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Univalence Criteria for General Integral Operators Using the Struve and Bessel Functions

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Abstract. In this paper we consider the class of Bessel functions and the class of Struve functions. We obtain some univalence criteria for two general integral operators.

1. Introduction and preliminaries

Let consider U the unit disc. Let $H(U)$ be the set of holomorphic functions in the unit disc U.
Consider $A = \{f \in H(U) : f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f$ is univalent in $U\}$.

Theorem 1.1. *[1] If the function f is regular in unit disc U, $f(z) = z + a_2z^2 + ...$ and*

$$(1-|z|^2)\cdot\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{1}$$

for all $z \in U$, then the function is univalent in U.

Theorem 1.2. *[4] If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$and $z \in U$ the following inequalities hold*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}\cdot g(\xi)}\right| \le \left|\frac{\xi - z}{1-\bar{z}\cdot\xi}\right| \tag{2}$$

and

$$|g'(z)| \le \frac{1-|g(z)|^2}{1-z^2} \tag{3}$$

the equalities hold in case $g(z) = \varepsilon\dfrac{z+u}{1+\bar{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

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Remark 1.3. *[2] For $z = 0$ from inequality (2) we obtain for every $\xi \in U$*

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi| \tag{4}$$

and hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|} \tag{5}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \tag{6}$$

for all $z \in U$.

Let us consider the second-order inhomogeneous differential equation(([**?**]), p.341)

$$z^2 w^{''}(z) + zw^{'}(z) + (z^2 - v^2)w(z) = \frac{4(\frac{z}{2})^{v+1}}{\sqrt{\pi}\Gamma(v + \frac{1}{2})} \tag{7}$$

whose homogeneous part is Bessel's equation, where v is an unrestricted real(or complex) number. The function H_v, which is called the Struve function of order v, is defined as a particular solution of (7). This function has the form

$$H_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2}) \cdot \Gamma(v + n + \frac{3}{2})} \cdot \left(\frac{z}{2}\right)^{2n+v+1} \quad \text{for all } z \in \mathbb{C} \tag{8}$$

We consider the transformation

$$g_v(z) = 2^v \sqrt{\pi}\Gamma(v + \frac{3}{2}) \cdot z^{\frac{-v-1}{2}} H_v(\sqrt{z}) \tag{9}$$

After some calculus we obtain

$$g_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{3}{2})\Gamma(v + \frac{3}{2})}{4^n \cdot \Gamma(n + \frac{3}{2})\Gamma(v + n + \frac{3}{2})} \cdot z^n \tag{10}$$

Using Theorem 2.1 ([5]) for our case with $b = c = 1, \kappa = v + \frac{3}{2}$ we obtain that:

Theorem 1.4. *[5],[3] If $v > \frac{\sqrt{3} - 7}{8}$ then the function g_v is univalent in U.*

The Bessel function of the first kind is defined by

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + v + 1)} \left(\frac{z}{2}\right)^{2n+v}. \tag{11}$$

We consider the transformation

$$f_v(z) = 2^v \Gamma(1 + v) z^{-\frac{v}{2}} J_v(\sqrt{z}) \tag{12}$$

After some calculus we obtain

$$f_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1+v)}{n! \Gamma(n+v+1) \cdot 4^n} \cdot z^n \tag{13}$$

Theorem 1.5. *[7],[9], [3] If $v > -2$ then $Re f_v'(z) < 0$ for $z \in U_1(0, 4(v+2))$ and f_v is univalent in $U_1(0, 4(v+2))$.*

2. Main results

Theorem 2.1. *Let f_{v_i} be Bessel functions, $z \in U, v_i \in (-2,-1), \alpha_i \in \mathbb{C}$ where*
$f_{v_i}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \Gamma(1+v_i)}{n! \cdot \Gamma(n+v_i+1) \cdot 4^n} \cdot z^n, i \in \{1,2,...,n\}$.
If

$$\left| \frac{z f_{v_i}'(z) - f_{v_i}(z)}{z f_{v_i}(z)} \right| \le 1, \text{ for all } i \in \{1,2,...,n\}, \ (\forall) z \in U \tag{14}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{15}$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \le \frac{1}{\max_{|z| \le 1} \left[(1-|z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |z||c|} \right]} \tag{16}$$

where $|c| = \frac{1}{32} \cdot \left| \frac{1}{(2+v_1)(1+v_1)} + \frac{1}{(2+v_2)(1+v_2)} + ... + \frac{1}{(2+v_n)(1+v_n)} \right|$ *then*
$G(z) = \int_0^z \left(\frac{f_{v_1}(t)}{t} \right)^{\alpha_1} \cdot \left(\frac{f_{v_2}(t)}{t} \right)^{\alpha_2} \cdot ... \cdot \left(\frac{f_{v_n}(t)}{t} \right)^{\alpha_n} dt \in S.$

Proof. We have $f_{v_i} \in S, i \in \{1,2,...,n\}$ and $\frac{f_{v_i}(z)}{z} \ne 0$.
For $z = 0$ we have $\int_0^z \left(\frac{f_{v_1(z)}}{z} \right)^{\alpha_1} \cdot \left(\frac{f_{v_2(z)}}{z} \right)^{\alpha_2} \cdot ... \cdot \left(\frac{f_{v_n}(z)}{z} \right)^{\alpha_n} = 1.$
Consider the function

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{G''(z)}{G'(z)}.$$

The function h has the form:

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{z \cdot f_{v_1}'(z) - f_{v_1}(z)}{z f_{v_1(z)}} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{z \cdot f_{v_n}'(z) - f_{v_n}(z)}{z f_{v_n(z)}}$$

We have:

$$h(0) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot a_{2,1} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot a_{2,n}$$

where $a_{2,1} = \frac{\Gamma(1+v_1)}{32 \cdot \Gamma(3+v_1)} = \frac{1}{32(2+v_1)(1+v_1)}$
$a_{2,2} = \frac{1}{32(2+v_2)(1+v_2)}$
$a_{2,n} = \frac{1}{32(2+v_n)(1+v_n)}$.

By using the relations (14) and (15) we obtain $|h(z)| < 1$ and
$h(0) = \dfrac{|\alpha_1 \cdot a_{2,1} + ... + \alpha_n \cdot a_{2,n}|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$ where

$$|c| = \frac{1}{32} \cdot \left| \frac{1}{(2+v_1)(1+v_1)} + \frac{1}{(2+v_2)(1+v_2)} + ... + \frac{1}{(2+v_n)(1+v_n)} \right|$$

Applying Remark 1.3 for the function h we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left| \frac{G''(z)}{G'(z)} \right| \leq \frac{|z| + |c|}{1 + |c| \cdot |z|}$$

$$\Longleftrightarrow \left| (1-|z|^2) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot (1-|z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c||z|},$$

for all $z \in U$.
Let's consider the function $H : [0, 1] \to \mathbb{R}$

$$H(x) = (1 - x^2)x\frac{x + |c|}{1 + |c|}; x = |z|$$

$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + |c|}{2 + |c|} > 0$ then $\max_{x \in [0,1]} H(x) > 0$.
We obtain

$$\left| (1-|z|^2) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \max_{|z| \leq 1} \left[(1-|z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |z||c|} \right].$$

Applying the condition (16) we obtain:

$$(1-|z|^2) \left| \frac{zF''(z)}{F'(z)} \right| \leq 1, \ (\forall) \ z \in U$$

and from Theorem 1.1 then $F \in S$. □

For $\alpha_1 = \alpha_2 = ... = \alpha_n$ in Theorem 2.1 we obtain the next corollary:

Corollary 2.2. *Let f_{v_i} be Bessel functions, $z \in U, v_i \in (-2, -1), \alpha_i \in \mathbb{C}$ where*
$f_{v_i}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \Gamma(1 + v_i)}{n! \cdot \Gamma(n + v_i + 1) \cdot 4^n} \cdot z^n, i \in \{1, 2, ..., n\}$.
If

$$\left| \frac{zf'_{v_i}(z) - f_{v_i}(z)}{zf_{v_i}(z)} \right| \leq 1, \ \text{ for all } i \in \{1, 2, ..., n\}, \ (\forall) z \in U \tag{17}$$

$$\max_{|z| \leq 1} \left[(1-|z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |z||c|} \right] \leq 1 \tag{18}$$

where $|c| = \frac{1}{32} \cdot \left| \frac{1}{(2+v_1)(1+v_1)} + \frac{1}{(2+v_2)(1+v_2)} + ... + \frac{1}{(2+v_n)(1+v_n)} \right|$ *then*
$F(z) = \int_0^z \frac{f_{v_1}(t)}{t} \cdot \frac{f_{v_2}(t)}{t} \cdot ... \cdot \frac{f_{v_n}(t)}{t} dt \in S$.

Theorem 2.3. *Let g_{v_i} be Struve functions, $z \in U, v_i \in (-2,-1), \alpha_i \in \mathbb{C}$ where*
$g_{v_i}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{3}{2})\Gamma(v+\frac{3}{2})}{4^n \cdot \Gamma(n+\frac{3}{2})\Gamma(v+n+\frac{3}{2})} \cdot z^n, i \in \{1,2,...,n\}.$
If

$$\left|\frac{zg'_{v_i}(z) - g_{v_i}(z)}{zg_{v_i}(z)}\right| \le 1, \text{ for all } i \in \{1,2,...,n\}, \ (\forall)z \in U \tag{19}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{20}$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \le \frac{1}{\max\limits_{|z|\le 1}\left[(1-|z|^2)\cdot|z|\cdot\frac{|z|+|c|}{1+|z||c|}\right]} \tag{21}$$

where $|c| = \frac{1}{15} \cdot \left|\frac{1}{(2v_1+3)(2v_1+5)} + \frac{1}{(2v_2+3)(2v_2+5)} + ... + \frac{1}{(2v_n+3)(2v_n+5)}\right|$ *then*
$G(z) = \int_0^z \left(\frac{g_{v_1}(t)}{t}\right)^{\alpha_1} \cdot \left(\frac{g_{v_2}(t)}{t}\right)^{\alpha_2} \cdot ... \cdot \left(\frac{g_{v_n}(t)}{t}\right)^{\alpha_n} dt \in S.$

Proof. We have $g_{v_i} \in S, i \in \{1,2,...,n\}$ and $\frac{g_{v_i}(z)}{z} \neq 0$.
For $z = 0$ we have $\left(\frac{g_{v_1(z)}}{z}\right)^{\alpha_1} \cdot \left(\frac{g_{v_2(z)}}{z}\right)^{\alpha_2} \cdot ... \cdot \left(\frac{g_{v_n}(z)}{z}\right)^{\alpha_n} = 1$.
Consider the function

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{G''(z)}{G'(z)}.$$

The function h has the form:

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{z \cdot g'_{v_1}(z) - g_{v_1}(z)}{zg_{v_1(z)}} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{z \cdot g'_{v_n}(z) - g_{v_n}(z)}{zg_{v_n(z)}}$$

We have:

$$h(0) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot b_{2,1} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot b_{2,n}$$

where $b_{2,1} = \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(v_1+\frac{3}{2})}{\Gamma(\frac{7}{2}) \cdot \Gamma(v_1+\frac{7}{2})} = \frac{1}{15(2v_1+3)(2v_1+5)}$
$b_{2,2} = \frac{1}{15(2v_2+3)(2v_2+5)}$
$b_{2,n} = \frac{1}{15(2v_n+3)(2v_n+5)}$.
By using the relations (19) and (20) we obtain $|h(z)| < 1$ and
$h(0) = \frac{|\alpha_1 \cdot b_{2,1} + ... + \alpha_n \cdot b_{2,n}|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$ where

$$|c| = \frac{1}{15} \cdot \left|\frac{1}{(2v_1+3)(2v_1+5)} + \frac{1}{(2v_2+3)(2v_2+5)} + ... + \frac{1}{(2v_n+3)(2v_n+5)}\right|.$$

Applying Remark 1.3 for the function h we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left|\frac{G''(z)}{G'(z)}\right| \leq \frac{|z| + |c|}{1 + |c| \cdot |z|}$$

$$\iff \left|(1 - |z|^2) \cdot z \cdot \frac{G''(z)}{G'(z)}\right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot (1 - |z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c||z|},$$

for all $z \in U$.
Let's consider the function $H : [0, 1] \to \mathbb{R}$

$$H(x) = (1 - x^2)x\frac{x + |c|}{1 + |c|}; x = |z|$$

$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + |c|}{2 + |c|} > 0$ then $\max\limits_{x\in[0,1]} H(x) > 0$.
We obtain

$$\left|(1 - |z|^2) \cdot z \cdot \frac{G''(z)}{G'(z)}\right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \max_{|z|\leq 1}\left[(1 - |z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |z||c|}\right].$$

Applying the condition (21) we obtain:

$$(1 - |z|^2)\left|\frac{zG''(z)}{G'(z)}\right| \leq 1, \ (\forall) \ z \in U$$

and from Theorem 1.1 then $G \in S$.
In Theorem 2.3 we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ and obtain the next corollary:

Corollary 2.4. *Let g_{v_i} be Struve functions, $z \in U_1, v_i \in (-2, -1), \alpha_i \in \mathbb{C}$ where*
$g_{v_i}(z) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n \cdot \Gamma(1 + v_i)}{n! \cdot \Gamma(n + v_i + 1) \cdot 4^n} \cdot z^n, i \in \{1, 2, ..., n\}$.
If

$$\left|\frac{zg'_{v_i}(z) - g_{v_i}(z)}{zg_{v_i}(z)}\right| \leq 1, \ \text{ for all } i \in \{1, 2, ..., n\}, \ (\forall) z \in U \tag{22}$$

$$\max_{|z|\leq 1}\left[(1 - |z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |z||c|}\right] \leq 1 \tag{23}$$

where $|c| = \frac{1}{15} \cdot \left|\frac{1}{(2v_1 + 3)(2v_1 + 5)} + \frac{1}{(2v_2 + 3)(2v_2 + 5)} + ... + \frac{1}{(2v_n + 3)(2v_n + 5)}\right|$ *then*
$G(z) = \int_0^z \frac{g_{v_1}(t)}{t} \cdot \frac{g_{v_2}(t)}{t} \cdot ... \cdot \frac{g_{v_n}(t)}{t} dt \in S$.

□

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