s)ds$ for $\beta > -1$ with $j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$, $\Gamma(\cdot)$ denotes the Gamma function, and $\widetilde{S}(t)z = \int_0^t S(s)zds$. In general, a local K-convoluted C-semigroup on X is called a K-convoluted C-semigroup on X if $T_0 = \infty$ (see [10,11]); a (local) K-convoluted C-semigroup on X is called a (local) K-convoluted semigroup on X if $C = I$ (identity operator on X) or a (local) α-times integrated C-semigroup on X if $K = j_{\alpha-1}$ for some $\alpha \ge 0$ (see [1,2,8,12,16, 21-25,27-32,36,38-39,42]). It is known that the theory of local α-times integrated C-semigroup is related to another family in $L(X)$ which is called a local α-times integrated C-cosine function (see [1,10,12,17,22,39]). Perturbation of local K-convoluted C-semigroups have been extensively studied by many authors (see [1,13-14,18-19,25,38-39 for the case $K = j_{\alpha-1}$ for some $\alpha \ge 0$, and [10] for the general case). Some basic properites of a nondegenerate (local) K-convoluted C-semigroup on X have been established by many authors in [2,5,33,36] for the case $K = j_{\alpha-1}$ with $\alpha = 0$, in [16,23] with $\alpha > 0$ is arbitrary, and in [10,15] for the general case. In section 2, we will apply the conclusion of [15, Theorem 3.7] to show that $\mathcal{T}$ is a subgenerator of a local K-convoluted $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w which depends contiouously differentiable on (x, y), and satisfies $Bw + Aw' \in C([0, T_0), X)$ (see Theorem 2.3 below). Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and $\mathcal{D}$ is a fixed subspace of $D(B) \times D(A)$ that is dense in $X \times X$. We then show that $\mathcal{T}$ is a subgenerator of an exponentially bounded K-convoluted $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ so that $\lambda \in \rho_C(A, B)$ and

$$\|[\hat{K}(\lambda)\lambda(\lambda^2 - \lambda A - B)^{-1}C]^{(k)}\|, \|[\hat{K}(\lambda)\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}}]^{(k)}\| \le \frac{Mk!}{(\lambda - \omega)^{k+1}} \tag{3}$$

for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$ if and only if there exist $M, \omega > 0$ so that for each pair $x, y \in D(B) \cap D(A)$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and $Bw + Aw' \in C([0, \infty), X)$ (see Corollary 2.4 and Theorem 2.7 below). Here $\rho_C(A, B)=\{\lambda \in \mathbb{F} \,|\, \lambda^2 - \lambda A - B$ is injective, $R(C) \subset R(\lambda^2 - \lambda A - B)$, and $(\lambda^2 - \lambda A - B)^{-1}C \in L(X)\}$. When $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty, we can apply a modification of [15, Corollary 3.6] (see Theorem 2.2 below) to obtain that $\mathcal{T}$ is the generator of a local K-convoluted $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w with $Bw + Aw' \in C([0, T_0), X)$ (see Theorem 2.9 below). In section 3, we will apply the modifications of [13, Theorems 2.10, 2.12 and Theorems 3.1-3.2] concerning the bounded and unbounded perturbations of a local α-times integrated C-semigroup on X with or without the local Lipschitz continuity (see Theorems 3.1-3.2 and 3.15-3.16 below) and a basic property of local α-times integrated C-cosine function (see [10, Theorem 2.1.11]) to obtain two new equivalence conditions concerning the generations of a local α-times integrated C-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and either a locally Lipschitz continuous local α-times integrated C-cosine function on X with subgenerator (resp., the generator) B for which A may not be bounded (see Theorems 3.4-3.5 and 3.7-3.8 below) or a local α-times integrated C-semigroup on X with subgenerator (resp., the generator) A for which B may not be bounded (see Theorems 3.12-3.13 and 3.17-3.18 below). Under some suitable assumptions, which can be used to show those preceding equivalence conditions which are equivalent to B is the generator of a locally Lipschitz continuous local α-times integrated C-cosine function on X for which A may not be bounded (see Corollaries 3.9 and 3.10 below), and are also equivalent to A is the generator of a local α-times integrated C-semigroup on X for which B may not be bounded (see Corollaries 3.19 and 3.20 below).

2. Abstract Cauchy Problems

In this section, we consider the abstract Cauchy problem ACP(A,B,f,x,y) which were extensively studied for the case $f=0$ (see [3,4]). A function u is called a (strong) solution of ACP(A,B,f,x,y) if $u \in C^1([0,T_0),X)$ satisfies ACP(A,B,f,x,y) (that is $u(0)=x$, $u'(0)=y$, and for a.e. $t\in(0,T_0)$, $u'(t)$ is differentiable and $u'(t)\in D(A)$, and $u''(t)=Au'(t)+Bu(t)+f(t)$ for a.e. $t\in(0,T_0)$). In the following, we always assume that $C\in L(X)$ is injective, K_0 a kernel on $[0,T_0)$, and both A and B are biclosed linear operators in X (that is $x\in D(A)$, $y\in D(B)$ and $Ax+By=z$ whenever $x_n\in D(A)$, $y_n\in D(B)$ with $x_n\to x$, $y_n\to y$ and $Ax_n+By_n\to z$), $CA\subset AC$ and $CB\subset BC$.

Lemma 2.1. *Assume that $\mathcal{D}$ is a subspace of $D(B)\times D(A)$. Then for each $(x,y)\in\mathcal{D}$ ACP$(\mathcal{T},KC\begin{pmatrix} x \\ y \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ has a unique solution $\begin{pmatrix} u \\ v \end{pmatrix}$ in $C([0,T_0),[\mathcal{T}])$ if and only if for each $(x,y)\in\mathcal{D}$ ACP$(A,B,K_0CBx+KCy,0,0)$ has a unique solution w with $Bw+Aw'\in C([0,T_0),X)$. In this case, $w=j_0*v$. In particular, $w\in C^1([0,T_0),[D(A)])\cap C([0,T_0),[D(B)])$ if either A or B is bounded. Here $\mathcal{T}=\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.*

Proof. Since the biclosedness of A and B with $CA\subset AC$ and $CB\subset BC$ implies that $\mathcal{T}$ is a closed linear operator in $X\times X$ with $\mathbf{C}\mathcal{T}\subset\mathcal{T}\mathbf{C}$. Suppose that $(x,y)\in\mathcal{D}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ denotes the unique solution of ACP$(\mathcal{T},K\mathbf{C}\begin{pmatrix} x \\ y \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ in $C([0,T_0),[\mathcal{T}])$. Then v and $Bu+Av$ are continuous on $[0,T_0)$, and $u'=v+KCx$ and $v'=Bu+Av+KCy$ a.e. on $[0,T_0)$, so that $u=j_0*v+K_0Cx$ on $[0,T_0)$, $j_0*v(t)\in D(B)$ for all $t\in[0,T_0)$, and $v'=Bj_0*v+K_0CBx+Av+KCy$ a.e. on $[0,T_0)$. Hence, $w=j_0*v$ is a solution of ACP$(A,B,K_0CBx+KCy,0,0)$ with $Bw+Aw'\in C([0,T_0),X)$. The uniqueness of solutions of ACP$(A,B,K_0CBx+KCy,0,0)$ follows from the fact that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution of ACP$(\mathcal{T},\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ in $C([0,T_0),[\mathcal{T}])$. Conversely, suppose that $(x,y)\in\mathcal{D}$ and w denotes the unique solution of ACP$(A,B,K_0CBx+KCy,0,0)$ with $Bw+Aw'\in C([0,T_0),X)$. We set $u=w+K_0Cx$ and $v=w'$ on $[0,T_0)$. Then $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}\in D(B)\times D(A)=D(\mathcal{T})$ for all $t\in[0,T_0)$ and $\mathcal{T}\begin{pmatrix} u \\ v \end{pmatrix}$ is continuous on $[0,T_0)$, and for a.e. $t\in(0,T_0)$ $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is differentiable and $\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} w'(t)+K(t)Cx \\ w''(t) \end{pmatrix}=\begin{pmatrix} w'(t)+K(t)Cx \\ Aw'(t)+Bw(t)+K_0(t)CBx+K(t)Cy \end{pmatrix}=\begin{pmatrix} v(t)+K(t)Cx \\ Av(t)+Bu(t)+K(t)Cy \end{pmatrix}=\mathcal{T}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}+K(t)\mathbf{C}\begin{pmatrix} x \\ y \end{pmatrix}$, and so $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of ACP$(\mathcal{T},K\mathbf{C}\begin{pmatrix} x \\ y \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ in $C([0,T_0),[\mathcal{T}])$. The uniqueness of solutions of ACP$(\mathcal{T},K\mathbf{C}\begin{pmatrix} x \\ y \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix})$ in $C([0,T_0),[\mathcal{T}])$ follows from the fact that 0 is the unique solution of ACP$(A,B,0,0,0)$.

□

By slightly modifying the proof of [15, Theorem 3.7], the next theorem concerning the well-posedness of ACP(A,f,x) is attained, and so its proof is omitted.

Theorem 2.2. *Assume that D is dense in X for some subspace D of $D(A)$. Then the following are equivalent :*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X;*

(ii) *for each $x\in D$ ACP$(A,KCx,0)$ has a unique solution $u(\cdot;Cx)$ in $C([0,T_0),[D(A)])$ which depends continuously on x (that is $\{u(\cdot;Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $[0,T_0)$ whenever $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D,\|\cdot\|)$).*

In this case, $u(\cdot, Cx) = S(\cdot)x$.

Just as an application of Theorem 2.2, the next theorem concerning the well-posedness of ACP(A, B, f, x, y) is also attained.

Theorem 2.3. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$. Then $\mathcal{T}$ is a subgenerator of a local K-convoluted $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w which depends continuously differentiable on (x, y) (that is $\{w_n(\cdot)\}_{n=1}^{\infty}$ and $\{w_n'(\cdot)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$ whenever $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(B), \|\cdot\|)$ and $\{y_n\}_{n=1}^{\infty}$ a Cauchy sequence in $(D(A), \|\cdot\|)$), and $Bw + Aw' \in C([0, T_0), X)$. Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and w_n denotes the unique solution of ACP$(A, B, K_0CBx_n + KCy_n, 0, 0)$.*

Proof. Since for each $(x, y) \in \mathcal{D}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ is the unique solution of ACP$(\mathcal{T}, KC\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ in $C([0, T_0), [\mathcal{T}])$ if and only if for each $(x, y) \in \mathcal{D}$ $u = w + K_0Cx$ and $v = w'$ on $[0, T_0)$, and w is the unique solution of ACP$(A, B, K_0CBx + KCy, 0, 0)$ with $Bw + Aw' \in C([0, T_0), X)$. By Theorem 2.2, we also have $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{S}(\cdot)\begin{pmatrix} x \\ y \end{pmatrix}$. Consequently, $\mathcal{T}$ is a subgenerator of a local K-convoluted $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w which depends continuously differentiable on (x, y).

□

Corollary 2.4. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$, and K_0 exponentially bounded. Then $\mathcal{T}$ is a subgenerator of an exponentially bounded K-convoluted $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ such that for each $(x, y) \in \mathcal{D}$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and $Bw + Aw' \in C([0, \infty), X)$.*

Lemma 2.5. *(see [20])Assume that $\lambda \in \rho_C(\mathcal{T})$ ($\mathcal{C}$-resolvent set of $\mathcal{T}$). Then*

(i) *$\lambda \in \rho_C(A, B)$;*

(ii) *$(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}$ are closable, and their closures are bounded and have the same domain;*

(iii)
$$(\lambda - \mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix}$$

on $D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})}) \times X$, and on $X \times X$ if $D(B) \cap D(A)$ is dense in X.

Lemma 2.6.*(see [20])Assume that $\lambda \in \rho_C(A, B)$. Then*

(i) *$\lambda - \mathcal{T}$ is injective;*

(ii) *$(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}$ are closable and their closures have the same domain, and*
$$(\lambda - \mathcal{T})\begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} = \mathcal{C}$$
on $D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})}) \times X$;

(iii) *$\lambda \in \rho_C(\mathcal{T})$ and*
$$(\lambda - \mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix},$$
if $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} \in L(X)$.

In particular, the conclusion of (iii) holds when A or B in $L(X)$, or $D(B) \cap D(A)$ is dense in X with $AB = BA$ on $D(B) \cap D(A)$.

Since $\hat{K}(\lambda)\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} = [\hat{K}(\lambda)\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}}]\frac{1}{\lambda} + \hat{K}_0(\lambda)C$ and $\hat{K}(\lambda)(\lambda^2 - \lambda A - B)^{-1}C = [\hat{K}(\lambda)\lambda(\lambda^2 - \lambda A - B)^{-1}C]\frac{1}{\lambda}$, we can combine Lemma 2.5 with Lemma 2.6 and [10, Theorem 2.2.5] to obtain the next new Miyadera-Feller-Phillips-Hille-Yosida type theorem concerning the generation of an exponentially bounded K-convoluted $\mathcal{C}$-semigroup on $X \times X$.

Theorem 2.7. *Assume that $D(B)\cap D(A)$ is dense in X, K_0 exponentially bounded, and $\hat{K}(\lambda)\neq 0$ for λ large enough. Then $\mathcal{T}$ is a subgenerator of an exponentially bounded K-convoluted C-semigroup on $X\times X$ if and only if there exist $M,\omega>0$ such that $\lambda\in\rho_C(A,B)$ and (3) holds for all $\lambda>\omega$ and $k\in\mathbb{N}\cup\{0\}$.*

Just as a result in [26, Theorem 2] for the case of C_0-semigroup and a result in [20, Corollary 2.10] for the case of local C-semigroup, we can combine Corollary 2.4 with Theorem 2.7 to obtain the next corollary.

Corollary 2.8. *Assume that $D(B)\cap D(A)$ is dense in X, K_0 exponentially bounded, and $\hat{K}(\lambda)\neq 0$ for λ large enough. Then the following ststements are equivalent:*

(i) *$\mathcal{T}$ is a subgenerator of an exponentially bounded K-convoluted C-semigroup on $X\times X$;*

(ii) *There exist $M,\omega>0$ such that $\lambda\in\rho_C(A,B)$ and (3) holds for all $\lambda>\omega$ and $k\in\mathbb{N}\cup\{0\}$;*

(iii) *There exist $M,\omega>0$ such that for each pair $x,y\in D(B)\cap D(A)$ $ACP(A,B,K_0CBx+KCy,0,0)$ has a unique solution w with $\|w(t)\|,\|w'(t)\|\leq Me^{\omega t}(\|x\|+\|y\|)$ for all $t\geq 0$ and $Bw+Aw'\in C([0,\infty),X)$.*

Combining Lemma 2.1 with [15, Corollary 3.6], the next theorem is also attained.

Theorem 2.9. *Assume that $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty. Then $\mathcal{T}$ is the generator of a local K-convoluted C-semigroup on $X\times X$ if and only if for each $(x,y)\in D(B)\times D(A)$ $ACP(A,B,K_0CBx+KCy,0,0)$ has a unique solution w with $Bw+Aw'\in C([0,T_0),X)$.*

3. Generation of Local α-Times Integrated C-Semigroups and C-Cosine Functions on X

Just as in the proofs of [18, Theorem 2.7 and Theorem 2.9] , we can modify the proofs of [14, Theorem 2.12 and Theorem 3.2] to obtain next two theorems, and so their proofs are omitted.

Theorem 3.1. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous nondegenerate local α-times integrated C-semigroup on X for some $\alpha\geq 1$. Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)\subset X$. Then $A+B$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous nondegenerate local α-times integrated C-semigroup on X, if either $\alpha=1$ or $\alpha>1$ with $C^{-1}Ax\in D(B^{l-1})$ for all $x\in\overline{D(B)}$. Here l denotes the smallest nonnegative integer that is larger than or equal to α.*

Theorem 3.2. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous nondegenerate local α-times integrated C-semigroup on X for some $\alpha\geq 1$. Assume that A is a bounded linear operator from $[D(B)]$ into $R(C)$ such that $A+B$ is a closed linear operator. Then $A+B$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous nondegenerate local α-times integrated C-semigroup on X, if $C^{-1}Ax\in D(B^l)$ for all $x\in D(B)$. Here l denotes the smallest nonnegative integer that is larger than or equal to α.*

Lemma 3.3. *Let A be a bounded linear operator from X into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$, $v=\begin{pmatrix} B \\ A\end{pmatrix}$ and $y\in D(A)$. Assume that $y_1=C^{-1}Ay$ and $a_1=\begin{pmatrix} 0 \\ y_1\end{pmatrix}$. Then*

(i) *$a_1\in D(\mathcal{T})$ and $\mathcal{T}a_1=\begin{pmatrix} y_1 \\ y_2\end{pmatrix}$ with $y_2=v\cdot a_1$;*

(ii) *For each $n\in\mathbb{N}$ with $n\geq 2$, we have $a_1\in D(\mathcal{T}^n)$ if and only if $a_k=\begin{pmatrix} y_{k-1} \\ y_k\end{pmatrix}\in D(\mathcal{T})$ and $\mathcal{T}a_k=a_{k+1}$ for all $2\leq k\leq n$ if and only if y_1, $y_k=v\cdot a_{k-1}\in D(B)$ for all $2\leq k\leq n-1$.*

Theorem 3.4. *Let $\mathcal{T}$ be a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X\times X$ for some $\alpha\geq 1$. Assume that A is a bounded linear operator from X into $R(C)$. Then B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X, if for each $y\in X$ we have $y_k\in D(B)$ for all $1\leq k\leq l-1$. Here y_k is given as in Lemma 3.3 and l denotes the smallest nonnegative integer that is larger than or equal to α.*

Proof. Suppose that $\begin{pmatrix} 0 & I \\ B & A\end{pmatrix}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X\times X$. Then it is also a subgenerator (resp., the generator) of a locally Lipschitz continuous local $(\alpha+1)$-times integrated C-semigroup on $X\times X$, and so $\begin{pmatrix} 0 & I \\ B & 0\end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local $(\alpha+1)$-times integrated C-semigroup on $X\times X$. Hence, B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X.

□

Theorem 3.5. *Let $\mathcal{T}$ be a subgenerator (resp., the generator) of a local α-times integrated $\mathbf{C}$-semigroup on $X \times X$ for some $\alpha \geq 1$. Assume that A is a bounded linear operator from $[D(B)]$ into $R(C)$. Then B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X, if for each $y \in D(B)$ we have $y_k \in D(B)$ for all $1 \leq k \leq l$. Here y_k is given as in Lemma 3.3 and l denotes the smallest nonnegative integer that is larger than or equal to α.*

Lemma 3.6. *Let A be a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$, $\mathcal{S} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$, and $y \in D(A)$. Assume that $y_1 = C^{-1}Ay$ and $a_1 = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$. Then*

(i) *$a_1 \in D(\mathcal{S})$ and $\mathcal{S}a_1 = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$;*

(ii) *For each $m \in \mathbb{N}$ with $m \geq 2$, we have $a_1 \in D(\mathcal{S}^{2m})$ if and only if $y_1 \in D(B^m)$ if and only if $a_1 \in D(\mathcal{S}^{2m+1})$.*

Theorem 3.7. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X for some $\alpha \geq 1$. Assume that $D(B)$ is dense in X and A is a bounded linear operator from X into $R(C)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local α-times integrated $\mathbf{C}$-semigroup on $X \times X$, if $R(C^{-1}A) \subset D(B^m)$. Here m denotes the smallest nonnegative integer that is larger than or equal to $\frac{\alpha}{2}$.*

Proof. Suppose that D(B) is dense in X and B a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X. Then $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local $(\alpha + 1)$-times integrated $\mathbf{C}$-semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local $(\alpha + 1)$-times integrated $\mathbf{C}$-semigroup on $X \times X$. Hence, it is also a subgenerator (resp., the generator) of a local α-times integrated $\mathbf{C}$-semigroup on $X \times X$.
□

Theorem 3.8. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous local α-times integrated C-cosine function on X for some $\alpha \geq 1$. Assume that $D(B)$ is dense in X and A is a bounded linear operator from $[D(B)]$ into $R(C)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local α-times integrated $\mathbf{C}$-semigroup on $X \times X$, if $R(C^{-1}A) \subset D(B^m)$. Here m denotes the smallest nonnegative integer that is larger than or equal to $\frac{\alpha}{2}$.*

Combining Theorems 2.9 and 3.1 with Theorems 3.4 and 3.7, the next corollary is also attained.

Corollary 3.9. *Assume that $\rho(A, B)$ is nonempty and $A \in L(X)$. Then the following are equivalent :*

(i) *$\mathcal{T}$ is the generator of a local K-convoluted $\mathbf{C}$-semigroup on $X \times X$;*

(ii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w in* $C([0, T_0), [D(B)])$.

Moreover, (i)-(ii) imply

(iii) *B is the generator of a locally Lipschitz continuous local α-times integrated C-cosine function on X, if $K = j_{\alpha-1}$ for some $1 \leq \alpha \leq 2$, $R(A) \subset R(C)$ and $R(C^{-1}A) \subset D(B^{l-1})$, and $(i) - (iii)$ are equivalent if the assumption of $D(B)$ is dense in X is also added.*

Similarly, we can combine Theorems 2.9 and 3.2 with Theorems 3.5 and 3.8 to obtain next corollary.

Corollary 3.10. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent :*

(i) *$\mathcal{T}$ is the generator of a local K-convoluted $\mathbf{C}$-semigroup on $X \times X$;*

(ii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w with $Bw + Aw' \in$* $C([0, T_0), X)$.

Moreover, (i)-(ii) are equivalent to

(iii) *B is the generator of a locally Lipschitz continuous local once integrated C-cosine function on X,*

if $K = j_0$ and A is a bounded linear operator from $[D(B)]$ into R(C) with R($C^{-1}A$) ⊂ D(B).

Lemma 3.11. *Let B be a bounded linear operator from $\overline{D(A)}$ into $R(C)$ or a bounded linear operator from $[D(A)]$ into $R(C)$, $v = \begin{pmatrix} B \\ A \end{pmatrix}$ and $x \in D(B)$. Assume that $x_1 = C^{-1}Bx$ and $b_1 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$. Then*

(i) $b_1 \in D(\mathcal{T})$ *and* $\mathcal{T}b_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *with* $x_2 = v \cdot b_1$ *if and only if* $x_1 \in D(A)$;

(ii) *For each* $n \in \mathbb{N}$ *with* $n \geq 2$, *we have* $b_1 \in D(\mathcal{T}^n)$ *if and only if* $b_k = \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \in D(\mathcal{T})$ *and* $\mathcal{T}b_k = b_{k+1}$ *for all* $1 \leq k \leq n$ *if and only if* $x_1, x_k = v \cdot b_{k-1} \in D(A)$ *for all* $2 \leq k \leq n$ *(if and only if* $x_1 \in D(A^2)$ *for* $n = 2$*).*

Theorem 3.12. *Let* $\mathcal{T}$ *be a subgenerator (resp., the generator) of a local* α*-times integrated* C*-semigroup on* $X \times X$ *for some* $\alpha > 0$. *Assume that* B *is a bounded linear operator from* $\overline{D(A)}$ *into* $R(C)$. *Then* A *is a subgenerator (resp., the generator) of a local* α*-times integrated* C*-semigroup on* X, *if for each* $x \in X$ *we have* $x_k \in D(A)$ *for all* $1 \leq k \leq l$. *Here* x_k *is given as in Lemma 3.11 and* l *denotes the smallest nonnegative integer that is larger than or equal to* α.

Proof. Clearly, $C\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}C$ on $X \times D(A)$ (resp., $C^{-1}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}C=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$) is equivalent to $CA = AC$ on $D(A)$ (resp., $C^{-1}AC = A$). Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X\times X$. Then $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup $\mathcal{S}(\cdot)$ on $X\times X$. For each pair $x, y \in X$, we set $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}=j_0*\mathcal{S}(t)\begin{pmatrix} x \\ y \end{pmatrix}$ for all $0 \leq t < T_0$. Then $\begin{pmatrix} u \\ v \end{pmatrix} \in C^1([0, T_0), X \times X) \cap C([0, T_0), [\mathcal{T}])$, $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}+\begin{pmatrix} j_\alpha(t)Cx \\ j_\alpha(t)Cy \end{pmatrix}=\begin{pmatrix} v(t) \\ Av(t) \end{pmatrix}+\begin{pmatrix} j_\alpha(t)Cx \\ j_\alpha(t)Cy \end{pmatrix}$ for all $0 \leq t < T_0$, so that $u(0) = 0 = v(0)$, $u'(t) = v(t) + j_\alpha(t)Cx$ and $v'(t) = Av(t) + j_\alpha(t)Cy$ for all $0 \leq t < T_0$. Hence, v is a solution of ACP$(A, j_\alpha Cy, 0)$ in $C^1([0, T_0), X) \cap C([0, T_0), [D(A)])$, $u(0) = 0$, and $u' = v + j_\alpha Cx$ on $[0, T_0)$. To show that A is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on X, we remain only to show that 0 is the unique solution of ACP$(A, 0, 0)$ in $C^1([0, T_0), X) \cap C([0, T_0), [D(A)])$ (see [15, Corollary 3.6]). To this end, suppose that v is a solution of ACP$(A, 0, 0)$ in $C^1([0, T_0), X) \cap C([0, T_0), [D(A)])$. We set $u = j_0 * v$, then $u(0) = 0 = v(0)$ and $\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} v(t) \\ Av(t) \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ for all $0 \leq t < T_0$. The uniqueness of solutions of ACP$(A, 0, 0)$ follows from the uniqueness of solutions of ACP$(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$.
□

Theorem 3.13. *Let* $\mathcal{T}$ *be a subgenerator (resp., the generator) of a local* α*-times integrated* C*-semigroup on* $X \times X$ *for some* $\alpha > 0$. *Assume that* B *is a bounded linear operator from* $[D(A)]$ *into* $R(C)$. *Then* A *is a subgenerator (resp., the generator) of a local* α*-times integrated* C*-semigroup on* X, *if for each* $x \in X$ *we have* $x_k \in D(A)$ *for all* $1 \leq k \leq l + 1$. *Here* x_k *is given as in Lemma 3.11 and* l *denotes the smallest nonnegative integer that is larger than or equal to* α.

Lemma 3.14. *Let* B *be bounded linear operator from* $\overline{D(A)}$ *into* $R(C)$ *or a bounded linear operator from* $[D(A)]$ *into* $R(C)$, $\mathcal{S} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$, *and* $x \in D(B)$. *Assume that* $x_1 = C^{-1}Bx$ *and* $b_1 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$. *Then*

(i) $b_1 \in D(\mathcal{S})$ *and* $\mathcal{S}b_1 = \begin{pmatrix} x_1 \\ Ax_1 \end{pmatrix} = b_2$ *if and only if* $x_1 \in D(A)$;

(ii) *For each* $n \in \mathbb{N}$ *with* $n \geq 2$, *we have* $b_1 \in D(\mathcal{S}^n)$ *if and only if* $b_1, b_k = \mathcal{S}b_{k-1} \in D(\mathcal{S})$ *for all* $2 \leq k \leq n$ *if and only if* $x_1 \in D(A^n)$.

Just as in the proofs of [18, Theorem 2.8 and Theorem 2.10] , we can modify the proofs of [14, Theorem 2.10 and Theorem 3.1] to obtain next two theorems, and so their proofs are omitted.

Theorem 3.15. *Let* A *be a subgenerator (resp., the generator) of a nondegenerate local* α*-times integrated* C*-semigroup on* X *for some* $\alpha > 0$. *Assume that* B *is a bounded linear operator from* $\overline{D(A)}$ *into* $R(C)$. *Then* $A + B$ *is a subgenerator (resp., the generator) of a nondegenerate local* α*-times integrated* C*-semigroup on* X, *if* $C^{-1}Bx \in D(A^l)$ *for all* $x \in \overline{D(A)}$. *Here* l *denotes the smallest nonnegative integer that is larger than or equal to* α.

Theorem 3.16. *Let* A *be a subgenerator (resp., the generator) of a nondegenerate local* α*-times integrated* C*-semigroup on* X *for some* $\alpha > 0$. *Assume that* B *is a bounded linear operator from* $[D(A)]$ *into* $R(C)$ *such that* $A + B$ *is a closed linear operator. Then* $A + B$ *is a subgenerator (resp., the generator) of a nondegenerate local* α*-times integrated* C*-semigroup on* X, *if* $C^{-1}Bx \in D(A^{l+1})$ *for all* $x \in D(A)$. *Here* l *denotes the smallest nonnegative integer that is larger than or equal to* α.

Theorem 3.17. *Let* A *be a subgenerator (resp., the generator) of a local* α*-times integrated* C*-semigroup on* X *for some* $\alpha > 0$. *Assume that* B *is a bounded linear operator from* $\overline{D(A)}$ *into* $R(C)$. *Then* $\mathcal{T}$ *is a subgenerator (resp.,* *the generator) of a local α-times integrated C-semigroup on $X \times X$, if $R(C^{-1}B) \subset D(A^l)$. Here l denotes the smallest nonnegative integer that is larger than or equal to α.*

Proof. Suppose that A is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup $S(\cdot)$ on X. To show that $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X \times X$, we need only to show that for each pair $x, y \in X$ ACP($\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} j_\alpha Cx \\ j_\alpha Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$) has a unique solution in $C^1([0, T_0), X \times X) \cap C([0, T_0), [\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}])$. Let us achieve this. For each pair $x, y \in X$, we set $v(t) = j_0 * S(t)y$ and $u(t) = j_0 * v(t) + j_{\alpha+1}Cx$ for all $0 \le t < T_0$. Then $u(0) = 0 = v(0)$, and $v'(t) = S(t)y = Av(t) + j_\alpha(t)Cy$ and $u'(t) = v(t) + j_\alpha(t)Cx$ for all $0 \le t < T_0$, so that $\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) + j_\alpha(t)Cx \\ Av(t) + j_\alpha(t)Cy \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} j_\alpha(t)Cx \\ j_\alpha(t)Cy \end{pmatrix}$ for all $0 \le t < T_0$. Hence, $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of ACP($\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} j_\alpha Cx \\ j_\alpha Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$) in $C^1([0, T_0), X \times X) \cap C([0, T_0), [\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}])$. The uniqueness of solutions of ACP($\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$) in $C^1([0, T_0), X \times X) \cap C([0, T_0), [\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}])$ follows from the uniqueness of solutions of ACP$(A, 0, 0)$. Consequently, $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X \times X$, which implies that $\mathcal{T}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X \times X$. □

Theorem 3.18. *Let A be a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on X for some $\alpha > 0$. Assume that B is a bounded linear operator from $[D(A)]$ into $R(C)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local α-times integrated C-semigroup on $X \times X$, if $R(C^{-1}B) \subset D(A^{l+1})$. Here l denotes the smallest nonnegative integer that is larger than or equal to α.*

Combining Theorems 2.9 and 3.1 with Theorems 3.12 and 3.17, the next corollary is also attained.

Corollary 3.19. *Assume that $\rho(A, B)$ is nonempty and $B \in L(X)$. Then the following are equivalent :*

(i) *$\mathcal{T}$ is the generator of a local K-convoluted C-semigroup on $X \times X$;*

(ii) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w in $C^1([0, T_0), [D(A)])$.*

Moreover, (i)-(ii) are equivalent to

(iii) *A is the generator of a local α-times integrated C-semigroup on X,*

if $K = j_{\alpha-1}$ for some $0 < \alpha \le 2$, $R(B) \subset R(C)$, and $R(C^{-1}B) \subset D(A^l)$.

Similarly, we can combine Theorems 2.9 and 3.1 with Theorems 3.13 and 3.18 to obtain next corollary.

Corollary 3.20. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent :*

(i) *$\mathcal{T}$ is the generator of a local K-convoluted C-semigroup on $X \times X$;*

(ii) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, K_0CBx + KCy, 0, 0)$ has a unique solution w with $Bw + Aw' \in C([0, T_0), X)$.*

Moreover, (i)-(ii) are equivalent to

(iii) *A is the generator of a local α-times integrated C-semigroup on X,*

if $K = j_{\alpha-1}$ for some $0 < \alpha \le 1$ and B is a bounded linear operator from $[D(A)]$ into $R(C)$ with $R(C^{-1}B) \subset D(A^2)$.

We end this paper with a simple illustrative example. Let $X = C_b(\mathbb{R})$(or $L^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $Y = UC_b(\mathbb{R})$(or $C_0(\mathbb{R})$)$= \overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [28,39] that for each $\alpha > \frac{1}{2}$, A generates an exponentially bounded, norm continuous α-times integrated semigroup $S(\cdot)$ on X which is defined by $(S(t)f)(x) = \frac{1}{\sqrt{2\pi}}(\widetilde{\phi_{\alpha,t}} * f)(t)$ for all $f \in X$ and $t \ge 0$ if the polynomial $p(x) = \sum_{j=0}^{k} a_j (ix)^j$ satisfies $\sup_{x \in \mathbb{R}} Re(p(x)) < \infty$. Here $\widetilde{\phi_{\alpha,t}}$ denotes the inverse Fourier transform of $\phi_{\alpha,t}$ with $\phi_{\alpha,t}(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{p(x)s} ds$. An application of Corollary 3.19 shows that for each bounded linear operator $B : X \to D(A)$, $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ generates an exponentially bounded, norm continuous α-times integrated semigroup on $X \times X$ when $\alpha \le 2$.

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