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Perturbation Bounds for the Metric Projection of a Point onto a Linear Manifold in Reflexive Strictly Convex Banach Spaces

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Abstract. In this paper, by using some recent perturbation bounds for the Moore–Penrose metric generalized inverse, we present some results on the perturbation analysis for projecting a point onto a linear manifold in reflexive strictly convex Banach spaces. The main results have two parts, part one covers consistent operator equations and part two covers the general so-called ill posed operator equations.

1. Introduction

Let X and Y be Banach spaces. Let $B(X, Y)$ be the Banach space consisting of all bounded linear operators from X to Y. For $A \in B(X, Y)$, let $\mathcal{N}(A)$ (resp. $\mathcal{R}(A)$) denote the kernel (resp. range) of A. Consider the following problem for projecting a point onto a linear manifold: For the given $A \in B(X, Y)$ with $\mathcal{R}(A)$ closed, $b \in Y$ and $p \in X$, find a vector $x^* \in X$ satisfying

$$\begin{aligned} &\|p - x^*\| = \inf_{x \in S} \|p - x\| \\ &\text{subject to} \quad S = \{x \in X : \|Ax - b\| = \inf_{z \in X} \|Az - b\|\}. \end{aligned} \tag{1.1}$$

The collection of all vectors $x \in X$ satisfying the constraint in (1.1) will be called the feasible set and its elements will be called feasible solutions of (1.1). Solving the problem (1.1) is important in many applications. For example, when $p = 0$, then the problem (1.1) is just the usual minimal norm least squares problem; if $b = 0$, then the problem (1.1) is that of projecting the vector p to the null space $\mathcal{N}(A)$, which is a key step in the interior-point projective algorithm for linear programming initiated with Karmarkar's pioneering work [12]. When X and Y are finite dimensional vector spaces or infinite dimensional Hilbert spaces, it is well-known that the unique optimal solution x^* to the problem (1.1) exists and unique, indeed, $x^* = A^\dagger b + (I - A^\dagger A)p$, where $A^\dagger$ is the Moore–Penrose orthogonal projection generalized inverse of A.

Now, we give A (resp. b and p) a small perturbation δA (resp. δb and δp). Put $\bar{A} = A + \delta A$, $\bar{b} = b + \delta b$ and $\bar{p} = p + \delta p$. Then the problem (1.1) is perturbed to the following:

$$\begin{aligned} &\|\bar{p} - y^*\| = \inf \|\bar{p} - y\| \\ &\text{subject to} \quad \tilde{S} = \{x \in X : \|\bar{A}y - \bar{b}\| = \inf_{z \in X} \|\bar{A}z - \bar{b}\|\}. \end{aligned} \tag{1.2}$$

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Obviously, if $\bar{A}^\dagger$ exists, then the problem (1.2) also has a unique optimal solution $y^* = \bar{A}^\dagger\bar{b} + (I - A^\dagger A)\bar{p})$. When X and Y are finite dimensional vector spaces or infinite dimensional Hilbert spaces, the problem (1.1) and its perturbation problem (1.2) have been considered by many authors in the literature (see [3, 6, 7, 18]). In their paper [21], by using the so-called stable perturbation [4] of operators (i.e., $\mathcal{R}(\bar{A}) \cap \mathcal{N}(A^+) = \{0\}$), the authors first got the following important perturbation results of the Moore–Penrose orthogonal projection generalized inverse in Hilbert spaces. More precisely, let $A \in B(X, Y)$ with $\mathcal{R}(A)$ closed, suppose that $\|A^\dagger\|\|\delta A\| < 1$ and $\mathcal{R}(\bar{A}) \cap \mathcal{N}(A^\dagger) = \{0\}$, then, the authors proved that (see [21, Proposition 7]) $\bar{A}^\dagger$ exists and

$$\|\bar{A}^\dagger - A^\dagger\| \leq \frac{1+\sqrt{5}}{2}\frac{\|A^\dagger\|^2\|\delta A\|}{1-\|A^\dagger\|\|\delta A\|}. \tag{1.3}$$

The perturbation bound (1.3) above has many applications, especially in solving problem (1.1) and its perturbation problem (1.2). As a consequence, they obtained (see [21, Proposition 8]) the following perturbation estimate for problems (1.1) and (1.2):

$$\begin{aligned}\frac{\|y^* - x^*\|}{\|x^*\|} \leq &\frac{\kappa}{1-\kappa\epsilon_A}\left(\epsilon_A + \frac{\|b - Ax^*\|}{\|A\|\|x^*\|}\kappa\epsilon_A + \frac{\|b\|}{\|A\|\|x^*\|}\epsilon_b\right)\\ &+ \frac{\|p - x^*\|}{\|x^*\|}\kappa\epsilon_A + \frac{\|p\|}{\|x^*\|}\epsilon_p,\end{aligned} \tag{1.4}$$

where $\epsilon_b = \frac{\|\delta b\|}{\|b\|}$, $\epsilon_A = \frac{\|\delta A\|}{\|A\|}$, $\epsilon_p = \frac{\|\delta p\|}{\|p\|}$ and $\kappa = \|A\|\|A^\dagger\|$ be the condition number of A.

Over the years, generalizations of the perturbation bound (1.3) have been considered in many papers (see [5, 9, 13, 19]). In recent years, by using the so-called the Moore–Penrose metric generalized inverse [16], certain results on extending the perturbation bound (1.3) to operators on Banach spaces are also considered by many authors (see [8, 14]). Motivated by some results in Hilbert spaces, and based on our recent perturbation results of the Moore–Penrose metric generalized inverse [2], in this paper, we will make a further study on the problem (1.1) and its perturbation problem (1.2) in reflexive and strictly convex Banach spaces, as a consequence, we present certain extensions of the perturbation bound (1.4). Some particular cases and applications will be also considered.

2. Preliminaries

In this section, we recall some concepts and basic results will be used in this paper. We first present the definition of set-valued metric projection.

Definition 2.1 ([15]). *Let $M \subset X$ be a subset. The set-valued mapping $P_M : X \to M$ defined by*

$$P_M(x) = \{s \in M \mid \|x - s\| = \operatorname{dist}(x, M)\}, \qquad \forall x \in X$$

is called the set-valued metric projection, where $\operatorname{dist}(x, M) = \inf_{z\in X} \|x - z\|$.

For $M \subset X$, if $P_M(x)$ is nonempty and contains at most a singleton for each $x \in X$, then M is called a Chebyshev set. We denote by π_M any selection for the set-valued mapping P_M, i.e., any single-valued mapping $\pi_M : \mathcal{D}(\pi_M) \to M$ with the property that $\pi_M(x) \in P_M(x)$ for any $x \in \mathcal{D}(\pi_M)$, where $\mathcal{D}(\pi_M) = \{x \in X : P_M(x) \neq \emptyset\}$. For the particular case, when M is a Chebyshev set, the mapping π_M is called the metric projector from X onto M.

Remark 2.2 ([15]). *It is well-known that if X is reflexive and strictly convex Banach space, then every closed convex subset in X is a Chebyshev set, and the metric projector is just the linear orthogonal projector in Hilbert space.*

We need the following important properties of the metric projection.

Lemma 2.3 ([15]). *Let X be a Banach space and L be a Chebyshev subspace of X. Then the metric projection π_L is quasi-additive on L. Moreover, $\|x - \pi_L(x)\| \leq \|x\|$ for any $x \in X$, i.e., $\|\pi_L\| \leq 2$.*

Definition 2.4. *Let $M \subset X$ be a subset and let $A: X \to Y$ be a mapping. Then we call A is quasi-additive on M if A satisfies*

$$A(x+z) = A(x) + A(z), \qquad \forall\, x \in X,\ \forall\, z \in M.$$

If A is quasi-additive on $\mathcal{R}(A)$, then we will simply say A is a quasi-linear operator. In general, quasi-linear operator is not a linear operator.

Now, we present the definition of the Moore–Penrose metric generalized inverse.

Definition 2.5 ([16]). *Let $A \in B(X,Y)$. Suppose that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are Chebyshev subspaces of X and Y, respectively. If there exists a bounded homogeneous operator $A^M: Y \to X$ such that:*

(1) $AA^MA = A$; (2) $A^MAA^M = A^M$;

(3) $A^MA = I_X - \pi_{\mathcal{N}(A)}$; (4) $AA^M = \pi_{\mathcal{R}(A)}$.

Then A^M is called the Moore–Penrose metric generalized inverse of A, where $\pi_{\mathcal{N}(A)}$ and $\pi_{\mathcal{R}(A)}$ are the metric projectors onto $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.

When X and Y are Hilbert spaces, then from Definition 2.5, we see obviously that the Moore–Penrose metric generalized inverse A^M of A is indeed the Moore–Penrose orthogonal projection generalized inverse $A^\dagger$ of A under usual sense. It is well-known that the theory of the Moore–Penrose metric generalized inverses has its genetic in the context of the so-called ill-posed linear problems. Please see [16, 17] for more information about the Moore–Penrose metric generalized inverses and related knowledge. Here we only need the following result which characterizes the existence of the Moore–Penrose metric generalized inverse in a reflexive and strictly convex Banach space.

Proposition 2.6 ([17, Corollary 2.1]). *Let X, Y be reflexive strictly convex Banach Spaces, let $A \in B(X,Y)$ with $\mathcal{R}(A)$ is closed. Then there exists a unique Moore–Penrose metric generalized inverse $A^M$of A.*

In our recent paper [2], mainly based on Proposition 2.6, by using the concept of quasi-additivity and the so-called generalized Neumman lemma [5], we have obtained the following perturbation results of the Moore–Penrose metric generalized inverse in reflexive and strictly convex Banach spaces.

Lemma 2.7 ([2, Theorem 3.3]). *Let X, Y be reflexive strictly convex Banach spaces, let A, $\delta A \in B(X,Y)$ with $\mathcal{R}(A)$ closed. Put $\bar{A} = A + \delta A$. Suppose that there exist two constants $\lambda_1, \lambda_2 \in (-1,1)$ such that $\|\delta Ax\| \le \lambda_1\|Ax\| + \lambda_2\|\bar{A}x\|$, then*

(1) *A^M exists and $\mathcal{N}(\bar{A}) = \mathcal{N}(A)$.*

In addition, if A^M is quasi-additive on $\mathcal{R}(A)$, then

(2) *$\bar{A}^M$ exists. Moreover,*

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|A\|} \le \|\bar{A}^M\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^M\|,$$

$$\|\bar{A}^M - A^M\| \le \|A^M\|\|\pi_{\mathcal{R}(\bar{A})} - \pi_{\mathcal{R}(A)}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A^M\|^2\|\delta A\|. \tag{2.1}$$

We should indicate that the error estimate formula (2.1) is not presented in [2, Theorem 3.3]. But, under our assumption, we can obtain this estimate easily. In fact, since $\mathcal{N}(\bar{A}) = \mathcal{N}(A)$, we have $\bar{A}^M\bar{A} = A^MA$, and then

$$\begin{aligned}\|\bar{A}^M - A^M\| &= \|\bar{A}^M\bar{A}\bar{A}^M - A^MAA^M\| = \|A^MA\bar{A}^M - A^MAA^M\| \\ &= \|A^M\bar{A}\bar{A}^M - A^MAA^M - A^M\delta A\bar{A}^M\| \\ &\le \|A^M\bar{A}\bar{A}^M - A^MAA^M\| + \|A^M\delta A\bar{A}^M\| \\ &\le \|A^M\|\|\pi_{\mathcal{R}(\bar{A})} - \pi_{\mathcal{R}(A)}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A^M\|^2\|\delta A\|.\end{aligned}$$

The perturbed bounds obtained in Lemma 2.7 will be used in the following section.

3. Main Results

Unless stated otherwise, in the remainder of this paper, for convenience, we always assume that X and Y are reflexive and strictly convex Banach spaces. Similar as in Hilbert spaces, we can prove the following existence and uniqueness result for the problem (1.1).

Lemma 3.1. *Suppose that $\mathcal{R}(A) \subset Y$ is closed, then, the unique optimal solution to the problem* (1.1) *exists, and can be expressed as*

$$x^* = A^M b + \pi_{\mathcal{N}(A)} p.$$

Proof. Since X, Y are reflexive strictly convex Banach spaces and $\mathcal{R}(A)$ is closed, it follows from [20, Proposition 2.3.7] that the problem (1.1) has solutions. Moreover, from Definition 2.5, we see that the feasible solution x to (1.1) can be expressed as $A^M b + \pi_{\mathcal{N}(A)} u$ for any vector $u \in X$, and then from Remark 2.2, we know that the optimal solution to (1.1) exists and uniquely. Using Definition 2.5 again, we see that $x^* = A^M b + \pi_{\mathcal{N}(A)} p$. □

In the remainder of this paper, the optimal solutions to (1.1) and (1.2) will be denoted by x^* and y^*, respectively. For convenience, in this section, we always let $\epsilon_b = \dfrac{\|\delta b\|}{\|b\|}$, $\epsilon_A = \dfrac{\|\delta A\|}{\|A\|}$, and $\kappa = \|A\|\|A^M\|$. Firstly, we assume that both linear operator equations $Ax = b$ and $\bar{A}y = \bar{b}$ are consistent, that is, assume that $b \in \mathcal{R}(A)$ and $\bar{b} \in \mathcal{R}(\bar{A})$. We always assume that $x \neq 0$ whenever $\|x\|$ appears in the denominator.

Theorem 3.2. *Let $A,\ \delta A \in B(X, Y)$ with $\mathcal{R}(A)$ closed. Assume that A^M is quasi-additive on $\mathcal{R}(A)$ and $\mathcal{R}(\delta A)$. Put $\bar{A} = A + \delta A$. Suppose that $b \in \mathcal{R}(A)$ and $\bar{b} \in \mathcal{R}(\bar{A})$,*

(1) *If $\|A^M\|\|\delta A\| < 1$, then for any solution y to the problem* (1.2)*, there is a feasible solution x to the problem* (1.1) *such that*

$$\frac{1}{1+\kappa\epsilon_A}\left(\frac{\|A^M \delta b\|}{\|A^M b\| + 2\|y\|} - \kappa\epsilon_A\right) \leq \frac{\|y-x\|}{\|x\|} \leq \frac{\kappa}{1-\kappa\epsilon_A}(\epsilon_b + \epsilon_A). \tag{3.1}$$

(2) *Suppose that there exist two constants $\lambda_1, \lambda_2 \in (-1, 1)$ such that $\|\delta Ax\| \leq \lambda_1\|Ax\| + \lambda_2\|\bar{A}x\|$, and $\bar{A}^M$ is quasi-additive on $\mathcal{R}(A)$. Then*

$$\frac{\|y^* - x^*\|}{\|x^*\|} \leq \kappa\left(\|\pi_{\mathcal{R}(\bar{A})} - \pi_{\mathcal{R}(A)}\| + \frac{1+\lambda_2}{1-\lambda_1}(\kappa\epsilon_A + \epsilon_b)\right) + 2\frac{\|\delta p\|}{\|x^*\|}. \tag{3.2}$$

In addition, if $\mathcal{R}(\bar{A}) = \mathcal{R}(A)$, then

$$\frac{\|y^* - x^*\|}{\|x^*\|} \leq \frac{1+\lambda_2}{1-\lambda_1}(\kappa\epsilon_A + \epsilon_b)\kappa + 2\frac{\|\delta p\|}{\|x^*\|}.$$

Proof. (1) Let x be the metric projection of y onto the feasible set of (1.1), i.e., $x = A^M b + \pi_{\mathcal{N}(A)} y$. Noting that $\bar{A}y = \bar{b}$ and A^M is quasi-additive on $\mathcal{R}(A)$ and $\mathcal{R}(\delta A)$, we get

$$\begin{aligned} y - x &= A^M A y - A^M b = A^M(Ay - b) \\ &= A^M(\bar{A}y - b - \delta A y) \\ &= A^M(\delta b - \delta A y) \\ &= A^M(\delta b - \delta A x) - A^M \delta A(y - x), \end{aligned}$$

which implies that $(I_X + A^M\delta A)(y - x) = A^M(\delta b - \delta Ax)$. Since $\|A^M\|\|\delta A\| < 1$, we know that $I_X + A^M\delta A$ is invertible. Hence,

$$y - x = (I_X + A^M\delta A)^{-1} A^M(\delta b - \delta Ax).$$

Now, noting that $\|A\|\|x\| \geq \|Ax\| = \|b\|$, we have

$$\begin{aligned}\frac{\|y-x\|}{\|x\|} &= \frac{\|(I_X + A^M\delta A)^{-1}A^M(\delta b - \delta Ax)\|}{\|x\|}\\ &\leq \frac{1}{1-\|A^M\delta A\|}\frac{\|A^M(\delta b - \delta Ax)\|}{\|x\|}\\ &\leq \frac{\|A^M\|\|A\|}{1-\|A^M\|\|\delta A\|}\frac{\|\delta b\| + \|\delta A\|\|x\|}{\|A\|\|x\|}\\ &\leq \frac{\kappa}{1-\kappa\epsilon_A}(\epsilon_b + \epsilon_A),\end{aligned}$$

which gives the right inequality of (3.1).

On the other hand, from Lemma 2.3, we know $\|\pi_{\mathcal{N}(A)}\| \leq 2$, and then

$$\|x\| \leq \|A^M b\| + \|\pi_{\mathcal{N}(A)} y\| \leq \|A^M b\| + 2\|y\|,$$

thus, the left one in (3.1) is from

$$\begin{aligned}\frac{\|y-x\|}{\|x\|} &\geq \frac{\|A^M(\delta b - \delta Ax)\|}{\|I_X + A^M\delta A\|\|x\|}\\ &\geq \frac{1}{1+\|A^M\|\|\delta A\|}\frac{\|A^M\delta b\| - \|A^M\|\|\delta A\|\|x\|}{\|x\|}\\ &\geq \frac{1}{1+\kappa\epsilon_A}\left(\frac{\|A^M\delta b\|}{\|A^M b\| + 2\|y\|} - \kappa\epsilon_A\right).\end{aligned}$$

(2) From Lemma 2.7, we know $\bar{A}^M$ exists, then from Lemma 3.1, we have

$$x^* = A^M b + (I_X - A^M A)p, \qquad y^* = \bar{A}^M\bar{b} + (I_X - \bar{A}^M\bar{A})\bar{p}.$$

Subtracting the second equality from the first equality above, we have

$$y^* - x^* = \bar{A}^M\bar{b} - A^M b + \delta p - \bar{A}^M\bar{A}\bar{p} + A^M Ap. \tag{3.3}$$

From Lemma 2.7 (1), we have $\mathcal{N}(\bar{A}) = \mathcal{N}(A)$, which means that $\bar{A}^M\bar{A} = A^M A$. Now, from (3.3), and noting that A^M and $\bar{A}^M$ are quasi-additive on $\mathcal{R}(A)$, $b \in \mathcal{R}(A)$, we have

$$\begin{aligned}y^* - x^* &= \bar{A}^M\bar{b} - A^M b + \delta p - A^M A\delta p\\ &= (\bar{A}^M - A^M)b + \bar{A}^M\delta b + (I_X - A^M A)\delta p.\end{aligned} \tag{3.4}$$

Noting that $\|b\| = \|Ax^*\| \leq \|A\|\|x^*\|$, i.e., $\frac{1}{\|x^*\|} \leq \frac{\|A\|}{\|b\|}$. From Lemma 2.3, we also have $\|I_X - A^M A\| = \|\pi_{\mathcal{N}(A)}\| \leq 2$. Now, by Lemma 2.7 (2) and (3.4),

$$\begin{aligned}\frac{\|y^* - x^*\|}{\|x^*\|} &= \frac{\|(\bar{A}^M - A^M)b + \bar{A}^M\delta b + (I_X - A^M A)\delta p\|}{\|x^*\|}\\ &\leq \frac{\|A\|}{\|b\|}\left(\|\bar{A}^M - A^M\|\|b\| + \|\bar{A}^M\|\|\delta b\|\right) + 2\frac{\|\delta p\|}{\|x^*\|}\\ &\leq \|A\|\|A^M\|\|\pi_{\mathcal{R}(\bar{A})} - \pi_{\mathcal{R}(A)}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A\|\|A^M\|^2\|\delta A\|\\ &\qquad + \frac{1+\lambda_2}{1-\lambda_1}\|A\|\|A^M\|\frac{\|\delta b\|}{\|b\|} + 2\frac{\|\delta p\|}{\|x^*\|}\\ &= \kappa\left(\|\pi_{\mathcal{R}(\bar{A})} - \pi_{\mathcal{R}(A)}\| + \frac{1+\lambda_2}{1-\lambda_1}(\kappa\epsilon_A + \epsilon_b)\right) + 2\frac{\|\delta p\|}{\|x^*\|}.\end{aligned} \tag{3.5}$$

Finally, if $\mathcal{R}(\bar{A}) = \mathcal{R}(A)$, then our desirable result follows from (3.5) and Lemma 2.7 (2). □

Now we consider the problems (1.1) and (1.2) for some general cases, that is, we drop the assumption that $b \in \mathcal{R}(A)$ and $\bar{b} \in \mathcal{R}(\bar{A})$ in Theorem 3.2.

Theorem 3.3. *Let $A,\ \delta A \in B(X, Y)$ with $\mathcal{R}(A)$ closed. Assume that A^M is quasi-additive on $\mathcal{R}(A)$ and $\mathcal{R}(\delta A)$. Put $\bar{A} = A + \delta A$.*

(1) *If $\|A^M\|\|\delta A\| < 1$, then for any solution y to the problem (1.2), there is a feasible solution x to the problem (1.1) such that*

$$\frac{\|y-x\|}{\|x\|} \leq \frac{\kappa}{1-\kappa\epsilon_A}\left(\frac{\|Ax-b\|+2(\|b\|+\|\delta b\|)}{\|A\|\|x\|}+2\epsilon_A\right). \tag{3.6}$$

(2) *If there exist two constants $\lambda_1, \lambda_2 \in (-1, 1)$ such that $\|\delta Ax\| \leq \lambda_1\|Ax\| + \lambda_2\|\bar{A}x\|$, then*

$$\frac{\|y^*-x^*\|}{\|x^*\|} \leq \frac{\kappa\|b\|}{\|A\|\|x^*\|}\left(\frac{1+\lambda_2}{1-\lambda_1}(1+\epsilon_b)+1\right)+2\frac{\|\delta p\|}{\|x^*\|}. \tag{3.7}$$

Proof. (1) Let $x = A^M b + \pi_{\mathcal{N}(A)}y$. Then, we see that x is a feasible solution to the problem (1.1). Let $\tilde{r} = \bar{A}y - \bar{b}$ be the residual of y. Then, since y is an extremal solution of (1.2), we get that

$$\|\tilde{r}\| = \|\bar{A}y - \bar{b}\| \leq \|\bar{A}x - \bar{b}\| \leq \|Ax - b\| + \|\delta b - \delta Ax\|. \tag{3.8}$$

On the other hand, noting that A^M is quasi-additive on $\mathcal{R}(\delta A)$ and $\mathcal{R}(A)$, we have

$$\begin{aligned} y - x &= A^M Ay - A^M b = A^M(\bar{A} - \delta A)y - A^M b \\ &= A^M\bar{A}y - A^M\delta Ay - A^M b \\ &= A^M(\bar{A}y - \bar{b}) - A^M\delta A(y-x) - A^M\delta Ax + A^M\bar{b} - A^M b \\ &= A^M\tilde{r} - A^M\delta A(y-x) - A^M\delta Ax + A^M\bar{b} - A^M b, \end{aligned}$$

which implies that

$$(I_X + A^M\delta A)(y-x) = A^M\tilde{r} - A^M\delta Ax + A^M\bar{b} - A^M b. \tag{3.9}$$

Therefore, by (3.8) and (3.9), we have

$$\begin{aligned} \frac{\|y-x\|}{\|x\|} &\leq \frac{\|(I_X + A^M\delta A)^{-1}\|\|A^M\tilde{r} - A^M\delta Ax + A^M\bar{b} - A^M b\|}{\|x\|} \\ &\leq \frac{\|A^M\|}{1-\|A^M\delta A\|}\frac{\|Ax-b\|+\|\delta Ax-\delta b\|+\|\delta A\|\|x\|+2\|b\|+\|\delta b\|}{\|x\|} \\ &\leq \frac{\|A^M\|}{1-\|A^M\delta A\|}\frac{\|Ax-b\|+2(\|b\|+\|\delta b\|+\|\delta A\|\|x\|)}{\|x\|} \\ &\leq \frac{\|A^M\|\|A\|}{1-\|A^M\|\|\delta A\|}\left(\frac{\|Ax-b\|+2(\|b\|+\|\delta b\|)}{\|A\|\|x\|}+2\frac{\|\delta A\|}{\|A\|}\right) \\ &= \frac{\kappa}{1-\kappa\epsilon_A}\left(\frac{\|Ax-b\|+2(\|b\|+\|\delta b\|)}{\|A\|\|x\|}+2\epsilon_A\right). \end{aligned}$$

This proves (3.6).

(2) By Lemma 3.1, we know $x^* = A^M b + \pi_{\mathcal{N}(A)}p$ and $y^* = \bar{A}^M\bar{b} + \pi_{\mathcal{N}(\bar{A})}\bar{p}$. But, from Lemma 2.7, we have $\pi_{\mathcal{N}(\bar{A})} = \pi_{\mathcal{N}(A)}$, which implies that

$$y^* - x^* = \bar{A}^M\bar{b} - A^M b + \pi_{\mathcal{N}(A)}\delta p.$$

Therefore, using Lemma 2.7 (2), we have

$$\begin{aligned}
\frac{\|y^* - x^*\|}{\|x^*\|} &\le \frac{\|\bar{A}^M\bar{b} - A^M b + \pi_{\mathcal{N}(A)}\delta p\|}{\|x^*\|} \\
&\le \frac{\|\bar{A}^M\|(\|b\| + \|\delta b\|) + \|A^M\|\|b\| + 2\|\delta p\|}{\|x^*\|} \\
&\le \frac{1+\lambda_2}{1-\lambda_1}\frac{\|A\|\|A^M\|(\|b\| + \|\delta b\|)}{\|A\|\|x^*\|} + \frac{\|A\|\|A^M\|\|b\|}{\|A\|\|x^*\|} + 2\frac{\|\delta p\|}{\|x^*\|} \\
&= \frac{\kappa}{\|A\|\|x^*\|}\left(\frac{1+\lambda_2}{1-\lambda_1}(\|b\| + \|\delta b\|) + \|b\|\right) + 2\frac{\|\delta p\|}{\|x^*\|} \\
&= \frac{\kappa\|b\|}{\|A\|\|x^*\|}\left(\frac{1+\lambda_2}{1-\lambda_1}(1+\epsilon_b) + 1\right) + 2\frac{\|\delta p\|}{\|x^*\|}.
\end{aligned}$$

This completes the proof. $\square$

Remark 3.4. *Under the same conditions of Theorem 3.3, if, in addition, $b \in \mathcal{R}(A)$, then*

$$\frac{\|y - x\|}{\|x\|} \le \frac{2\kappa}{1 - \kappa\epsilon_A}(\epsilon_b + \epsilon_A).$$

In fact, if $b \in \mathcal{R}(A)$, then $Ax = b$. Moreover, in this case, (3.8) *and* (3.9) *imply that $\|\tilde{r}\| \le \|\delta b - \delta Ax\|$ and*

$$(I_X + A^M\delta A)(y - x) = A^M\tilde{r} + A^M(\delta b - \delta Ax). \tag{3.10}$$

Thus, from (3.6) *and* (3.10)*, and noting that $\|b\| = \|Ax\| \le \|A\|\|x\|$, we can get our result.*

Remark 3.5. *Generally, $\bar{A}^M$ is not a bounded linear operator, so $\bar{A}^M\bar{b} \ne \bar{A}^M b + \bar{A}^M\delta b$. But, under the same conditions of Theorem 3.3, if, in addition, $\mathcal{R}(\bar{A}) = \mathcal{R}(A)$, and $\bar{A}^M$ is quasi-additive on $\mathcal{R}(A)$, $b \in \mathcal{R}(A)$, then $\bar{A}^M\bar{b} = \bar{A}^M b + \bar{A}^M\delta b$ and $\|b\| = \|Ax^*\| \le \|A\|\|x^*\|$. In this case, using Lemma 2.7, we can get a better perturbed bound as in Theorem 3.2.*

As immediate consequences of the above Theorem 3.3, we have the following corollaries.

Corollary 3.6. *Under the same conditions of Theorem 3.3, if, in addition, $b = 0$, and $\delta b = 0$, that is, the problem* (1.1) *is that of projecting p to the null space of A, then*

$$\frac{\|y^* - x^*\|}{\|x^*\|} \le 2\frac{\|\delta p\|}{\|x^*\|}.$$

Corollary 3.7. *Under the same conditions of Theorem 3.3, if, in addition, $p = 0$, and $\delta p = 0$, that is, the problem* (1.1) *is the best approximate solution problem, then*

$$\frac{\|y^* - x^*\|}{\|x^*\|} \le \frac{\kappa\|b\|}{\|A\|\|x^*\|}\left(\frac{1+\lambda_2}{1-\lambda_1}(1+\epsilon_b) + 1\right).$$

4. Concluding Remark

By using some recent perturbation results for the Moore–Penrose metric generalized inverse, the perturbation analysis for the metric projection of a point onto a linear manifold in reflexive strictly convex Banach spaces has been presented in this paper. The main results in our paper have two parts. Part one covers consistent operator equations and part two covers the general so-called ill posed operator equations, To our knowledge, the results here are the first one for Banach space operators up to now, and the results for two special cases of our problem–the best approximate solution problem and the projection of a vector to the null space of an operator are just the consequences of our general analysis.

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