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Relationship between Entire Functions and Their Derivatives Sharing Small Function except a Set

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Abstract. The paper is mainly devoted to deriving the relationship between an entire function and its derivative when they share one small function except possibly a set, which is related to the famous Brück conjecture. In addition, two propositions of infinite products are obtained. The first one is the growth property of a certain infinite product. The second one is the property of entire solutions of the differential equation which concerns infinite products.

1. Introduction and main results

It was Rubel and Yang [22] who firstly studied the relationship between an entire function and its derivative when these functions share two values. They proved that if entire functions $f-e_i$ and $f'-e_i$ have the same zeros counting multiplicities (CM), where e_i $(i=1,2)$ is a finite constant, then $f=f'$. From then on, many outstanding works have been obtained, see [15, 21]. In 1996, Brück [3] also considered the related problem and posed the following famous conjecture. The present paper mainly concerns this conjecture. It says that:

Brück conjecture. Let f be a nonconstant entire function such that the hyper order is finite but not a positive integer. If $f-a$ and $f'-a$ have same zeros with the same multiplicities (CM), where a is a finite value, then $f'-a=c(f-a)$, where c is a nonzero constant.

Here, the order $\rho(f)$ and hyper order $\sigma_2(f)$ are defined as

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r},\quad \sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of f.

When $a=0$, Brück himself proved the conjecture. Since then, many authors devoted to studying this conjecture. In 1998, Gundersen-Yang [12] affirmed the conjecture for the case f is of finite order. Later,

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Chen-Shon [7] got that the conjecture also holds when f is of hyper order strictly less than 1/2. Cao [4] further proved that the conjecture is right if the hyper order of f is 1/2. Recently, there is another research direction on the conjecture. That is to weaken the condition of sharing value, see e.g. [8, 19]. Below, the meromorphic function a is called a small function of f if $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$ outside a set of r of finite Lebesgue measure. By the notation $N_L(r,\ f-a, f'-a)$ (see Definition 1 below), Wang [24, Theorem 1.2] generalized some previous results, and her result can be described as follows.

Theorem A. Let f be an entire function of finite order, and let a be a small function of f. If $f-a$ and $f'-a$ have the same zeros ignoring multiplicities (IM), and

$$s = \max\{\limsup_{r\to+\infty} \frac{\log N_L(r,\ F,G)}{\log r},\ \ \limsup_{r\to+\infty} \frac{\log N_L(r,\ G,F)}{\log r}\} < 1, \tag{1}$$

where $F = f-a$ and $G = f'-a$, then, $f'-a = h(z)(f-a)$, where h is a meromorphic function of order no more than s.

Definition 1. Let z_0 be a common zero of F and G with multiplicity p and q, respectively. Let $n_L(r,F,G)$ be the number of this point z_0 with $|z_0| < r$ and $p > q$, each point counted $p-q$ times, where z_0 runs over the zeros of F. And denote by $N_L(r,\ F,G)$ the counting function of $n_L(r,F,G)$.

In order to state our main result, we introduce a new notation, (see e.g. [2, 18]).

Definition 2. Let F and G be two meromorphic functions, and $m_F(\rho)$ (resp. $m_G(\rho)$) the multiplicity of ρ as zero of F (resp. G). Let $D(F,\ G)$ be the set of the point ρ which runs over the zeros of FG, counting with $|m_F(\rho) - m_G(\rho)|$ times. If $|m_F(\rho) - m_G(\rho)| = 0$, then $D(F,\ G)$ does not contain ρ. It is mentioned that if F and G have the same zeros counting multiplicities, then $D(F,G) = \{\emptyset\}$.

The size of a set Λ is measured by the counting function $n(r,\Lambda)$, the number of these points in $\Lambda \cap \{z : |z| < r\}$ counted with multiplicities. And the order $\rho(\Lambda)$ of Λ is defined as

$$\rho(\Lambda) = \limsup_{r\to\infty} \frac{\log n(r,\Lambda)}{\log r}.$$

The set $D(F,G)$ is called the exceptional set. It follows from Theorem A that

$$n(r, D(F,G)) = n_L(r, f-a, f'-a) + n_L(r, f'-a, f-a) = O(r^t), \tag{2}$$

where t is a positive number less than 1, since for arbitrary small $\varepsilon > 0$

$$\begin{aligned}
(2r)^{s+\varepsilon} &\geq N_L(2r,\ f-a, f'-a) \\
&= \int_0^{2r} \frac{n_L(t, f-a, f'-a) - n_L(0, f-a, f'-a)}{t} dt + n_L(0, f-a, f'-a)\log(2r) \\
&\geq \int_r^{2r} \frac{n_L(t, f-a, f'-a)}{t} dt \geq n_L(r, f-a, f'-a)/2.
\end{aligned}$$

Clearly, $\rho(D(F,G)) < 1$ in Theorem A. One would like the exceptional set $D(f-a, f'-a)$ to be as large as possible, such as $\rho(D(F,G)) = 1$. The present paper is devoted to considering the size of exceptional set in Theorem A. By adapting the concept of convergence type (see e.g. [13, Hayman, p.17]), we prove the following theorem.

Main Theorem. Let f be an entire function of finite order, let a be a small function of f, and let $\mathcal{G} = D(f-a, f'-a)$. If

$$\int_0^{+\infty} \frac{n(t,\mathcal{G})}{t^2} dt < \infty, \tag{3}$$

then, $f' - a = Az^k \frac{\Pi_1}{\Pi_2}(f - a)$, where A is a nonzero constant, k is an integer, and

$$\Pi_i = \sum_{a_v \in G_i} (1 - \frac{z}{a_v}), \ \ (i = 1, 2)$$

is a infinite product with set $G_i \subset \mathcal{G}$ $(i = 1, 2)$.

Remark 1. It is mentioned that the infinite product Π_i converges to an entire function since the condition (3). It turns out that $\mathcal{G} = D(f - a, f' - a)$ can be as large as a set of order 1 convergence type. So, Main theorem is a generalization of Theorem A in some sense. In particular, if $f - a$ and $f' - a$ have the same zeros with same multiplicities (CM), then $D(f - a, f' - a) = \emptyset$. So, the main result yields $f' - a = c(f - a)$, where c is a nonzero constant. Thus, it also confirms that Brück conjecture holds if f is of finite order.

For the proof of Main theorem, we need two propositions of infinite products (see in Section 2), which have their own rights.

2. Two propositions of infinite products

Before giving the propositions, we firstly introduce the following notation.

Definition 3. Let $m(H)$ (resp. $\lambda(H)$) denotes the linear measure (resp. the logarithmic measure) of a set H. By $\mathcal{X}_H(t)$, we denote the characteristic function of H. Then, the upper and the lower logarithmic density of H are defined

$$\overline{\log dens} H = \limsup_{r \to +\infty} \frac{\int_1^r (\mathcal{X}_H(t))/t dt}{\log r}, \ \ \underline{\log dens} H = \liminf_{r \to +\infty} \frac{\int_1^r (\mathcal{X}_H(t))/t dt}{\log r}.$$

Now, we show the proposition 1 as follows.

Proposition 1. Let G be a set of nonzero points satisfying $\int_0^{+\infty} \frac{n(t,G)}{t^{p+1}} dt < \infty$ with an integer $p \geq 1$, and let $E(z, p - 1) = (1 - z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^{p-1}}{p-1}}$. Then the infinite product

$$\Pi(z) = \prod_{z \in G} E(\frac{z}{a_v}, p - 1)$$

is an entire function. Furthermore, for $|z| = r$ large enough and arbitrary $\varepsilon > 0$,

(1) we have,

$$\log |\Pi(z)| \leq \varepsilon r^p.$$

(2) there exists a set $\overline{E}$ with arbitrary small upper logarithmic density $\overline{\log dens E}$ such that

$$\log |\Pi(z)| \geq -\varepsilon r^p,$$

holds for all $|z| = r \notin \overline{E}$.

Remark 2. It is well known that one important result concerning infinite products is the Hadamard' factorization [26, Theorem 2.7]. It states that if f is a meromophic function of finite order, then it has representation as $f(z) = z^k \frac{\Pi_1}{\Pi_2} e^Q$, where k is an integer, Q is a polynomial, and Π_i is a infinite product, which is also called the canonical product of the zeros or poles of f. This can be regarded as a generalization of the Fundamental Theorem of Algebra. So, Proposition 1 may contribute to the estimate of the infinite products in Hadamard' factorization.

Based on Proposition 1, we below consider a differential equation which concerns a certain infinite product.

Proposition 2. Let f be an entire function, let a be a small function of f and the order $\rho(a)$ be finite, and let Q be a nonconstant polynomial. Suppose that Π_1, Π_2 are two infinite products defined as in Proposition 1 with $p \leq \deg Q = m$. If f satisfies the following differential equation

$$\frac{f'-a}{f-a} = \frac{\Pi_1}{\Pi_2} e^Q, \tag{4}$$

then the order of f is infinite.

Remark 3. If the differential equation (4) is extended to $\frac{f'-a}{f-a} = z^k \frac{\Pi_1}{\Pi_2} e^Q$, where k is an integer, then the conclusion of Proposition 2 still holds.

3. The proofs of propositions and main theorem

Proof. [Proof of Proposition 1.] According to the mind in [17, Li, Theorem 1], we firstly prove (1). Let a_v be the element of G, if there are any, repeated according to the multiplicities. By the assumption, one then has

$$\sum_{v=1}^{\infty} |a_v|^{-p} = p \int_0^{+\infty} \frac{dn(t,G)}{t^p} = p^2 \int_0^{+\infty} \frac{n(t,G)}{t^{p+1}} dt < \infty. \tag{5}$$

We then make use of the following result [13, p.27].

Lemma 1. If $G = \{a_v\}$ is a sequence of nonzero complex numbers such that $\sum_{v=1}^{\infty} |a_v|^{-p}$ converges, then $\Pi(z) = \Pi_{z \in G} E(\frac{z}{a_v}, p-1)$ is an entire function, whose zero set is G, and satisfies the following estimate

$$\log|\Pi(z)| \leq pA(p)\{|z|^{p-1} \int_0^{|z|} \frac{n(t,G)}{t^p} dt + |z|^p \int_{|z|}^{+\infty} \frac{n(t,G)}{t^{p+1}} dt\},$$

where $A(p)$ is a positive fixed constant.

Note that $\int_0^{+\infty} \frac{n(t,G)}{t^{p+1}} dt < \infty$. So, for any $\varepsilon > 0$, when $|z|$ is large enough, say $|z| = r \geq r_0 > 0$, then $\int_r^{+\infty} \frac{n(t,G)}{t^{p+1}} dt < \varepsilon$. Therefore,

$$\varepsilon > \int_r^{2r} \frac{n(t,G)}{t^{p+1}} dt \geq \frac{n(r,G)}{(2r)^{p+1}} \int_r^{2r} 1 dt \geq \frac{n(r,G)}{2^{p+1} r^p},$$

which implies $n(r,G) \leq 2^{p+1} r^p \varepsilon$. Furthermore, for $|z| = r$ large enough, one has

$$\begin{aligned} \log|\Pi(z)| &\leq pA(p)\{|z|^{p-1} \int_0^{r_0} \frac{n(t,G)}{t^p} dt + |z|^{p-1} \int_{r_0}^{|z|} \frac{n(t,G)}{t^p} dt + |z|^p \int_{|z|}^{+\infty} \frac{n(t,G)}{t^{p+1}} dt\} \\ &\leq pA(p)\{|z|^{p-1} \int_0^{r_0} \frac{n(t,G)}{t^p} dt + |z|^{p-1} \int_{r_0}^{|z|} \frac{2^{p+1} t^p \varepsilon}{t^p} dt + \varepsilon |z|^p\} \\ &\leq pA(p)[2^{p+1} + k_0] \varepsilon |z|^p, \end{aligned}$$

where k_0 is a positive constant. It is the desired result (1).

Now, we prove (2). It follows from Proposition 1 that $\log M(r,\Pi) \leq \varepsilon r^p$ for $|z| = r$ large enough, where $M(r,\Pi)$ is the maximus modulus of Π on the circle $|z| = r$, that is $M(r,\Pi) = \max\{|\Pi(z)| : |z| = r\}$. Let us employ the Minimum Modulus Theorem of the entire function, see e.g. [1, p.362, 4.5.14].

The Minimum Modulus Theorem. Let f be holomorphic in the disc $B(0, 2eR)$ and continuous in the closure of the disc. Assume that $f(0) = 1$ and let τ be a constant such that $0 < \tau < \frac{3e}{2}$. Then, in the disc $|z| \leq R$, and outside a collections of closed disc $D_1, \cdots, D_q$ the sum of whose radii does not exceed $4\tau R$, we have

$$\log|f(z)| \geq -(2 + \log\frac{3e}{2\tau})\log M(2eR, f).$$

Set $h \geq 0$ be an integer. Note that $\Pi(0) = 1$. Then, for $|z| \leq R = 2^{h+1}$, applying the above lemma to Π, one has,

$$\log|\Pi(z)| \geq -(2 + \log\frac{3e}{2\tau})\log M(2eR, \Pi),$$

outside a collections of closed disc $D_1, \cdots, D_q$ the sum of whose radii does not exceed $4\tau R$. Define the set Y_h as

$$Y_h = \{r : \textit{there exist } z \in \cup_{j=1}^{q} D_j \textit{ such that } |z| = r\}.$$

Then, for any $2^h \leq |z| = r \leq 2^{h+1}$ and $r \notin Y_h$, one has

$$\begin{aligned}\log|\Pi(z)| &\geq -(2 + \log\frac{3e}{2\tau})\log M(2eR, \Pi) \geq -(2 + \log\frac{3e}{2\tau})\varepsilon(2e2^{h+1})^p \\ &\geq -(2 + \log\frac{3e}{2\tau})\varepsilon(4e2^h)^p \geq -(2 + \log\frac{3e}{2\tau})\varepsilon(4er)^p \\ &\geq -A\varepsilon r^p,\end{aligned}$$

where $A = (2 + \log\frac{3e}{2\tau})(4e)^p$ is a fixed positive constant and independent with h and z. Then, due to the same way of Chiang and Feng in [5], we will prove (2) below. Set

$$E_h = Y_h \cap [2^h,\ 2^{h+1}].$$

Then,

$$\int_{E_h} 1dt \leq \int_{Y_h} 1dt \leq 4\tau 2^{h+1}.$$

Set $\overline{E} = \cup_{h=0}^{\infty} E_h \cap (1, \infty)$. Then, we have for all z satisfying $|z| = r \notin \overline{E} \cup [0, 1]$, that

$$\log|\Pi(z)| \geq -A\varepsilon r^p.$$

For any $r > 1$, there exists nonnegative integer h such that $2^h \leq r \leq 2^{h+1}$. Then,

$$\begin{aligned}\int_{\overline{E}\cap[1,r]} \frac{1}{t}dt &\leq \int_{\overline{E}\cap[1,2^{h+1}]} \frac{1}{t}dt = \sum_{j=0}^{h} \int_{E_j} \frac{1}{t}dt \\ &\leq \sum_{j=0}^{h} \frac{1}{2^{j+1}} 4\tau 2^{j+1} \leq 4\tau(h+1) \leq 4\tau\frac{\log r}{\log 2} + 4\tau.\end{aligned}$$

Therefore,

$$\delta(\overline{E}) = \overline{\log dens}\ \overline{E} = \limsup_{r\to+\infty} \frac{\int_{\overline{E}\cap[1,r]} \frac{1}{t}dt}{\log r} \leq \frac{4\tau}{\log 2}.$$

Note that $0 < \tau < \frac{3e}{2}$. This is the desired result (2). □

Now, we turn to the proof of Proposition 2.

Proof. [Proof of Proposition 2.] Suppose that the order of f is finite. Below, we will derive a contradiction. Rewrite (4) as

$$\frac{\frac{f'}{f}-\frac{a}{f}}{1-\frac{a}{f}}\frac{\Pi_2}{\Pi_1}=e^Q. \tag{6}$$

By the Wiman-Valiron theory (see e.g. [14, 20]), there exists a subset $E_1 \in (1,+\infty)$ with finite logarithmic measure, and for some points $z_r = re^{i\theta}$ satisfying $|z_r| = r \notin E_1$, $M(r,f) = |f(z_r)|$ and

$$\frac{f'(z_r)}{f(z_r)}=\frac{v(r,f)}{r}(1+o(1))$$

as $r \to \infty$, where $v(r,f)$ denotes the central index of the function f. Here, recall a result of Wang and Yi, (see e.g. [23, Lemma 5]).

Lemma 2. Let f be a nonconstant entire function of finite order. Suppose that a is a nonzero small function of f. Then, there exists a set $E_5 \subset (1,\ \infty)$ satisfying $\underline{\log dens} E_5 = 1$, such that

$$\frac{\log^+ M(r,a)}{\log^+ M(r,f)} \to 0,\ \ \frac{M(r,a)}{M(r,f)} \to 0,$$

holds for $|z| = r \in E_5,\ r \to \infty$.

Let us turn back to the proof of Proposition 2. By Lemma 2, there exists a set $E \in (1,+\infty)$ satisfying $\underline{\log dens} E = 1$, such that

$$\frac{\log^+ M(r,a)}{\log^+ M(r,f)} \to 0,\ \ \frac{M(r,a)}{M(r,f)} \to 0, \tag{7}$$

holds for $|z| = r \in E$, as $r \to \infty$. Taking the principle branch of the logarithm of Eq (6) yields

$$\log\frac{\frac{f'}{f}-\frac{a}{f}}{1-\frac{a}{f}}\frac{\Pi_2}{\Pi_1}=\log e^Q = Q + i2k\pi = \mathrm{Re}Q + i(\mathrm{IM}Q+2k\pi), \tag{8}$$

where k is an integer depending on $\mathrm{IM}Q$ such that $\mathrm{IM}Q + 2k\pi \in (-\pi,\pi]$. Furthermore, one has by (8) that

$$|\mathrm{Re}Q| = |\log|\frac{\frac{f'}{f}-\frac{a}{f}}{1-\frac{a}{f}}\frac{\Pi_2}{\Pi_1}|| \le |\log|\frac{\frac{f'}{f}-\frac{a}{f}}{1-\frac{a}{f}}|| + |\log|\frac{\Pi_2}{\Pi_1}||. \tag{9}$$

By Proposition 1, for any positive ε and r large enough, there exists measure set E_2 with arbitrary small $\overline{\log dens} E_2$ such that

$$e^{-\varepsilon r^m} \le e^{-\varepsilon r^p} \le |\frac{\Pi_2}{\Pi_1}(z)| \le e^{\varepsilon r^p} \le e^{\varepsilon r^m}, \tag{10}$$

holds for $|z| = r \notin E_2$.

Here, we employ two results to handle this proposition, the first one is due to Wang and Laine [25, Lemma 2.4], the latter one is due to Gundersen [9, Corollary 2].

Lemma 3. Let f be an entire function of finite order ρ, and $f(re^{i\theta_r}) = M(r, f)$ for every r. Given $\zeta > 0$ and $0 < C(\rho, \zeta) < 1$, there exist a constant $0 < l_0 < 1$ and a set $E_\zeta \subset [0, \infty]$ of lower logarithmic density greater than $1 - \zeta$ such that

$$e^{-5\pi} M(r, f)^{1-C(\rho,\zeta)} \le |f(re^{i\theta})|$$

for all $r \in E_\zeta$ large enough and all θ such that $|\theta - \theta_r| \le l_0$.

Lemma 4. Let f be a transcendental meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty]$ with finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0, 1]$ and for all $k,\ j, 0 \le j < k$, we have

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \le |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Note that logarithmic density of $E,\ E_1,\ E_2,\ E_\zeta,\ H$. Then the upper logarithmic density of the set $E \cap E_\zeta \ge 1 - \zeta$, since

$$\overline{\log dens}(E \cap E_\zeta) \ge \overline{\log dens} E + \underline{\log dens} E_\zeta - \overline{\log dens}(E \cup E_\zeta) \ge 1 - \zeta.$$

Obviously, the upper logarithmic density $E \cap E_\zeta \backslash (E_1 \cup E_2 \cup H)$ is more than $1 - \zeta - \mu$, where μ is a small enough positive number, since the logarithmic density of E_2 is small enough. Note that ζ and μ can be chosen small enough, so the upper logarithmic density $E \cap E_\zeta \backslash (E_1 \cup E_2 \cup H)$ is close to 1.

We assume that $Q(z) = a_m z^m + \cdots + a_1 z + a_0$ with $a_m = \alpha e^{i\beta} \ne 0$. Now, we split the proof into two cases as follows.

Case 1. $\rho(f) > 1$.

Note that $\rho(f) = \limsup_{r\to+\infty} \frac{\log v(r,f)}{\log r} > 1$ and the upper logarithmic density of the set $E \cap E_\zeta \backslash (E_1 \cup E_2 \cup H)$ is close to 1. Then, there exists a sequence $\{r_n\} \in E \cap E_\zeta \backslash (E_1 \cup E_2 \cup H)$ such that $\frac{v(r_n,f)}{r_n} \to \infty$ as $r_n \to \infty$. Set $f(z_{r_n}) = M(r_n, f)$. Then, one gets $\frac{f'(z_{r_n})}{f(z_{r_n})} - \frac{a(z_{r_n})}{f(z_{r_n})} = \frac{v(r_n,f)}{r_n}(1 + o(1))$.

Assume $z_{r_n} = r_n e^{i\theta_n}$ with $r_n \to \infty$ and $\theta_n \to \theta_0 \in [0, 2\pi]$. Then $\mathrm{Re}(a_m z_{r_n}^m) = \mathrm{Re}(\alpha r_n^m e^{i(\beta+m\theta_n)}) = \alpha r_n^m \cos(\beta + m\theta_n)$. Next, we consider two subcases.

Subcase 1.1. $\cos(\beta + m\theta_0) \ne 0$.

Then, for n large enough, there exists a positive constant $A > 0$ such that

$$|\mathrm{Re}(Q(z_{r_n})| = (1 + o(1))|\mathrm{Re}(a_m z_{r_n}^m)| \ge A r_n^m.$$

Together with (9) and (10), one gives

$$|\log|\frac{\frac{f'}{f} - \frac{a}{f}}{1 - \frac{a}{f}}(z_{r_n})|| \ge |Re(Q(z_{r_n})| - |\log|\frac{\Pi_2}{\Pi_1}||(z_{r_n}) \ge A r_n^m - \varepsilon r_n^p \ge (A - \varepsilon) r_n^m. \tag{11}$$

On the other hand,

$$|\log|(\frac{f'}{f} - \frac{a}{f})(z_{r_n})|| = \log \frac{v(r_n, f)}{r_n} + O(1).$$

Combining this and (11) yields

$$(A - \varepsilon) r_n^m \le \log \frac{v(r_n, f)}{r_n} + O(1).$$

Taking the logarithm of both side of the above inequality yields

$$m\log r_n \le \log\log v(r_n,f) + \log\log r_n + O(1),$$

which implies that the order of f is infinite, a contradiction.

Subcase 1.2. $\cos(\beta + m\theta_0) = 0$.

Below, we introduce a method of Wang in [24] to handle this subcase. In view of $\cos(\beta + m\theta) \not\equiv 0$, without loss of generality, we assume that

$$\cos(\beta + m\theta) > 0,\ \ \theta \in (\theta_0,\ \ \theta_0 + \frac{\pi}{m}).$$

Note that $\theta_n \to \theta_0$. Then, as n large enough, $|\theta_n - \theta_0| \le l_0 < \frac{\pi}{m}$. Choose now θ_n^* such that $l_0/2 \le \theta_n^* - \theta_n \le l_0$. Then, we can assume that $\theta_n^* \to \theta_0^*$ as $n \to \infty$. Obviously, $\cos(\beta + m\theta_0^*) > 0$. Furthermore, it follows from Lemma 3 that

$$e^{-5\pi}M(r_n,f)^{1-C} \le |f(z_n^*)|,$$

with $z_n^* = r_n e^{i\theta_n^*}$. Considering (7), it is easy to see

$$\frac{\log M(r_n,a)}{(1-c)\log M(r_n,f) - 5\pi} \to 0,$$

as $n \to \infty$, which implies

$$\frac{|a(z_n^*)|}{|f(z_n^*)|} \le \frac{M(r_n,a)}{e^{-5\pi}M(r_n,f)^{1-C}} \to 0, \tag{12}$$

as $n \to \infty$. Then, for n large enough, the same discussion as above yields

$$\mathrm{Re}(Q(z_n^*)) = (1 + o(1))\mathrm{Re}(a_m Q(z_n^*)^m) \ge Br_n^m,$$

where B is a positive constant. Then, for n large enough, we have

$$|\frac{\frac{f'}{f} - \frac{a}{f}}{1 - \frac{a}{f}}(z_n^*)| = |e^Q\frac{\Pi_1}{\Pi_2}(z_n^*))| = e^{\mathrm{Re}(Q(z_n^*))}|\frac{\Pi_1}{\Pi_2}(z_n^*)| \ge e^{(B-\varepsilon)r_n^m}. \tag{13}$$

On the other hand, it follows from (12) and Lemma 4 that

$$|\frac{\frac{f'}{f} - \frac{a}{f}}{1 - \frac{a}{f}}(z_n^*)| \le (1 + o(1))[|\frac{f'}{f}(z_n^*)| + |\frac{a}{f}(z_n^*)|] \le (1 + o(1))|z_n|^{(\rho(f)-1+\varepsilon)},$$

which contradicts the above estimate (13).

Case 2. $\rho(f) \le 1$.

We claim that the order of f must be 1. Otherwise, assume that $\rho(f) < 1$. Note that a is a small function of f. Then, $\rho(a) < 1$ and $\frac{f'-a}{f-a}$ is of order less than 1. It contradicts the equation

$$\frac{f'-a}{f-a} = \frac{\Pi_1}{\Pi_2}e^Q,$$

since $\deg Q = m \ge 1$. So, we assume that $\rho(f) = 1$ in the following discussion.

For $|z_r| = r \in E\backslash(E_1 \cup E_2)$, we have, from (7), that

$$\frac{\log^+ M(r,a)}{\log^+ M(r,f)} \to 0.$$

Thus, without loss of generality, we assume, for all $r \in E\backslash(E_1 \cup E_2)$ and any $\varepsilon > 0$, that $\frac{\log^+ M(r,a)}{\log^+ M(r,f)} < \varepsilon$. This implies

$$M(r,a) < M(r,f)^{\varepsilon}, \ \ and \ \ \frac{M(r,a)}{M(r,f)} < \frac{1}{M(r,f)^{1-\varepsilon}}. \tag{14}$$

Note that $\rho(f) = \limsup\limits_{r\to+\infty} \frac{\log v(r,f)}{\log r} = 1$. Then, for r large enough,

$$\frac{\log v(r,f)}{\log r} \le 1+\varepsilon, \ \ and \ \ v(r,f) \le r^{1+\varepsilon}.$$

Now, we introduce a connection between the growth of the central index $v(r,f)$ and the maximus modulus $M(r,f)$. It can be seen in [14, Theorems 1.9 and 1.10] or [16, p.11, Satz 4.3 and 4.4]. It is stated as:

Lemma 5. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu(r)$ be the maximum term, i.e. $\mu(r,g) = \max\{|a_n|r^n; n = 1, \ ,2\ , \cdots\}$, $v(r,g)$ be the central index, i.e. $v(r,g) = \max\{m : \mu(r,g) = |a_m|r^m\}$, then for $r < R$,

$$M(r,g) < \mu(r,g)\{v(R,g) + \frac{R}{R-r}\}.$$

Then, applying Lemma 5 to the function f, one has, for r large enough and $R = 2r$,

$$M(r,f) \le \mu(r,f)[v(2r,f)+2] \le |a_{v(r,f)}|r^{v(r,f)}[v(2r,f)+2].$$

Taking the principle branch of the logarithm of the above inequality shows

$$\begin{aligned} \log M(r,f) &\le v(r,f)\log r + \log v(2r,f) + C \\ &\le v(r,f)\log r + \log(2r)^{1+\varepsilon} + C \le 2v(r,f)\log r, \end{aligned} \tag{15}$$

where C is a positive number.

Note that $\rho(f) = \limsup\limits_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} = 1$. So, for $\varepsilon > 0$, there exists a sequence $\{r_n\} \subset E \cap E_\zeta \backslash (E_1 \cup E_2 \cup H)$ (we still use the notation r_n) such that

$$\frac{\log\log M(r_n,f)}{\log r_n} \ge \frac{\rho(f)}{1+\varepsilon} = \frac{1}{1+\varepsilon},$$

which leads to

$$\log M(r_n,f) \ge r_n^{\frac{1}{1+\varepsilon}}, \ \ and \ \ M(r_n,f) \ge e^{r_n^{\frac{1}{1+\varepsilon}}}. \tag{16}$$

By the above inequality and together with (15), one gets

$$2v(r_n,f)\log r_n \ge \log M(r_n,f) \ge r_n^{\frac{1}{1+\varepsilon}}. \tag{17}$$

Combining (14) and (16) yields

$$|\frac{a(z_{r_n})}{f(z_{r_n})}| \le \frac{M(r_n,a)}{M(r_n,f)} < \frac{1}{M(r_n,f)^{1-\varepsilon}} \le \frac{1}{e^{(1-\varepsilon)r_n^{\frac{1}{1+\varepsilon}}}}. \tag{18}$$

Still set $|z_{r_n}| = r_n$ and $M(r_n, f) = f(z_{r_n})$. It follows from (17) that

$$|\frac{f'(z_{r_n})}{f(z_{r_n})}| = \frac{v(r_n, f)}{r_n}(1 + o(1)) \geq \frac{1}{2}\frac{1}{\log r_n} r_n^{\frac{1}{1+\varepsilon}-1} = \frac{1}{2}\frac{1}{\log r_n} r_n^{\frac{-\varepsilon}{1+\varepsilon}}. \tag{19}$$

By (18) and (19), one can see that $|\frac{a(z_{r_n})}{f(z_{r_n})}| = o(|\frac{f'(z_{r_n})}{f(z_{r_n})}|)$. Thus,

$$|\frac{f'(z_{r_n})}{f(z_{r_n})} - \frac{a(z_{r_n})}{f(z_{r_n})}| = |\frac{f'(z_{r_n})}{f(z_{r_n})}|(1 + o(1)).$$

A easy calculation yields

$$\frac{1}{2}\frac{1}{\log r_n} r_n^{\frac{-\varepsilon}{1+\varepsilon}} \leq |\frac{f'(z_{r_n})}{f(z_{r_n})}| = \frac{v(r_n, f)}{r_n}(1 + o(1)) \leq r_n^{\rho(f)+\varepsilon-1} = r_n^{\varepsilon}.$$

Furthermore,

$$\begin{aligned}
(\frac{-\varepsilon}{1+\varepsilon})\log r_n - \log\log r_n - \log 2 &= \log \frac{1}{2}\frac{1}{\log r_n} r_n^{\frac{-\varepsilon}{1+\varepsilon}} \\
&\leq \log \frac{v(r_n, f)}{r_n}(1 + o(1)) \\
&\leq \log r_n^{\varepsilon} = \varepsilon \log r_n,
\end{aligned}$$

which indicates

$$\begin{aligned}
|\log|\frac{f'(z_{r_n})}{f(z_{r_n})} - \frac{a(z_{r_n})}{f(z_{r_n})}|| &= |\log \frac{v(r_n, f)}{r_n}(1 + o(1))| \\
&\leq \max\{\varepsilon \log r_n, |\frac{-\varepsilon}{1+\varepsilon}|\log r_n + \log\log r_n\}.
\end{aligned} \tag{20}$$

Without loss of generality, still set $z_{r_n} = r_n e^{i\theta_n}$ with $r_n \to \infty$ and $\theta_n \to \theta_0 \in [0, 2\pi]$. Then $\mathrm{Re}(a_m z_{r_n}^m) = \mathrm{Re}(\alpha r_n^m e^{i(\beta + m\theta_n)}) = \alpha r_n^m \cos(\beta + m\theta_n)$.

Next, we also split the proof into two subcases as follows.

Subcase 2.1. $\cos(\beta + m\theta_0) \neq 0$.

As above, one has for n large enough, there exists a positive constant $C > 0$ such that

$$|\mathrm{Re}(Q(z_{r_n})| = (1 + o(1))|\mathrm{Re}(a_m z_{r_n}^m)| \geq C r_n^m.$$

Together with (9) and (10), one gives

$$|\log|\frac{\frac{f'}{f} - \frac{a}{f}}{1 - \frac{a}{f}}(z_{r_n})|| \geq |\mathrm{Re}(Q(z_{r_n})| - |\log|\frac{\Pi_2}{\Pi_1}||(z_{r_n}) \geq C r_n^m - \varepsilon r^p \geq (C - \varepsilon) r_n^m. \tag{21}$$

Combining this and (20) yields

$$(C - \varepsilon) r_n^m \leq \max\{\varepsilon \log r_n, |\frac{-\varepsilon}{1+\varepsilon}|\log r_n + \log\log r_n\},$$

which is impossible.

Subcase 2.2. $\cos(\beta + m\theta_0) = 0$.

With the same argument of Subcase 1.2, one can derive a contradiction. Here, we omit the details.

Thus, we complete the proof of Proposition 2. □

Proof. [Proof of Main theorem.] Based on the propositions 1 and 2, we give the proof of the main result. In fact, by Hadamard' factorization [26, Theorem 2.7], one has

$$\frac{f'-a}{f-a} = z^k \frac{\Pi_1}{\Pi_2} e^Q,$$

where Q is a polynomial and k is an integer. Note that f is of finite order. Then, it follows from Proposition 2 that Q is of degree 0, say $Q = A$. Therefore, one gets the conclusion of Main theorem. □

To conclude this paper, we give two natural further studies which are related to the main results. One is the size of possible exceptional set $\mathcal{G}$ in Main theorem. We would like $\mathcal{G}$ to be as large as possible, such as $n(r, \mathcal{G}) = o(r)$. Unfortunately, our method in the paper does not work, since (5) may not converges for $p = 1$. The other one is to generalize some differential equations to those concern infinite products. For example, the differential equation

$$f'' + A_1(z)e^{az} + A_2e^{bz} = H,$$

where A_1, A_2, H are three entire functions with order less than 1, and a, b are two constants. It is related to a famous differential equation question posed by Gundersen in [11]. We refer to [6, 10, 25] for some results of the above differential equation. It is natural to generalize the above differential equation to $f'' + \Pi_1(z)e^{az}f' + \Pi_2e^{bz}f = \Pi_3$, where the infinite product Π_i is defined as in the main theorem.

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