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Busemann-Petty Problem for the *i*-th Radial Blaschke-Minkowski Homomorphisms

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Abstract. Schuster introduced the notion of radial Blaschke-Minkowski homomorphism and considered its Busemann-Petty problem. In this paper, we further study the Busemann-Petty problem for the radial Blaschke-Minkowski homomorphisms and give the affirmative and negative forms of Busemann-Petty problem for the *i*-th radial Blaschke-Minkowski homomorphisms.

1. Introduction

The setting for this paper is Euclidean n-space $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. For the n-dimensional volume of body K, we write $V(K)$.

If K is a compact star shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n\backslash\{0\} \to [0,\infty)$, is defined by (see [5])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. The set of star bodies (about the origin) in $\mathbb{R}^n$ will be denoted by $\mathcal{S}^n_o$, for the set of all origin-symmetric star bodies we write $\mathcal{S}^n_{os}$.

Intersection bodies were first appeared in a paper by Busemann (see [2]) and were explicitly defined and named by Lutwak (see [19]). In 1988, Lutwak defined the notion of intersection bodies as follows: For $K \in \mathcal{S}^n_o$, the intersection body, IK, of K is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^\perp$, the hyperplane orthogonal to u, i.e. for all $u \in S^{n-1}$,

$$\rho(IK,u) = V_{n-1}(K \cap u^\perp).$$

Further, Lutwak ([19]) showed the following Busemann-Petty problem by intersection bodies:

Problem 1.1 (Busemann-Petty problem). *For $K, L \in \mathcal{S}^n_o$, is there the implication*

$$IK \subseteq IL \Rightarrow V(K) \leq V(L)?$$

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For the Problem 1.1, Lutwak ([19]) gave an affirmative answer if K is restricted to the class of intersection bodies and two negative answers if K is not origin-symmetric or L is not an intersection body.

Remark 1.1 If $K, L \in \mathcal{S}^n_{os}$, then Problem 1.1 is called the symmetric Busemann-Petty problem. Gardner ([4]) and Zhang ([28]) showed that the symmetric Busemann-Petty problem has an affirmative answer for $n \leq 4$ and a negative answer for $n \geq 5$.

Intersection bodies have been becoming the centered notion in the dual Brunn-Minkowski theory (see e.g. [4–14, 19, 24–27]). Based on the properties of intersection bodies, Schuster ([20]) introduced the radial Blaschke-Minkowski homomorphism which is a more general intersection operator as follows:

Definition 1.A. *A map $\Psi : \mathcal{S}^n_o \to \mathcal{S}^n_o$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:*

(1) Ψ is continuous;

(2) For all $K, L \in \mathcal{S}^n_o$, $\Psi(K \widetilde{+}_{n-1} L) = \Psi K \widetilde{+} \Psi L$, i.e. ΨK is radial Blaschke-Minkowski sum;

(3) $\Psi(\vartheta K) = \vartheta \Psi K$ for all $K \in \mathcal{S}^n_o$ and all $\vartheta \in SO(n)$.

Here $\widetilde{+}_{n-1}$ and $\widetilde{+}$ denote L_{n-1} and L_1 radial Minkowski addition, respectively; and $SO(n)$ is the group of rotations in n dimension,

Meanwhile, Schuster ([20]) showed that the radial Blaschke-Minkowski homomorphism satisfies the geometric inequalities such as Aleksandrov-Fenchel, Minkowski and Brunn-Minkowski type inequalities. In particular, Schuster ([20]) proved the following fact:

Theorem 1.A. *A map $\Psi : \mathcal{S}^n_o \to \mathcal{S}^n_o$ is a radial Blaschke-Minkowski homomorphism if and only if there is a non-negative measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that for $K \in \mathcal{S}^n_o$, $\rho(\Psi K, \cdot)$ is the convolution of $\rho(K, \cdot)^{n-1}$ and μ, i.e.,*

$$\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} * \mu. \tag{1.1}$$

Here $\widehat{e}$ denotes the pole point of S^{n-1} and $\mathcal{M}(S^{n-1}, \widehat{e})$ denotes the signed finite Borel measure space on S^{n-1} (see [20]).

According to (1.1), Schuster ([20]) defined the mixed radial Blaschke-Minkowski homomorphisms as follows:

Definition 1.B. *Let $\Psi : \mathcal{S}^n_o \to \mathcal{S}^n_o$ be a radial Blaschke-Minkowski homomorphism with non-negative generating measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$, defined a mixed operator $\Psi : \mathcal{S}^n_o \times \cdots \times \mathcal{S}^n_o \to \mathcal{S}^n_o$ by*

$$\rho(\Psi(K_1, \cdots, K_{n-1}), \cdot) = \rho(K_1, \cdot) \cdots \rho(K_{n-1}, \cdot) * \mu. \tag{1.2}$$

The body $\Psi(K_1, \cdots, K_{n-1})$ is called the mixed radial Blaschke-Minkowski homomorphism of $K_1, \cdots, K_{n-1} \in \mathcal{S}^n_o$.

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then write

$$\Psi_i(K, L) = \Psi(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i}) \ (i = 0, 1, \cdots, n-2),$$

which is called the mixed radial Blaschke-Minkowski homomorphism of K and L. If $L = B$ (B denotes the unit ball centered at the origin in $\mathbb{R}^n$), we call $\Psi_i K = \Psi_i(K, B)$ the i-th radial Blaschke-Minkowski homomorphism of K. Obviously, by (1.2) and notice $\rho(B, \cdot) = 1$, we know that for $i = 0, 1, \cdots, n-2$, $\rho(\Psi_i K, \cdot) = \rho(K, \cdot)^{n-i-1} * \mu$.

If we let i be real, then (1.1) can be extended to the following definition.

Definition 1.1. *For $K \in \mathcal{S}^n_o$, $0 \leq i < n-1$, the i-th radial Blaschke-Minkowski homomorphism, $\Psi_i K$, of K is given by*

$$\rho(\Psi_i K, \cdot) = \rho(K, \cdot)^{n-i-1} * \mu, \tag{1.3}$$

where $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$.

From (1.3), we have that for $c > 0$,

$$\Psi_i(cK) = c^{n-i-1} \Psi_i K. \tag{1.4}$$

In 2008, Schuster ([21]) considered the following Busemann-Petty problem for the radial Blaschke-Minkowski homomorphisms.

Problem 1.2. *Let $\Psi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be a radial Blaschke-Minkowski homomorphism. For $K, L \in \mathcal{S}_o^n$, is there the implication*

$$\Psi K \subseteq \Psi L \Rightarrow V(K) \leq V(L)?$$

Obviously, Problem 1.2 is a more general Busemann-Petty problem than Problem 1.1. For the Problem 1.2, Schuster ([21]) gave the following affirmative and negative answers, respectively.

Theorem 1.B. *Let $\Psi : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be a radial Blaschke-Minkowski homomorphism. If $K \in \Psi\mathcal{S}_o^n$ and $L \in \mathcal{S}_o^n$, then*

$$\Psi K \subseteq \Psi L \Rightarrow V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$. Here $\Psi\mathcal{S}_o^n$ denotes the range of Ψ.

Theorem 1.C. *Suppose that $\mathcal{S}_{os}^n \subseteq \mathcal{S}_o^n(\Psi)$, $L \in \mathcal{S}_{os}^n$, $\rho(L,\cdot) \in \mathcal{H}^n$ and $\rho(L,\cdot) > 0$ (i.e., L is polynomial). If $L \notin \Psi\mathcal{S}_o^n$, then there exists $K \in \mathcal{S}_{os}^n$, such that*

$$\Psi K \subset \Psi L.$$

But

$$V(K) > V(L).$$

Here $\mathcal{H}^n$ denotes the space of all finite sums of spherical harmonic of dimension n.

In 2011, Wang, Liu and He ([22]) extended the radial Blaschke-Minkowski homomorphisms to L_p space. In recent years, a lot of important results for the radial Blaschke-Minkowski homomorphisms and their L_p analogies were obtained (see e.g. [1, 3, 15–17, 21–23, 29–33]).

The main goal of this paper is to study the Busemann-Petty problem for the i-th radial Blaschke-Minkowski homomorphisms. First, we give an affirmative answer of the Busemann-Petty problem for the i-th radial Blaschke-Minkowski homomorphisms.

Theorem 1.1. *Let $K, L \in \mathcal{S}_o^n$, $0 \leq i < n-1$ and Ψ_i be the i-th radial Blaschke-Minkowski homomorphism. If $K \in \Psi_i\mathcal{S}_o^n$, then*

$$\Psi_i K \subseteq \Psi_i L \Rightarrow \widetilde{W}_i(K) \leq \widetilde{W}_i(L).$$

And $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ if and only if $K = L$. Here $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of $K \in \mathcal{S}_o^n$.

Obviously, the case $i = 0$ of Theorem 1.1 yields Theorem 1.B.

Next, the following negative forms of the Busemann-Petty problem for the i-th radial Blaschke-Minkowski homomorphisms are given:

Theorem 1.2. *Suppose that $\mathcal{S}_{os}^n \subseteq \mathcal{S}_o^n(\Psi_i)$ $(0 \leq i < n-1)$, $L \in \mathcal{S}_{os}^n$, $\rho(L,\cdot) \in \mathcal{H}^n$ and $\rho(L,\cdot) > 0$ (i.e., L is polynomial). If $L \notin \Psi_i\mathcal{S}_o^n$, then there exists $K \in \mathcal{S}_{os}^n$ such that*

$$\Psi_i K \subset \Psi_i L.$$

But

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

Here $\mathcal{S}_o^n(\Psi_i)$ denotes the injective set of Ψ_i.

Clearly, taking $i = 0$ in Theorem 1.2, we immediately get Theorem 1.C.

Theorem 1.3. *Let $K, L \in \mathcal{S}_o^n$, $0 \leq i < n-1$ and Ψ_i be an even i-th radial Blaschke-Minkowski homomorphism. If $K \notin \mathcal{S}_{os}^n$, then there exists $L \in \mathcal{S}_o^n$, such that*

$$\Psi_i K \subset \Psi_i L.$$

But

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

Let $i = 0$ in Theorem 1.3, we get a new negative form of the Busemann-Petty problem for the radial Blaschke-Minkowski homomorphisms.

Corollary 1.1. *Let $K, L \in \mathcal{S}_o^n$ and Ψ be an even radial Blaschke-Minkowski homomorphism. If $K \notin \mathcal{S}_{os}^n$, then there exists $L \in \mathcal{S}_o^n$, such that*

$$\Psi K \subset \Psi L.$$

But

$$V(K) > V(L).$$

The proofs of Theorems 1.1-1.3 are completed in Section 3.

2. Background Materials

2.1. *i-th radial Blaschke combinations and general i-th radial Blaschke bodies*

For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both 0), the radial Minkowski combination, $\lambda K \widetilde{+} \mu L \in \mathcal{S}_o^n$, of K and L is defined by (see [5])

$$\rho(\lambda K \widetilde{+} \mu L, \cdot) = \lambda\rho(K, \cdot) + \mu\rho(L, \cdot).$$

For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both 0), the radial Blaschke combination, $\lambda \cdot K \widehat{+} \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by (see [5])

$$\rho(\lambda \cdot K \widehat{+} \mu \cdot L, \cdot)^{n-1} = \lambda\rho(K, \cdot)^{n-1} + \mu\rho(L, \cdot)^{n-1}.$$

From the definitions of above two combinations, we easily see $\lambda \cdot K \widehat{+} \mu \cdot L = \lambda K \widetilde{+}_{n-1} \mu L$.

Now, in order to prove our results, we will extend the radial Blaschke combinations to the following i-th radial Blaschke combinations.

For $K, L \in \mathcal{S}_o^n$, $0 \leq i < n-1$ and $\lambda, \mu \geq 0$ (not both 0), the i-th radial Blaschke combination, $\lambda \cdot K \widehat{+}_i \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by

$$\rho(\lambda \cdot K \widehat{+}_i \mu \cdot L, \cdot)^{n-i-1} = \lambda\rho(K, \cdot)^{n-i-1} + \mu\rho(L, \cdot)^{n-i-1}. \tag{2.1}$$

Taking $i = 0$ in (2.1), then $\lambda \cdot K \widehat{+}_0 \mu \cdot L$ is the radial Blaschke combination $\lambda \cdot K \widehat{+} \mu \cdot L$.

If for $\tau \in [-1, 1]$, let

$$\lambda = f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad \mu = f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)} \tag{2.2}$$

and $L = -K$ in (2.1), then we write

$$\widehat{\nabla}_i^\tau K = f_1(\tau) \cdot K \widehat{+}_i f_2(\tau) \cdot (-K), \tag{2.3}$$

and called $\widehat{\nabla}_i^\tau K$ the general i-th radial Blaschke body of K. From (2.2) and (2.3), we easily see that $\widehat{\nabla}_i^1 K = K$, $\widehat{\nabla}_i^{-1} K = -K$ and

$$\widehat{\nabla}_i^0 K = \frac{1}{2} \cdot K \widehat{+}_i \frac{1}{2} \cdot (-K). \tag{2.4}$$

For the general i-th radial Blaschke bodies, by (2.2) we know that $f_1(\tau) + f_2(\tau) = 1$. Hence, if $K \in \mathcal{S}_{os}^n$, then $\widehat{\nabla}_i^\tau K \in \mathcal{S}_{os}^n$. If $K \notin \mathcal{S}_{os}^n$, then we have the following fact.

Theorem 2.1. *For $K, L \in \mathcal{S}_o^n$, $0 \leq i < n-1$. If $K \notin \mathcal{S}_{os}^n$, then for $\tau \in [-1, 1]$,*

$$\widehat{\nabla}_i^\tau K \in \mathcal{S}_{os}^n \Leftrightarrow \tau = 0. \tag{2.5}$$

Proof. If $\tau = 0$, by (2.4) we immediately get $\widehat{\nabla}_i^\tau K \in \mathcal{S}_{os}^n$.

Conversely, since $\rho_M(-u) = \rho_{-M}(u)$ for any $M \in \mathcal{S}_o^n$ and $u \in S^{n-1}$, thus if $\widehat{\nabla}_i^\tau K \in \mathcal{S}_{os}^n$, i.e., $\widehat{\nabla}_i^\tau K = -\widehat{\nabla}_i^\tau K$, then for all $u \in S^{n-1}$,

$$\rho_{\widehat{\nabla}_i^\tau K}^{n-i-1}(u) = \rho_{-\widehat{\nabla}_i^\tau K}^{n-i-1}(u) = \rho_{\widehat{\nabla}_i^\tau K}^{n-i-1}(-u),$$

by (2.3) we have

$$\rho_{f_1(\tau)\cdot K \widehat{+}_i f_2(\tau)\cdot(-K)}^{n-i-1}(u) = \rho_{f_1(\tau)\cdot K \widehat{+}_i f_2(\tau)\cdot(-K)}^{n-i-1}(-u).$$

This together with (2.1) yields

$$f_1(\tau)\rho_K^{n-i-1}(u) + f_2(\tau)\rho_{-K}^{n-i-1}(u) = f_1(\tau)\rho_K^{n-i-1}(-u) + f_2(\tau)\rho_{-K}^{n-i-1}(-u),$$

i.e.,

$$f_1(\tau)\rho_K^{n-i-1}(u) + f_2(\tau)\rho_{-K}^{n-i-1}(u) = f_1(\tau)\rho_{-K}^{n-i-1}(u) + f_2(\tau)\rho_K^{n-i-1}(u),$$

thus

$$[f_1(\tau) - f_2(\tau)][\rho_K^{n-i-1}(u) - \rho_{-K}^{n-i-1}(u)] = 0.$$

Since $K \notin \mathcal{S}_{os}^n$ implies $\rho_K^{n-i-1}(u) - \rho_{-K}^{n-i-1}(u) \neq 0$, thus we obtain

$$f_1(\tau) - f_2(\tau) = 0.$$

This and (2.2) give $\tau = 0$. □

2.2. Dual mixed quermassintegrals

In 1975, Lutwak ([18]) introduced the dual mixed volumes as follows: For $K_1, \cdots, K_n \in \mathcal{S}_o^n$, the dual mixed volume $\widetilde{V}(K_1, \cdots, K_n)$ is defined by

$$\widetilde{V}(K_1, K_2, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\rho_{K_2}(u)\cdots\rho_{K_n}(u)du. \tag{2.6}$$

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$ and $K_n = L$ in (2.6), then we write $\widetilde{W}_i(K, L) = \widetilde{V}(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{B, \cdots, B}_{i}, L)$ $(i = 0, 1, \cdots, n-2)$. If let i be real, then $\widetilde{W}_i(K, L)$ is called the dual mixed quermassintegrals whose representation is that for $K, L \in \mathcal{S}_o^n$ and $i \in \mathbb{R}$,

$$\widetilde{W}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i-1}(u)\rho_L(u)du. \tag{2.7}$$

If we let $K = L$ in (2.7), then it just is the dual quermassintegrals, $\widetilde{W}_i(K)$, of $K \in \mathcal{S}_o^n$ denoted by

$$\widetilde{W}_i(K) = \widetilde{W}_i(K, K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)du. \tag{2.8}$$

Further let $i = 0$ in (2.8), then we have the following polar coordinate formula for the volume of a body K:

$$V(K) = \widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)du.$$

For the above dual mixed quermassintegrals, the corresponding the Minkowski inequality is stated that (see [20]): *If $K, L \in \mathcal{S}_o^n$ and $0 \le i < n-1$, then*

$$\widetilde{W}_i(K, L) \le \widetilde{W}_i(K)^{\frac{n-i-1}{n-i}}\widetilde{W}_i(L)^{\frac{1}{n-i}}, \tag{2.9}$$

with equality if and only if K and L are dilatate.

3. Busemann-Petty Problem for the *i*-th Radial Blaschke-Minkowski Homomorphisms

This section is mainly devoted to prove Theorems 1.1, 1.2 and 1.3. We begin by proving the following lemma.

Lemma 3.1. *If $M, N \in \mathcal{S}_o^n$, $0 \le i, j < n-1$, then*

$$\widetilde{W}_j(M, \Psi_i N) = \widetilde{W}_i(N, \Psi_j M). \tag{3.1}$$

Proof. According to (1.3) and (2.7), we obtain that if $0 \leq i, j < n-1$, then

$$\begin{aligned}\widetilde{W}_j(M, \Psi_i N) &= \frac{1}{n}\int_{S^{n-1}} \rho_M^{n-j-1}(u)\rho_{\Psi_i N}(u)du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_M^{n-j-1}(u)\rho_N^{n-i-1}(u) * \mu du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_N^{n-i-1}(u)\rho_M^{n-j-1}(u) * \mu du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_N^{n-i-1}(u)\rho_{\Psi_j M}(u)du \\ &= \widetilde{W}_i(N, \Psi_j M).\end{aligned}$$

□

Proof of Theorem 1.1. Since $\Psi_i K \subseteq \Psi_i L$ $(0 \leq i < n-1)$, thus using (2.7) we know for any $M \in \mathcal{S}_o^n$ and $0 \leq j < n-1$,

$$\widetilde{W}_j(M, \Psi_i K) \leq \widetilde{W}_j(M, \Psi_i L).$$

This together with (3.1) yields

$$\widetilde{W}_i(K, \Psi_j M) \leq \widetilde{W}_i(L, \Psi_j M).$$

Because of $K \in \Psi_i \mathcal{S}_o^n$, taking $\Psi_j M = K$, then by (2.8) and inequality (2.9) we obtain

$$\widetilde{W}_i(K) \leq \widetilde{W}_i(L, K) \leq \widetilde{W}_i(L)^{\frac{n-i-1}{n-i}} \widetilde{W}_i(K)^{\frac{1}{n-i}},$$

i.e.,

$$\widetilde{W}_i(K) \leq \widetilde{W}_i(L).$$

According to the equality condition of inequality (2.9), we see that $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ if and only if K and L are dilatate. From this, let $K = cL$ $(c > 0)$ and together with $\widetilde{W}_i(K) = \widetilde{W}_i(L)$, we obtain $c = 1$. Therefore, $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ if and only if $K = L$ in Theorem 1.1. □

Proof of Theorem 1.2. Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ denote the generating measure of Ψ_i. Since $L \in \mathcal{S}_{os}^n$ and $\rho(L, \cdot) \in \mathcal{H}^n$, it follows from Schuster's conclusion (see [21], the proof of Theorem 4.4) that there exists an even function $f \in \mathcal{H}^n$, such that

$$\rho(L, \cdot) = f * \mu. \tag{3.2}$$

Here the function f must be negative, otherwise, there exists $L_0 \in \mathcal{S}_o^n$ such that $\rho(L_0, \cdot)^{n-i-1} = f$. This together (1.3) with (3.2) yields

$$\rho(\Psi_i L_0, \cdot) = \rho(L_0, \cdot)^{n-i-1} * \mu = f * \mu = \rho(L, \cdot),$$

i.e., $L = \Psi_i L_0$. This and $L \notin \Psi_i \mathcal{S}_o^n$ are contradictory.

From this, we can find a non-negative, even function $G \in \mathcal{H}^n$ and an even function $H \in \mathcal{H}^n$, such that

$$G = H * \mu. \tag{3.3}$$

Because of $L \in \mathcal{S}_{os}^n$ and $\rho(L, \cdot) > 0$, hence there exists $\varepsilon > 0$ and $K \in \mathcal{S}_{os}^n$, such that

$$\rho(K, \cdot)^{n-i-1} = \rho(L, \cdot)^{n-i-1} - \varepsilon H,$$

thus

$$\rho(K, \cdot)^{n-i-1} * \mu = \rho(L, \cdot)^{n-i-1} * \mu - \varepsilon H * \mu.$$

Therefore, by (1.3) and (3.3) we have

$$\rho(\Psi_i K, \cdot) = \rho(\Psi_i L, \cdot) - \varepsilon G.$$

This together $G \geq 0$ with $\varepsilon > 0$ gives

$$\Psi_i K \subset \Psi_i L.$$

But by (2.7), (2.8), (3.2) and (1.3), we obtain

$$\begin{aligned}
\widetilde{W}_i(L) - \widetilde{W}_i(K, L) &= \frac{1}{n}\int_{S^{n-1}} \rho_L^{n-i}(u)du - \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i-1}(u)\rho_L(u)du \\
&= \frac{1}{n}\int_{S^{n-1}} [\rho_L^{n-i-1}(u) - \rho_K^{n-i-1}(u)]\rho_L(u)du \\
&= \frac{1}{n}\int_{S^{n-1}} [\rho_L^{n-i-1}(u) - \rho_K^{n-i-1}(u)](f * \mu)du \\
&= \frac{1}{n}\int_{S^{n-1}} [(\rho_L^{n-i-1}(u) * \mu) - (\rho_K^{n-i-1}(u) * \mu)]f du \\
&= \frac{1}{n}\int_{S^{n-1}} [\rho_{\Psi_i L}(u) - \rho_{\Psi_i K}(u)]f du. \qquad (3.4)
\end{aligned}$$

Notice that $\Psi_i K \subset \Psi_i L$ and $f < 0$, then (3.4) gives

$$\widetilde{W}_i(L) - \widetilde{W}_i(K, L) < 0.$$

Hence, using Minkowski inequality (2.9) we have

$$\widetilde{W}_i(L) < \widetilde{W}_i(K, L) \leq \widetilde{W}_i(K)^{\frac{n-i-1}{n-i}}\, \widetilde{W}_i(L)^{\frac{1}{n-i}},$$

this and $0 \leq i < n-1$ yield

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

□

The proof of Theorem 1.3 needs the following lemmas.

Lemma 3.2. *If $K \in \mathcal{S}_o^n$, $0 \leq i < n-1$ and $\tau \in [-1, 1]$, then*

$$\widetilde{W}_i(\widehat{\nabla}_i^\tau K) \leq \widetilde{W}_i(K), \qquad (3.5)$$

with equality for $\tau \in (-1, 1)$ if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.5) becomes an equality.

Proof. According to (2.1) and (2.7), we have for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
\widetilde{W}_i(\lambda \cdot K \widehat{+}_i \mu \cdot L, Q) &= \frac{1}{n}\int_{S^{n-1}} \rho_{\lambda \cdot K \widehat{+}_i \mu \cdot L}^{n-i-1}(u)\rho_Q(u)du \\
&= \frac{\lambda}{n}\int_{S^{n-1}} \rho_K^{n-i-1}(u)\rho_Q(u)du + \frac{\mu}{n}\int_{S^{n-1}} \rho_L^{n-i-1}(u)\rho_Q(u)du \\
&= \lambda \widetilde{W}_i(K, Q) + \mu \widetilde{W}_i(L, Q).
\end{aligned}$$

Using inequality (2.9) we obtain

$$\widetilde{W}_i(\lambda \cdot K \widehat{+}_i \mu \cdot L, Q) \leq [\lambda \widetilde{W}_i(K)^{\frac{n-i-1}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-i-1}{n-i}}]\widetilde{W}_i(Q)^{\frac{1}{n-i}}$$

Let $Q = \lambda \cdot K \widehat{+}_i \mu \cdot L$ in above inequality and together with (2.8), then

$$\widetilde{W}_i(\lambda \cdot K \widehat{+}_i \mu \cdot L)^{\frac{n-i-1}{n-i}} \leq \lambda \widetilde{W}_i(K)^{\frac{n-i-1}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{n-i-1}{n-i}}. \qquad (3.6)$$

And the equality condition of inequality (2.9) implies that equality holds in (3.6) for $\lambda, \mu > 0$ if and only if K and L are dilatate (if $\lambda = 0$ or $\mu = 0$, then (3.6) becomes an equality).

From (3.6), (2.3) and (2.2), and together $f_1(\tau) + f_2(\tau) = 1$ with $\widetilde{W}_i(K) = \widetilde{W}_i(-K)$, we get that

$$\widetilde{W}_i(\widehat{\nabla}_i^\tau K) \leq \widetilde{W}_i(K),$$

this is just inequality (3.5).

Since $f_1(\tau), f_2(\tau) > 0$ with $\tau \in (-1, 1)$, from the equality condition of (3.6), we know that equality holds in (3.5) for $\tau \in (-1, 1)$ if and only if K and $-K$ are dilatate, that is K is origin-symmetric.

If $\tau = \pm 1$, then by $\widehat{\nabla}_i^{\pm 1} K = \pm K$ we see (3.5) becomes an equality. □

Lemma 3.3. *Let Ψ_i $(0 \leq i < n-1)$ be an even i-th radial Blaschke-Minkowski homomorphism. If $K \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, then*

$$\Psi_i(\widehat{\nabla}_i^\tau K) = \Psi_i K. \tag{3.7}$$

Proof. Since Ψ_i $(0 \leq i < n-1)$ is an even i-th radial Blaschke-Minkowski homomorphism, thus for any $K \in \mathcal{S}_o^n$, $\Psi_i(-K) = \Psi_i K$.

From this, according to (1.3), (2.1) and (2.3), we have

$$\begin{aligned}\rho(\Psi_i(\widehat{\nabla}_i^\tau K), \cdot) &= \rho(\widehat{\nabla}_i^\tau K, \cdot)^{n-i-1} * \mu \\ &= [f_1(\tau)\rho(K, \cdot)^{n-i-1} + f_2(\tau)\rho(-K, \cdot)^{n-i-1}] * \mu \\ &= f_1(\tau)\rho(K, \cdot)^{n-i-1} * \mu + f_2(\tau)\rho(-K, \cdot)^{n-i-1} * \mu \\ &= f_1(\tau)\rho(\Psi_i K, \cdot) + f_2(\tau)\rho(\Psi_i(-K), \cdot) \\ &= f_1(\tau)\rho(\Psi_i K, \cdot) + f_2(\tau)\rho(\Psi_i K, \cdot) = \rho(\Psi_i K, \cdot).\end{aligned}$$

This gives (3.7). □

Proof of Theorem 1.3. Since $K \notin \mathcal{S}_{os}^n$, thus by Lemma 3.2 we know that for $\tau \in (-1, 1)$,

$$\widetilde{W}_i(\widehat{\nabla}_i^\tau K) < \widetilde{W}_i(K).$$

Choose $\varepsilon > 0$ such that

$$\widetilde{W}_i((1+\varepsilon)\widehat{\nabla}_i^\tau K) < \widetilde{W}_i(K).$$

From this, let $L = (1+\varepsilon)\widehat{\nabla}_i^\tau K$, then $L \in \mathcal{S}_o^n$ (Theorem 2.1 gives that for $\tau = 0$, $L \in \mathcal{S}_{os}^n$; for $\tau \in (-1, 1)$ and $\tau \neq 0$, $L \in \mathcal{S}_o^n \backslash \mathcal{S}_{os}^n$) and $\widetilde{W}_i(L) < \widetilde{W}_i(K)$.

But by (1.4) and (3.7) we obtain

$$\Psi_i L = \Psi_i((1+\varepsilon)\widehat{\nabla}_i^\tau K) = (1+\varepsilon)^{n-i-1}\Psi_i(\widehat{\nabla}_i^\tau K) = (1+\varepsilon)^{n-i-1}\Psi_i K \supset \Psi_i K.$$ □

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