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Proper Weak Regular Splitting and its Application to Convergence of Alternating Iterations

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Abstract. Theory of matrix splittings is a useful tool for finding the solution of a rectangular linear system of equations, iteratively. The purpose of this paper is two-fold. Firstly, we revisit the theory of weak regular splittings for rectangular matrices. Secondly, we propose an alternating iterative method for solving rectangular linear systems by using the Moore-Penrose inverse and discuss its convergence theory, by extending the work of Benzi and Szyld [Numererische Mathematik 76 (1997) 309-321; MR1452511]. Furthermore, a comparison result is obtained which ensures the faster convergence rate of the proposed alternating iterative scheme.

1. Introduction

Many questions in science and engineering give rise to linear discrete ill-posed problems. In particular, the discretization of Fredholm integral equations of the first kind, and in particular deconvolution problems with a smooth kernel, lead to linear systems of equations of the form

$$Ax = b, \quad A \in \mathbb{R}^{m\times n}, \quad x \in \mathbb{R}^n, \ b \in \mathbb{R}^m, \tag{1}$$

with a matrix of ill-determined rank, where $\mathbb{R}^{m\times n}$ denotes the set of all real rectangular matrices. Linear systems of equations with a matrix of this kind are commonly referred to as linear discrete ill-posed problems. We consider equation (1) as a least-squares problem in the case of an inconsistent system. Similarly, singular linear systems of equations arise in many problems like the finite difference representation of Neumann problems, finite element electromagnetic analysis using edge elements and the computation of stationary probability vectors of stochastic matrices in the analysis of Markov chains, to name a few. In particular, we arrive at an M-matrix as the coefficient matrix A. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an *M-matrix* if $A = sI - B$, where $B \geq 0$ (see Section 2.2 for its meaning) and $s \geq \rho(B)$. It becomes a singular M-matrix when $s = \rho(B)$. The study of M-matrices has a long history. A systematic study of M-matrices was first initiated by Fiedler and Ptak [7]. Fifty equivalent conditions of an M-matrix are reported in the book by Berman and Plemmons [4]. An extensive theory of M-matrix has been developed for its role in the iterative methods. In both theoretical and practical cases, iterative methods play a vital role in solving a large sparse linear system of equations as either solvers or preconditioners. In this note, we propose an alternating iterative method using the theory of proper splittings.

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For $A \in \mathbb{R}^{m\times n}$, a *splitting* is an expression of the form $A = U - V$, where U and V are matrices of the same order as in A. The concept of splitting first arises from the iterative solution of the large linear system of equations (1). Standard iterative methods like the Jacobi, Gauss-Seidel and successive over-relaxation methods arise from different choices of U and V. Berman and Plemmons [3] proposed first an iterative method for solving (1). They used the Moore-Penrose inverse for computing the least squares solution in the inconsistent case. The matrix G satisfying the four matrix equations: $AGA = A$, $GAG = G$, $(AG)^T = AG$ and $(GA)^T = GA$ is called the *Moore-Penrose inverse* of A (here B^T denotes the transpose of B). It always exists and unique, and is denoted by $A^\dagger$. The advantage of the iterative technique for solving the rectangular system of linear equations is that it avoids the use of the normal system $A^TAx = A^Tb$ where A^TA is frequently ill-conditioned and influenced greatly by roundoff errors (see [8]).

Berman and Plemmons [3] introduced the notion of proper splitting for rectangular matrices, which we recall next. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper splitting* if $R(U) = R(A)$ and $N(U) = N(A)$, where the text $R(A)$ and $N(A)$ denote the range and null-space of a matrix A, respectively. The authors of [3] considered the following iteration scheme:

$$x^{i+1} = Hx^i + c, \tag{2}$$

where $A = U - V$ is a proper splitting, $H = U^\dagger V \in \mathbb{R}^{n\times n}$ is called the *iteration matrix* and $c = U^\dagger b$ to solve (1), iteratively. The same authors proved that the iteration scheme (2) converges to $A^\dagger b$, the least squares solution of minimum norm for any initial vector x^0 if and only if the spectral radius of H is less than 1 (see Corollary 1, [3]).

The authors of [3] also obtained several convergence criteria for different subclasses of proper splittings. Recently, Jena *et al.* [9] revisited the same theory. Certain necessary parts of the same theory are recalled and discussed in Section 3 of this paper. The above discussion extends the convergence theory of the iterative scheme:

$$x^{i+1} = U^{-1}Vx^i + U^{-1}b, \tag{3}$$

which is being used to solve the nonsingular linear system $Ax = b$.

On the other hand, the speed of the iteration schemes (2) and (3) is a subject of concern. In this direction, several works have been done in literature. Among these works, Benzi and Szyld [2] proposed the concept of the alternating iteration method for solving the linear system of the form $Ax = b$, iteratively where A may be nonsingular or singular. They considered two splittings of $A \in \mathbb{R}^{n\times n}$ such that $A = M - N = P - Q$, and proposed the scheme

$$x^{i+1/2} = M^{-1}Nx^i + M^{-1}b, \quad x^{i+1} = P^{-1}Qx^{i+1/2} + P^{-1}b,\ i = 0, 1, 2, \ldots. \tag{4}$$

Then, eliminating $x^{i+1/2}$, they obtained

$$x^{i+1} = P^{-1}QM^{-1}Nx^i + P^{-1}(QM^{-1} + I)b, \quad i = 0, 1, 2, \ldots. \tag{5}$$

Finally, they discussed the convergence theory of the above scheme using a weak regular splitting of A among other results. (Recall that a splitting $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is *weak regular* [12] if U^{-1} exists, $U^{-1} \geq 0$ and $U^{-1}V \geq 0$.) *The objective of the present paper is to introduce an alternating iteration technique and to develop its convergence theory for solving the rectangular linear system of equations (1).* By doing this, we will have another iteration scheme of the form (7) which converges faster than the iteration scheme (2).

To fulfill this objective, we organize the content of the paper as follows. In Section 2, we set up our notation and terminology. Furthermore, we collect some useful facts on projection, the Moore-Penrose inverse, proper splittings, spectral radius and its connection with non-negative matrices which will be used in deriving the main results in Section 3 and Section 4. The next Section recalls results on the theory of regular and weak regular splittings for rectangular matrices. It also contains two comparison results which will help us in detecting a better splitting between matrix splittings. The main contribution of this paper discussed in Section 4 is that we introduce the notion of an alternating iterative scheme for rectangular matrices by using the Moore-Penrose inverse. Then convergence and comparison results involving this scheme are reported. Finally, we end up with a concluding Section which compares our work with Benzi and Szyld's work.

2. Prerequisites

This section contains our notation and definitions, and also we recall some useful facts related to the Perron-Frobenius theory for non-negative matrices. Throughout the paper, all our matrices are real. Let L and M be complementary subspaces of $\mathbb{R}^n$, i.e., $L \oplus M = \mathbb{R}^n$. Let also $P_{L,M}$ be a projector on L along M. Then $P_{L,M}A = A$ if and only if $R(A) \subseteq L$ and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. If $L \perp M$, then $P_{L,M}$ will be denoted by P_L. The *spectral radius* of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A. It is known that $\rho(AB) = \rho(BA)$, where A and B are two matrices such that AB and BA are defined. We now recall some facts on generalized inverses, non-negative matrices and proper splittings below.

2.1. Generalized inverses

These are generalizations of the ordinary matrix inverse. Generalized inverses exist for all matrices while the ordinary matrix inverse does not exist. Some of the important generalized inverses are the Moore-Penrose inverse, the group inverse and the Drazin inverse. While the definition of the first one is introduced in page 2, the other two are presented next. The *Drazin inverse* of a matrix $A \in \mathbb{R}^{n\times n}$ is the unique solution $X \in \mathbb{R}^{n\times n}$ satisfying the equations: $A^k = A^kXA$, $X = XAX$ and $AX = XA$, where k is the index[1)] of A. It is denoted by A^D. But for $k = 1$, A^D is called as *group inverse* of A, and is denoted by $A^{\#}$. While $A^{\dagger}$ and A^D exist for any matrix A, $A^{\#}$ does not. It exists only for matrices of index 1. We refer to [1] for more details. In the case of nonsingular matrix A, $A^{\dagger} = A^{-1} = A^D = A^{\#}$. Some of the well-known properties of $A^{\dagger}$ which will be frequently used in this paper are: $R(A^T) = R(A^{\dagger})$; $N(A^T) = N(A^{\dagger})$; $AA^{\dagger} = P_{R(A)}$; $A^{\dagger}A = P_{R(A^T)}$. In particular, if $x \in R(A^T)$ then $x = A^{\dagger}Ax$.

2.2. Non-negative matrices

$A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is called *non-negative* if $A \geq 0$, where $A \geq 0$ means $a_{ij} \geq 0$ for each i, j, and there exists at least one pair of indices k, l for which $a_{k,l} > 0$. For $A, B \in \mathbb{R}^{m\times n}$, $A \leq B$ means $B - A \geq 0$. Similarly, $B > 0$ means all the entries of B are positive. The same notation and nomenclature are also used for vectors. A matrix $A \in \mathbb{R}^{m\times n}$ is called *semi-monotone* if $A^{\dagger} \geq 0$. Next four results deal with non-negativity and spectral radius, and are going to be used in Section 3 and Section 4.

Theorem 2.1. *(Theorem 2.20, [12])*
Let $B \in \mathbb{R}^{n\times n}$ and $B \geq 0$. Then
(i) B has a non-negative real eigenvalue equal to its spectral radius.
(ii) There exists a non-negative eigenvector for its spectral radius.

Theorem 2.2. *(Theorem 2.21, [12])*
Let $A, B \in \mathbb{R}^{n\times n}$ and $A \geq B \geq 0$. Then $\rho(A) \geq \rho(B)$.

Theorem 2.3. *(Theorem 3.15, [12])*
Let $X \in \mathbb{R}^{n\times n}$ and $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0$.

Theorem 2.4. *(Theorem 1.11, [4], Chapter 2)*
Let $B \in \mathbb{R}^{n\times n}$, $B \geq 0$ and $x > 0$ be such that $Bx - \alpha x \leq 0$. Then $\rho(B) \leq \alpha$.

[1)]The *index* of $A \in \mathbb{R}^{n\times n}$ is the least non-negative integer k such that rank(A^{k+1})=rank(A^k).

2.3. Proper splittings

Here, we recall some results on proper splittings which are useful in proving our main results. The first one contains a few properties of a proper splitting.

Theorem 2.5. *(Theorem 1, [3])*
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Then
(a) $A = U(I - U^{\dagger}V)$;
(b) $I - U^{\dagger}V$ is invertible;
(c) $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$.

If $A = U - V$ is a proper splitting of $A \in \mathbb{R}^{m\times n}$, then $U = A + V$ is also a proper splitting. Thus $I + A^{\dagger}V$ is invertible by Theorem 2.5 (b). Since FG and GF have same eigenvalues for any F and G such that both the product are defined, and $I + A^{\dagger}V$ is invertible, so -1 is not an eigenvalue of $VA^{\dagger}$. Hence $I + VA^{\dagger}$ is invertible. This fact can also be proved by considering the proper splitting $U^T = A^T + V^T$. The next lemma shows a relation between the eigenvalues of $U^{\dagger}V$ and $A^{\dagger}V$.

Lemma 2.6. *(Lemma 2.6, [11])*
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Let μ_i, $1 \le i \le s$ and λ_j, $1 \le j \le s$ be the eigenvalues of the matrices $U^{\dagger}V$ and $A^{\dagger}V$, respectively. Then for every j, we have $1 + \lambda_j \neq 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.

3. Proper Regular & Proper Weak Regular Splittings

In this section, the theory of proper regular and weak regular splittings is recalled first, and then some new results are proposed. We reproduce the definitions of proper regular splitting and proper weak regular splitting below.

Definition 3.1. *(Definition 1, [5] & Definition 1.1, [9])*
A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper regular splitting if it is a proper splitting such that $U^{\dagger} \ge 0$ and $V \ge 0$.

Definition 3.2. *(Definition 1.2, [9])*
A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper weak regular splitting if it is a proper splitting such that $U^{\dagger} \ge 0$ and $U^{\dagger}V \ge 0$.

The class of matrices having a fixed positive real number in all the entries always have proper regular and proper weak regular splittings. We next present an example of a proper splitting which is a proper weak regular splitting but not a proper regular splitting.

Example 3.3. *Let* $A = \begin{bmatrix} 9 & -8 & 15 \\ -6 & 6 & -10 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 10 \\ -3 & 4 & -5 \end{bmatrix} - \begin{bmatrix} -3 & 4 & -5 \\ 3 & -2 & 5 \end{bmatrix} = U - V$. *Then* $R(U) = R(A)$, $N(U) = N(A)$, $U^{\dagger} = \begin{bmatrix} 3/34 & 3/34 \\ 1/4 & 1/2 \\ 5/34 & 5/34 \end{bmatrix} \ge 0$ *and* $U^{\dagger}V = \begin{bmatrix} 0 & 3/17 & 0 \\ 3/4 & 0 & 5/4 \\ 0 & 5/17 & 0 \end{bmatrix} \ge 0$. *Thus $A = U - V$ is a proper weak regular splitting but not a proper regular splitting since $V \not\ge 0$.*

Jena *et al.* proved the following convergence criteria for the class of proper regular splittings.

Theorem 3.4. *(Theorem 1.3, [9])*
Let $A = U - V$ be a proper regular splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^{\dagger} \ge 0$ if and only if $\rho(U^{\dagger}V) < 1$.

Berman and Plemmons [3] initiated the study of convergence theory of iteration scheme (2) without terming the class of proper splittings $A = U - V$ as proper weak regular splittings. One of their results presented below characterizes semi-monotone matrices in the terms of this class of splittings.

Theorem 3.5. *(Theorem 3, [3])*
Let $A = U - V$ be a proper weak regular splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^{\dagger} \geq 0$ if and only if $\rho(U^{\dagger}V) < 1$.

Noted next result is proved in [9] which contains equivalent convergence conditions for iteration scheme (2).

Theorem 3.6. *(Theorem 3.1, [9])*
Let $A = U - V$ be a proper regular splitting of $A \in \mathbb{R}^{m\times n}$. If $A^{\dagger} \geq 0$, then
(a) $A^{\dagger} \geq U^{\dagger}$;
(b) $\rho(A^{\dagger}V) \geq \rho(U^{\dagger}V)$;
(c) $\rho(U^{\dagger}V) = \frac{\rho(A^{\dagger}V)}{1+\rho(A^{\dagger}V)} < 1$.

The conditions of the proper weak regular splitting still can be weakened by dropping the condition $U^{\dagger} \geq 0$, and the resultant splitting is known as *proper nonnegative (proper weak) splitting* (Definition 3.1, [10]). A convergence result for a proper nonnegative splitting is obtained below.

Lemma 3.7. *(Lemma 3.4, [10])*
Let $A = U - V$ be a proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$ and $A^{\dagger}U \geq 0$. Then $\rho(U^{\dagger}V) = \frac{\rho(A^{\dagger}U)-1}{\rho(A^{\dagger}U)} < 1$.

We remark that the above result is also true for the proper weak regular splitting. Next result further adds a few more equivalent conditions to the above Lemma for a proper weak regular splitting.

Theorem 3.8. *Let $A = U - V$ be a proper weak regular splitting of $A \in \mathbb{R}^{m\times n}$. Then*

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g).$$

(a) $A^{\dagger}U \geq 0$;
(b) $\rho(U^{\dagger}V) = \frac{\rho(A^{\dagger}U)-1}{\rho(A^{\dagger}U)}$;
(c) $\rho(U^{\dagger}V) < 1$;
(d) $(I - U^{\dagger}V)^{-1} \geq 0$;
(e) $A^{\dagger}V \geq 0$;
(f) $A^{\dagger}V \geq U^{\dagger}V$;
(g) $\rho(U^{\dagger}V) = \frac{\rho(A^{\dagger}V)}{1+\rho(A^{\dagger}V)} < 1$.

Proof. $(a) \Rightarrow (b)$: Follows from the proof of Lemma 3.7.
$(b) \Rightarrow (c)$: Obvious.
$(c) \Rightarrow (d)$: The conditions $\rho(U^{\dagger}V) < 1$ and $U^{\dagger}V \geq 0$ together yields that $(I - U^{\dagger}V)^{-1} = \sum_{k=0}^{\infty}(U^{\dagger}V)^k \geq 0$, by Theorem 2.3.
$(d) \Rightarrow (e)$: By Theorem 2.5 (c), we obtain $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$. Post-multiplying V both the sides, we get $A^{\dagger}V = (I - U^{\dagger}V)^{-1}U^{\dagger}V$. Hence $A^{\dagger}V \geq 0$ as $(I - U^{\dagger}V)^{-1} \geq 0$ and $U^{\dagger}V \geq 0$.
$(e) \Rightarrow (f)$: We have $A^{\dagger}V = (I - U^{\dagger}V)^{-1}U^{\dagger}V$ by Theorem 2.5 (c). Pre-multiplying $I - U^{\dagger}V$ both the sides, we obtain $(I - U^{\dagger}V)A^{\dagger}V = U^{\dagger}V$ which implies $A^{\dagger}V - U^{\dagger}V = U^{\dagger}VA^{\dagger}V$. Thus $A^{\dagger}V \geq U^{\dagger}V$ as $U^{\dagger}V \geq 0$ and $A^{\dagger}V \geq 0$.
$(f) \Rightarrow (g)$: Observe that $A^{\dagger}V \geq 0$ as $U^{\dagger}V \geq 0$. Let λ be any eigenvalue of $A^{\dagger}V$ and $f(\eta) = \frac{\eta}{1+\eta}$, $\eta \geq 0$. Then f is a strictly increasing function. Let μ be any eigenvalue of $U^{\dagger}V$. We now have $\mu = \frac{\lambda}{1+\lambda}$ by Lemma 2.6. Hence, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(A^{\dagger}V)$. As a result, the maximum value of μ is $\rho(U^{\dagger}V)$. Thus $\rho(U^{\dagger}V) = \frac{\rho(A^{\dagger}V)}{1+\rho(A^{\dagger}V)} < 1$. □

The rate of convergence of the iteration scheme (2) depends on the spectral radius of the iteration matrix $U^{\dagger}V$. Hence, the spectral radius of the iteration matrix plays a vital role in the comparison of the speed of the convergence of different iterative schemes of the same linear system given in (1). Next result compares the spectral radii of the iteration matrices between a proper regular splitting and a proper weak regular splitting arising out of the same coefficient matrix A.

Theorem 3.9. *Let $A = B - C$ be a proper weak regular splitting and $A = U - V$ be a proper regular splitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If $A \geq 0$ and $B^{\dagger} \geq U^{\dagger}$, then*

$$\rho(B^{\dagger}C) \leq \rho(U^{\dagger}V) < 1.$$

Proof. By Theorem 3.4 and Theorem 3.5, we have $\rho(U^{\dagger}V) < 1$ and $\rho(B^{\dagger}C) < 1$. Also $\rho(U^{\dagger}V)$ and $\rho(B^{\dagger}C)$ are strictly increasing functions of $\rho(A^{\dagger}V)$ and $\rho(A^{\dagger}C)$, so it suffices to show that

$$\rho(A^{\dagger}V) \geq \rho(A^{\dagger}C).$$

But $I + A^{\dagger}C$ and $I + VA^{\dagger}$ are both invertible as $A = B - C = U - V$ are proper splittings. The conditions $A = B-C$ is a proper weak regular splitting and $\rho(B^{\dagger}C) < 1$ imply that $A^{\dagger}C \geq 0$ by Theorem 2.3 and Theorem 2.5 (c) which in turn yields $I + A^{\dagger}C \geq 0$. Clearly, $I + VA^{\dagger} \geq 0$. Now $B^{\dagger} \geq U^{\dagger}$ results $A^{\dagger}(I + VA^{\dagger}) \geq (I + A^{\dagger}C)A^{\dagger}$ i.e., $A^{\dagger}VA^{\dagger} \geq A^{\dagger}CA^{\dagger}$. Then, post-multiplying it by V, we have

$$(A^{\dagger}V)^2 \geq A^{\dagger}CA^{\dagger}V.$$

Again, post-multiplying $A^{\dagger}VA^{\dagger} \geq A^{\dagger}CA^{\dagger}$ by A, we get $A^{\dagger}VA^{\dagger}A = A^{\dagger}V \geq A^{\dagger}CA^{\dagger}A = A^{\dagger}C$. So

$$A^{\dagger}VA^{\dagger}C \geq (A^{\dagger}C)^2.$$

Therefore, by Theorem 2.2, we have

$$\rho^2(A^{\dagger}V) \geq \rho(A^{\dagger}VA^{\dagger}C) = \rho(A^{\dagger}CA^{\dagger}V) \geq \rho^2(A^{\dagger}C).$$

Hence $\rho(A^{\dagger}V) \geq \rho(A^{\dagger}C)$. Thus

$$\rho(B^{\dagger}C) \leq \rho(U^{\dagger}V) < 1.$$

□

We now present a result which replaces the condition $A \geq 0$ in the above theorem by row sums of $U^{\dagger}$ are positive.

Theorem 3.10. *Let $A = B - C$ be a proper weak regular splitting and $A = U - V$ be a proper regular splitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If $B^{\dagger} \geq U^{\dagger}$ and row sums of $U^{\dagger}$ are positive, then*

$$\rho(B^{\dagger}C) \leq \rho(U^{\dagger}V) < 1.$$

Proof. We have $\rho(U^{\dagger}V) < 1$ and $\rho(B^{\dagger}C) < 1$, by Theorem 3.4 and Theorem 3.5, respectively. As $U^{\dagger}V \geq 0$, by Theorem 2.1, there exists $x \geq 0$ such that $U^{\dagger}Vx = \rho(U^{\dagger}V)x$. So $x \in R(U^T) = R(B^T)$. Therefore $Ux = \frac{1}{\rho(U^{\dagger}V)}UU^{\dagger}Vx = \frac{1}{\rho(U^{\dagger}V)}Vx$. Now $Ax = (U - V)x = U(I - U^{\dagger}V)x = (1 - \rho(U^{\dagger}V))Ux = (\frac{1}{\rho(U^{\dagger}V)} - 1)Vx \geq 0$ as $V \geq 0$ and $\rho(U^{\dagger}V) < 1$. Then the condition $B^{\dagger} \geq U^{\dagger}$ yields $B^{\dagger}Ax \geq U^{\dagger}Ax$, i.e., $B^{\dagger}(B - C)x \geq U^{\dagger}(U - V)x$ which in turn implies that $x - B^{\dagger}Cx \geq x - U^{\dagger}Vx$. Hence $B^{\dagger}Cx \leq U^{\dagger}Vx = \rho(U^{\dagger}V)x$. By replacing A by $A - \epsilon J$ and V by $V + \epsilon J$, where all the entries of J are 1, and ϵ is a small positive real number, we can assume that $x > 0$. Thus $\rho(B^{\dagger}C) \leq \rho(U^{\dagger}V) < 1$, by Theorem 2.4. □

We remark that the above result is also true if we replace the condition 'row sums of $U^{\dagger}$ are positive' by 'no row of $U^{\dagger}$ is zero' as the conditions '$U^{\dagger} \geq 0$' and 'no row of $U^{\dagger}$ is zero' yield 'row sums of $U^{\dagger}$ are positive'. The above proof adopts a similar technique as in the proof of Lemma (Section 3, [6]). Notice that $U^{\dagger}(V + \epsilon J) > 0$ may not be possible always unless row sums of $U^{\dagger}$ are positive. Hence we have assumed the condition row sums of $U^{\dagger}$ are positive. This fact is shown through an example below.

Example 3.11. *Let* $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$. *We have* $R(U) = R(A)$, $N(U) = N(A)$, $V \geq 0$ *and* $U^\dagger = \begin{bmatrix} 0 & 0 \\ 1/25 & 2/25 \\ 1/50 & 1/25 \end{bmatrix} \geq 0$. *Hence* $A = U - V$ *is a proper regular splitting. But for* $\epsilon = 0.01$, *we have* $U^\dagger(V + \epsilon J) = \begin{bmatrix} 0 & 0 & 0 \\ 3/2500 & 1003/2500 & 503/2500 \\ 3/5000 & 535/2667 & 218/2167 \end{bmatrix} \geq 0$.

One can use the comparison results to pick the best splitting among any finite number of splittings. However, the major drawback of this theory is the following: *it is time-consuming and needs many computation.* To avoid this situation and to get a finer process, we now proceed to introduce alternating iteration scheme for rectangular matrices replacing the ordinary matrix inverse by the Moore-Penrose inverse, and then discuss its convergence theory.

4. Application to Convergence of Alternating Iterations

Let $A = M - N = U - V$ be two proper splittings of $A \in \mathbb{R}^{m\times n}$. We now propose

$$x^{i+1/2} = M^\dagger N x^i + M^\dagger b, \quad x^{i+1} = U^\dagger V x^{i+1/2} + U^\dagger b, \quad i = 0, 1, 2, \dots, \tag{6}$$

as the general class of iterative method for finding the solution of (1) with the initial approximation x^0. In the case of nonsingular M and U, the above equation reduces to equation (8) of section 3, [2] (i.e., equation (4) of this paper). Not only that many well-known methods belong to such a class, and are also discussed in the same section of [2].

In order to study the convergence of the above scheme, we construct a single splitting $A = B - C$ associated with the iteration matrix by eliminating $x^{i+1/2}$ from (6). So, we have

$$x^{i+1} = U^\dagger V M^\dagger N x^i + U^\dagger(VM^\dagger + I)b, \quad i = 0, 1, 2, \dots, \tag{7}$$

where $H = U^\dagger V M^\dagger N$ is the iteration matrix of the new iterative scheme (7).

Recall that the convergence of the individual splittings $A = M - N$ and $A = U - V$ does not imply the convergence of the alternating iterative scheme (7). Example 3.1, [2] is in this direction, and is obtained below for the sack of completeness and ready reference.

Example 4.1. *(Example 3.1, [2])*
Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ *and* $U = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$. *Then* $A = M - N = U - V$ *are two convergent proper splittings, but* $\rho(H) = \rho(U^\dagger V M^\dagger N) = 1$.

Convergence of the iteration scheme (7) is addressed in the next result.

Theorem 4.2. *Let* $A = M - N = U - V$ *be two proper weak regular splittings of a semi-monotone matrix* $A \in \mathbb{R}^{m\times n}$. *Then* $\rho(H) = \rho(U^\dagger V M^\dagger N) < 1$.

Proof. We have $H = U^\dagger V M^\dagger N = U^\dagger(U - A)M^\dagger(M - A) = U^\dagger U - U^\dagger A - M^\dagger A + U^\dagger A M^\dagger A$. Since $A = M - N = U - V$ are two proper splittings, so $R(U) = R(M) = R(A)$ and $N(U) = N(M) = N(A)$. Hence $M^\dagger M = U^\dagger U = A^\dagger A$. We then have $H = U^\dagger U - U^\dagger A - M^\dagger A + U^\dagger A M^\dagger A$. Again, $U^\dagger A M^\dagger = U^\dagger(U - V)M^\dagger = U^\dagger U M^\dagger - U^\dagger V M^\dagger = M^\dagger M M^\dagger - U^\dagger V M^\dagger = M^\dagger - U^\dagger V M^\dagger$. But $U^\dagger V M^\dagger \geq 0$ as $A = M - N = U - V$ are two proper weak regular splittings. So $M^\dagger \geq M^\dagger - U^\dagger V M^\dagger \geq U^\dagger A M^\dagger$ which results $(I - U^\dagger A)M^\dagger \geq 0$. The condition $U^\dagger \geq 0$ yields $U^\dagger + (I - U^\dagger A)M^\dagger = U^\dagger + M^\dagger - U^\dagger A M^\dagger \geq 0$. This implies $A^\dagger - U^\dagger - M^\dagger + U^\dagger A M^\dagger \leq A^\dagger$ which can be rewritten as $U^\dagger U A^\dagger - U^\dagger A A^\dagger - M^\dagger A A^\dagger + U^\dagger A M^\dagger A A^\dagger \leq A^\dagger$. We then have $(U^\dagger U - U^\dagger A - M^\dagger A + U^\dagger A M^\dagger A)A^\dagger \leq A^\dagger$, i.e., $HA^\dagger \leq A^\dagger$. Thus $(I - H)A^\dagger \geq 0$.

As $H \geq 0$, we have $0 \leq (I + H + H^2 + H^3 + \cdots + H^m)(I - H)A^\dagger = (I - H^{m+1})A^\dagger \leq A^\dagger$ for each $m \in \mathbb{N}$. So, the partial sums of the series $\sum_{m=0}^{\infty} H^m$ is uniformly bounded. Hence $\rho(H) < 1$. □

Next example shows that the converse of Theorem 4.2 is not true.

Example 4.3. *Let* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 4 & 0 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ *and* $U = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. *Then* $A = M - N = U - V$ *are two proper splittings. Also* $\rho(H) = \rho(U^{\dagger}VM^{\dagger}N) = 3/8 < 1$. *But* $M^{\dagger} = \begin{bmatrix} 1/4 & -1/6 \\ -1/4 & 1/3 \\ 0 & 1/6 \end{bmatrix} \ngeq 0$ *and* $U^{\dagger} = \begin{bmatrix} 5/12 & -1/6 \\ -1/3 & 1/3 \\ 1/12 & 1/6 \end{bmatrix} \ngeq 0$, *i.e.,* $A = M - N = U - V$ *are not proper weak regular splittings.*

It is of interest to know the type of splitting $B - C$ of A that yields the iterative scheme (7)(i.e., $x^{i+1} = Hx^{i} + B^{\dagger}b$ with $H = B^{\dagger}C$). This can be restated as what can we say about the type of the induced splitting $A = B - C$ which is induced by $H = U^{\dagger}VM^{\dagger}N$. The same problem is settled partially by the next result under the assumptions of a few conditions.

Theorem 4.4. *Let* $A = M - N = U - V$ *be two proper weak regular splittings of a semi-monotone matrix* $A \in \mathbb{R}^{m\times n}$. *Then the unique splitting* $A = B - C$ *induced by* H *with* $B = M(M + U - A)^{\dagger}U$ *is a proper weak regular splitting if* $R(M + U - A) = R(A)$ *and* $N(M + U - A) = N(A)$.

Proof. From equation (7), we have $B^{\dagger} = U^{\dagger}(VM^{\dagger} + I)$. By substituting $V = U - A$, we get $B^{\dagger} = U^{\dagger} + U^{\dagger}UM^{\dagger} - U^{\dagger}AM^{\dagger} = U^{\dagger}MM^{\dagger} + U^{\dagger}UM^{\dagger} - U^{\dagger}AM^{\dagger} = U^{\dagger}(M+U-A)M^{\dagger}$. Since $R(M+U-A) = R(A)$, $N(M+U-A) = N(A)$ and $A = M-N = U-V$ are proper splittings, we have $(M+U-A)(M+U-A)^{\dagger} = P_{R(M+U-A)} = P_{R(A)} = P_{R(U)} = P_{R(M)}$ and $(M + U - A)^{\dagger}(M + U - A) = P_{R((M+U-A)^T)} = P_{R(A^T)} = P_{R(M^T)} = P_{R(U^T)}$. Let $X = M(M + U - A)^{\dagger}U$, then $B^{\dagger}X = U^{\dagger}(M+U-A)M^{\dagger}M(M+U-A)^{\dagger}U = U^{\dagger}P_{R(U)}U = U^{\dagger}U$. So $B^{\dagger}X$ is symmetric and $B^{\dagger}XB^{\dagger} = B^{\dagger}$. Similarly, it can be shown that $XB^{\dagger}$ is symmetric and $XB^{\dagger}X = X$. Hence $X = (B^{\dagger})^{\dagger} = B = M(M + U - A)^{\dagger}U$.

Next to show that $R(B) = R(A)$ and $N(B) = N(A)$. First, we prove that $N(U) = N(A) = N(B)$. Clearly, $N(U) \subseteq N(B)$. Let $Bx = 0$. Pre-multiplying $M^{\dagger}$ to $Bx = 0$ and using $M^{\dagger}M = P_{R(A^T)} = P_{R((M+U-A)^T)}$, we obtain $(M + U - A)^{\dagger}Ux = 0$. Again, pre-multiplying $(M + U - A)$ and using the fact $(M + U - A)(M + U - A)^{\dagger} = P_{R(M+U-A)} = P_{R(U)}$, we get $x \in N(U)$. So $N(B) \subseteq N(U)$. We next have to prove that $R(A) = R(B)$, i.e., $N(M^T) = N(A^T) = N(B^T)$. Since $B = M(M + U - A)^{\dagger}U$, so $N(M^T) \subseteq N(B^T)$. Hence we need to show the other way, i.e., $N(B^T) \subseteq N(M^T)$. Let $x \in N(B^T)$. Then $(M(M + U - A)^{\dagger}U)^T x = 0$. Pre-multiplying $(U^{\dagger})^T$, we get $(UU^{\dagger})^T[(M+U-A)^{\dagger}]^T M^T x = 0$, i.e., $x^T M(M+U-A)^{\dagger}UU^{\dagger} = x^T M(M+U-A)^{\dagger} = 0$. Again, post-multiplying $(M + U - A)$, we get $x^T MM^{\dagger}M = 0$. Thus $M^T x = 0$, i.e., $N(B^T) \subseteq N(M^T)$.

We have $B^{\dagger} = M^{\dagger} + U^{\dagger} - U^{\dagger}AM^{\dagger} = A^{\dagger} - U^{\dagger}UM^{\dagger}MA^{\dagger} + U^{\dagger}UM^{\dagger}AA^{\dagger} + U^{\dagger}AM^{\dagger}MA^{\dagger} - U^{\dagger}AM^{\dagger}AA^{\dagger} = A^{\dagger} - (U^{\dagger}UM^{\dagger} - U^{\dagger}AM^{\dagger})(MA^{\dagger} - AA^{\dagger}) = A^{\dagger} - (U^{\dagger}(U - A)M^{\dagger}(M - A)A^{\dagger}) = A^{\dagger} - U^{\dagger}VM^{\dagger}NA^{\dagger} = (I - H)A^{\dagger}$. Next to prove that $A = B - C$ is a proper splitting, i.e., to show that $A = B - C$, $R(B) = R(A)$ and $N(B) = N(A)$. We have already shown the last two conditions, so we have to prove only $A = B - C$. By Theorem 4.2, we have $\rho(H) < 1$ and so $I - H$ is invertible. Let $X = A(I - H)^{-1}$. Then $XB^{\dagger} = AA^{\dagger}$ which results $XB^{\dagger}$ is symmetric and $XB^{\dagger}X = X$. Again $B^{\dagger}X = (I-H)A^{\dagger}A(I-H)^{-1} = (A^{\dagger}A - HA^{\dagger}A)(I-H)^{-1} = (A^{\dagger}A - A^{\dagger}AH)(I-H)^{-1} = A^{\dagger}A$ which yields $B^{\dagger}X$ is symmetric and $B^{\dagger}XB^{\dagger} = A^{\dagger}A(I-H)A^{\dagger} = (I-H)A^{\dagger}AA^{\dagger} = B^{\dagger}$. Hence $X = (B^{\dagger})^{\dagger} = B = A(I-H)^{-1}$ and $C = B - A$. Now $B^{\dagger}C = B^{\dagger}B - B^{\dagger}A = B^{\dagger}B - (I - H)A^{\dagger}A = H$. Thus $A = B - C$ is a proper splitting. Next, we have to prove that the proper splitting $A = B - C$ is unique. Suppose that there exists another induced splitting $A = \bar{B} - \bar{C}$ such that $H = \bar{B}^{\dagger}\bar{C}$. Then $\bar{B}H = \bar{B}\bar{B}^{\dagger}\bar{C} = \bar{C} = \bar{B} - A$. So $\bar{B} = A + \bar{B}H$, i.e., $\bar{B}(I - H) = A$. This reveals that $\bar{B} = A(I - H)^{-1} = B$ and therefore, H induces the unique proper splitting $A = B - C$.

Finally, $B^{\dagger} = U^{\dagger} + U^{\dagger}UM^{\dagger} - U^{\dagger}AM^{\dagger} = U^{\dagger} + (M^{\dagger} - U^{\dagger}AM^{\dagger}) = U^{\dagger} + U^{\dagger}VM^{\dagger} \geq 0$ since $A = M - N = U - V$ are proper weak regular splittings and $M^{\dagger} - U^{\dagger}AM^{\dagger} = U^{\dagger}VM^{\dagger}$. Also $B^{\dagger}C = U^{\dagger}VM^{\dagger}N \geq 0$. Hence $A = B - C$ with $B = M(M + U - A)^{\dagger}U$ is a proper weak regular splitting. □

Another question comes to picture now, i.e., among these splittings which will converge faster. More specifically, we want to know the rate of convergence of the induced splitting for the iterative scheme (7). If the induced splitting $A = B - C$ will not converge faster than the individual splittings $A = M - N$ and $A = U - V$, then the proposed alternating iteration method will not be useful. In this direction, we next present a result which compares the rate of convergence of the induced splitting with the individual splitting.

Theorem 4.5. *Let $A \in \mathbb{R}^{m\times n}$ and $A \geq 0$. Let $A = M-N = U-V$ be two proper regular splittings of a semi-monotone matrix A such that $R(M+U-A) = R(A)$ and $N(M+U-A) = N(A)$. Then $\rho(H) \leq min\{\rho(U^{\dagger}V), \rho(M^{\dagger}N)\} < 1$, where $H = U^{\dagger}VM^{\dagger}N$.*

Proof. Let H be the iteration matrix corresponding to the induced splitting $A = B - C$. Then, by Theorem 4.4, $A = B - C$ is a proper weak regular splitting. Using the conditions $U^{\dagger}VM^{\dagger} \geq 0$ and $U^{\dagger}NM^{\dagger} \geq 0$, we have

$$B^{\dagger} = U^{\dagger}(M+U-A)M^{\dagger} = U^{\dagger}MM^{\dagger} + U^{\dagger}VM^{\dagger} = U^{\dagger} + U^{\dagger}VM^{\dagger} \geq U^{\dagger}$$

and

$$B^{\dagger} = U^{\dagger}(M+U-A)M^{\dagger} = M^{\dagger} + U^{\dagger}NM^{\dagger} \geq M^{\dagger}.$$

Now, applying Theorem 3.9 to the splittings $A = B - C$ and $A = U - V$, we have

$$\rho(H) \leq \rho(U^{\dagger}V) < 1.$$

Again, applying the same theorem to the splittings $A = B - C$ and $A = M - N$, we obtain

$$\rho(H) \leq \rho(M^{\dagger}N) < 1.$$

Hence $\rho(H) \leq \min\{\rho(U^{\dagger}V), \rho(M^{\dagger}N)\} < 1$. □

In words, the above theorem says that the spectral radius of the product of the iteration matrices $U^{\dagger}V$ and $M^{\dagger}N$ cannot exceed the spectral radius of either factor under the assumption of some conditions. The converse of Theorem 4.5 does not hold. This is illustrated by the following example.

Example 4.6. *Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 1 & -2 & 3 \\ -4 & -6 & -8 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -6 & 9 \\ 5 & 15/2 & 10 \end{bmatrix}$. Then $A = M - N = U - V$ are two proper splittings with $\rho(H) = \rho(U^{\dagger}VM^{\dagger}N) = 9/10 < 1$. But $A = M - N = U - V$ are not proper regular splittings as $N = \begin{bmatrix} 0 & 0 & 0 \\ -6 & -9 & -12 \end{bmatrix} \ngeq 0$ and $V = \begin{bmatrix} 2 & -4 & 6 \\ 3 & 9/2 & 6 \end{bmatrix} \ngeq 0$. Also $A \ngeq 0$.*

We next produce an example which states that the condition proper regular splitting cannot be dropped.

Example 4.7. *Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $A = M-N = U-V$ are two splittings with $\rho(H) = \rho(U^{\dagger}VM^{\dagger}N) = 1$. But $A = U - V$ is not a proper regular splitting as $V = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ngeq 0$. Then $\rho(H) = 1 \nleq min\{\rho(U^{\dagger}V) = 2, \rho(M^{\dagger}N) = 1/2\} \nless 1$.*

However, we have a few examples which show that Theorem 4.5 is also true even if $A \ngeq 0$. One such example is provided below.

Example 4.8. *Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \end{bmatrix}$. Then $A = M-N = U-V$ are two proper regular splittings with $\rho(H) = \rho(U^{\dagger}VM^{\dagger}N) = 7/40 = 0.175 \leq min\{\rho(U^{\dagger}V) = 1/2 = 0.5, \rho(M^{\dagger}N) = 2/5 = 0.4\} < 1$.*

Note that Theorem 4.5 also holds for $A = U - V$ is a proper weak regular splitting. This suggests the following question.

Can we drop the condition $A \geq 0$ from Theorem 4.5 ?

The answer is partially affirmative if we use of Theorem 3.10 instead of Theorem 3.9. The same result is stated below.

Theorem 4.9. *Let $A = M - N = U - V$ be two proper regular splittings of a semi-monotone matrix A such that $R(M + U - A) = R(A)$ and $N(M + U - A) = N(A)$. Suppose that row sums of $U^\dagger$ and $M^\dagger$ are positive. Then $\rho(H) \leq min\{\rho(U^\dagger V), \rho(M^\dagger N)\} < 1$, where $H = U^\dagger V M^\dagger N$.*

Finally, we conclude this section with a problem which appears to be open:

Can we drop the conditions "row sums of $U^\dagger$ and $M^\dagger$ are positive" from Theorem 4.9 ?

5. Conclusions

The notion of the alternating iterative method for singular and rectangular linear systems is introduced. The present work extends the work of Benzi and Szyld [2] to the rectangular(singular) case. The following three main results are obtained among others.

- Sufficient conditions for the convergence of alternating iteration scheme is provided (Theorem 4.2). This coincides with the first objective of Theorem 3.2, [2] in the case of nonsingular matrices.
- The induced splitting is shown to be a proper weak regular splitting under a few assumptions. This result not only partially fulfills the 2nd objective of Theorem 3.2, [2] in rectangular matrix setting but also extends Theorem 3.4, [2].
- Theorem 4.5 describes that the induced splitting is a better choice among the individual splittings which generalizes Theorem 4.1, [2] for non-negative A.

The numerical benchmark of the alternating iterative method indicates that the rate of convergence of the proposed alternating iterative method is not higher than the rate of convergence of the usual iterative method. A problem for future study is also proposed in the last part of Section 4. Not only that if we consider

$$X^{i+1} = U^\dagger V M^\dagger N X^i + U^\dagger (V M^\dagger + I), \quad i = 0, 1, 2, \ldots,$$

then this scheme will converge to the Moore-Penrose inverse of A.

In the case of a real singular matrix, let m be the degree of the minimal polynomial for A. If $b \in R(A^k)$, then the linear system $Ax = b$ has a unique Krylov solution $x = A^D b \in K_{m-k}(A, b)$, where k is the index of A. Scope exists to extend this work to compute $A^D b$, and the Drazin inverse of A as computing Drazin inverse of a matrix is still a challenging problem.

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