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Module Amenability, Module Character Biprojectivity and Module Character Biflatness of Lau Product of Two Banach Algebras

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Abstract. Let $\mathfrak{A}$ be a Banach algebra and T be an $\mathfrak{A}$-module homomorphism from $\mathfrak{A}$-bimodule B into $\mathfrak{A}$-bimodule A. We investigate module amenability (resp. module approximate amenability), module character amenability (resp. module character approximate amenability), module character biprojectivity and module character biflatness of $A \times_{T_{\mathfrak{A}}} B$ for every two Banach $\mathfrak{A}$-bimodule A and B.

1. Introduction

Let A be a Banach algebra and let $\phi \in \Delta(A)$. Kaniuth, Lau and Pym in [12], have recently introduced and studied the interesting notion of φ-amenability. Specifically, A is called (left) ϕ-amenable if there exists $m \in A^{**}$ such that $m(\phi) = 1$ and $m(f.a) = \phi(a)m(f)$ for all $a \in A$ and $f \in A^*$. Also, the notion of (right) character amenability was introduced and studied by Monfared in [16]. He called a Banach algebra A character amenable if it is both left and right character amenable.

Moreover, the notion of approximate character amenability of Banach algebras was introduced by Aghababa, Shi and Wu in [19]. They called a Banach algebra A is approximately right character amenable, if for every $\phi \in \Delta(A) \cup \{0\}$ and every A-bimodule X, that the left module action is $a.x = \phi(a)x$ ($a \in A, x \in X$), every derivation $D : A \longrightarrow X^*$ is approximately inner. Approximately left character amenability is defined similarly and A is called approximately character amenable if it is both approximately left and right character amenable. Also, various notions of character amenability such as module approximate amenability and module character inner amenability of Banach algebras are introduced and investigated in [7] and [21].

The notion of Biprojective Banach algebras and biflatness Banach algebras were introduced by A. Ya. Helemskii in [10, 11]. A Banach algebra A is called biprojective if the map $\Delta : A\widehat{\otimes}A \longrightarrow A$ has a bounded right inverse which is an A-bimodule map (i.e. a bounded linear map which preserves the module operations). A Banach algebra A is said to be biflat if the adjoint $\Delta^* : A^* \longrightarrow (A\widehat{\otimes}A)^*$ of Δ has a bounded left inverse which is an A-bimodule map.

Let A and B be Banach algebras with spectrum $\Delta(B) \neq 0$. Let $\theta \in \Delta(B)$, then the direct product $A \times B$ equipped with the algebra multiplication

$$(a, b).(c, d) = (ac + \theta(d)a + \theta(b)c, bd), \qquad (a, c \in A, b, d \in B),$$

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and the l^1-norm is a Banach algebras which is called the θ-Lau product of A and B and is denoted by $A \times_\theta B$. This type of product was first introduced by Lau [14] for certain class of Banach algebras and was extended by M. S. Monfared [17] for the general case. Many basic properties of $A \times_\theta B$ such as Biflatness and biprojectivity are investigated in [17] and [13].

Recently, in the case where A is commutative, a new extension of Lau product introduced and has been studied by Bhatt and Dabhi [4]. In [1], the authors, by slight change in the definition of multiplication given by Bhatt and Dabhi on $A \times_\varphi B$, investigated the two notions, biprojectivity and biflatness on $A \times_\varphi B$ for an arbitrary Banach algebra A, where φ is a Banach algebra homomorphism from B into A.

The notion of module amenability was introduced by Amini [2] for a class of Banach algebras that are modules over another Banach algebra with compatible actions. Let $\phi \in \Delta(\mathfrak{A}) \cup \{0\}$ and consider the set Ω_A of linear maps $\varphi : A \to \mathfrak{A}$ such that

$$\varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(\alpha.a) = \varphi(a.\alpha) = \phi(\alpha)\varphi(a) \quad (a, b \in A, \alpha \in \mathfrak{A}).$$

The concept of (φ, ϕ)-module amenability and module character amenability for Banach algebra A, where $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$ were introduced by Bodaghi and Amini in [6](see also [8]). In [5], the authors have introduced and investigated the concept of module biprojectivity and module biflatness of Banach algebras.

The purpose of the present paper is to investigate module amenability (resp. module approximate amenability), module character amenability (resp. module character approximate amenability), module character biprojectivity and module character biflatness of $A \times_{T_\mathfrak{A}} B$ in terms of the same properties for A and B.

We briefly summarize the results in this paper. In section 3 we investigate relations between module amenability (resp. module approximate amenability) of A, B and $A \times_{T_\mathfrak{A}} B$. We also prove that $A \times_{T_\mathfrak{A}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module amenable and $\big((0, \psi), \phi\big)$-module amenable (resp. $\big((\varphi, \varphi \circ T), \phi\big)$-module approximate amenable and $\big((0, \psi), \phi\big)$-module approximate amenable) for every $\varphi \in \Omega_A$, $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if both A and B are module character amenable (resp. module approximate character amenable).

In section 4 we show that under some conditions $A \times_{T_\mathfrak{A}} B$ is $(\varphi, \varphi \circ T)$-module biprojective (resp. $(0, \psi)$-module biprojective) if and only if A is φ-module biprojective (resp. B is ψ-module biprojective) and show same results for approximate property. Finally, we prove that $A \times_{T_\mathfrak{A}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module biflat and $\big((0, \psi), \phi\big)$-module biflat (resp. $\big((\varphi, \varphi \circ T), \phi\big)$-module approximate biflat and $\big((0, \psi), \phi\big)$-module approximate biflat)for every $\varphi \in \Omega_A$, $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if both A and B are module character biflat (resp. module character approximate biflat).

2. Preliminaries

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and there is a constant $k > 0$ such that

$$||a.x|| \le k||a||||x||, \quad ||x.a|| \le k||a||||x|| \quad (a \in A, x \in X).$$

Let $\mathfrak{A}$ and A be Banach algebras such that A be a Banach $\mathfrak{A}$-bimodule with compatible actions

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \qquad (a, b \in A, \alpha \in \mathfrak{A}),$$

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a) \quad (a \in A, \alpha \in \mathfrak{A}, x \in X),$$

and similar for the right or two-sided actions. Then we say that X is a Banach A-$\mathfrak{A}$-module. A Banach A-$\mathfrak{A}$-module X is called commutative A-$\mathfrak{A}$-module, if $\alpha.x = x.\alpha$ $(\alpha \in \mathfrak{A}, x \in X)$. Note that in general, A is not a Banach A-$\mathfrak{A}$-module because A need not satisfy the compatibility conditions $a.(\alpha.b) = (a.\alpha).b$ $(a, b \in A, \alpha \in \mathfrak{A})$.

If X is a (commutative) Banach A-$\mathfrak{A}$-module , then so is X^*, whenever the actions of A and $\mathfrak{A}$ on X^* define by

$$\langle \alpha.f, x\rangle = \langle f, x.\alpha\rangle, \ \langle a.f, x\rangle = \langle f, x.a\rangle \quad (a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*),$$

and similarly for the right actions.

Let X and Y be two A-$\mathfrak{A}$-modules, then a bounded map $h : X \to Y$ is called A-$\mathfrak{A}$-module map if $h(x \pm y) = h(x) \pm h(y)$ and

$$h(\alpha.x) = \alpha.h(x), \ \ h(x.\alpha) = h(x).\alpha, \ \ h(a.x) = a.h(x), \ \ h(x.a) = h(x).a,$$

for $x, y \in X, a \in A$, and $\alpha \in \mathfrak{A}$.

Let $A\widehat{\otimes}A$ be the projective tensor product of A and A which is a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule by the following actions:

$$\alpha.(a \otimes b) = (\alpha.a) \otimes b, \ \ c.(a \otimes b) = (ca) \otimes b \qquad (\alpha \in \mathfrak{A}, a, b, c \in A),$$

similarly for the right actions. Let $I_{A\widehat{\otimes}A}$ be the closed ideal of $A\widehat{\otimes}A$ generated by elements of the form

$$\{a.\alpha \otimes b - a \otimes \alpha.b \ \ |\alpha \in \mathfrak{A}, a, b \in A\}. \tag{1}$$

Consider the map $\omega_A \in \mathcal{L}(A\widehat{\otimes}A, A)$ defined by $\omega_A(a \otimes b) = ab$ and extended by linearity and continuity. Let J_A be the closed ideal of A generated by

$$\omega(I_{A\widehat{\otimes}A}) = \{(a.\alpha)b - a(\alpha.b) \ \ |a, b \in A, \alpha \in \mathfrak{A}\}. \tag{2}$$

Then, the module projective tensor product $A\widehat{\otimes}_{\mathfrak{A}}A$, which is $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ by [20], and the quotient Banach algebra A/J_A are both Banach A-bimodules and Banach $\mathfrak{A}$-bimodules. Also, A/J_A is A-$\mathfrak{A}$-module with compatible actions when A acts on A/J_A canonically.

Define $\tilde{\omega}_A \in \mathcal{L}(A\widehat{\otimes}_{\mathfrak{A}}A, A/J_A)$ by $\tilde{\omega}_A(a \otimes b + I_{A\widehat{\otimes}A}) = ab + J_A$ and extend by linearity and continuity. Obviously, $\tilde{\omega}_A$ is A-$\mathfrak{A}$-bimodule map. Moreover, $\tilde{\omega}^*$, the first adjoints of $\tilde{\omega}$ is also A-$\mathfrak{A}$-module map.

Let X be a Banach A-$\mathfrak{A}$-module. A bounded map $D : A \to X$ is called an $\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b), \ \ D(ab) = D(a).b + b.D(b) \qquad (a, b \in A),$$

and

$$D(\alpha.a) = \alpha.D(a), \ \ D(a.\alpha) = D(a).\alpha \quad (a \in A, \alpha \in \mathfrak{A}).$$

Although D in general is not linear, but still its boundedness implies its norm continuity. For every $x \in X$, we define ad_x by $ad_x(a) = a.x - x.a$ $(a \in A)$. A $\mathfrak{A}$-module derivation D is said to be inner (resp. approximate inner) if there exists $x \in X$ (resp. there exists a bounded net $(x_\alpha) \subset X$) such that $D(a) = ad_x(a)$ $(a \in A)$ (resp. $D(a) = \lim_\alpha ad_{x_\alpha}(a)(a \in A)$). A Banach algebra A is called $\mathfrak{A}$-module amenable (resp. $\mathfrak{A}$-module approximate amenable) if for any commutative Banach A-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D : A \to X^*$ is inner (resp. approximate inner)(see [2] and [18]).

3. Module amenability of $A \times_{T_{\mathfrak{A}}} B$

Suppose that A and B are Banach algebra and Banach $\mathfrak{A}$-bimodule with compatible actions. Then, an $\mathfrak{A}$-module homomorphism is a Banach algebra homomorphism $T : A \to B$ with

$$T(\alpha.a) = \alpha.T(a), \ \ T(a.\alpha) = T(a).\alpha \qquad (a \in A, \alpha \in \mathfrak{A}).$$

We denote by $\mathrm{Hom}_{\mathfrak{A}}(A, B)$, the all $\mathfrak{A}$-module homomorphism from A into B.

Let $T \in \mathrm{Hom}_{\mathfrak{A}}(B, A)$ with $\|T\| \leq 1$. The cartesian product space $A \times B$ equipped with the following algebra multiplication

$$(a_1, b_1).(a_2, b_2) = (a_1a_2 + a_1T(b_2) + T(b_1)a_2, b_1b_2), \ \ (a_1, a_2 \in A, b_1, b_2 \in B),$$

and the norm $\|(a,b)\| = \|a\|_A + \|b\|_B$, defines a Banach algebra which is denoted by $A \times_T B$. Furthermore, $A \times_T B$ is a Banach A-bimodule under the module actions

$$a'.(a,b) = (a',0).(a,b), \quad (a,b).a' = (a,b).(a',0) \qquad (a,a' \in A, b \in B),$$

and is $\mathfrak{A}$-bimodule under the module actions

$$\alpha.(a,b) = (\alpha.a, \alpha.b), \quad (a,b).\alpha = (a.\alpha, b.\alpha) \qquad (a \in A, b \in B, \alpha \in \mathfrak{A}).$$

For this reason we denote $A \times_T B$ by $A \times_{T_{\mathfrak{A}}} B$ which we call it the module Lau product of A and B.

We note that the dual space $(A \times_{T_{\mathfrak{A}}} B)^*$ can be identified with $A^* \times B^*$, via

$$\langle (f,g),(a,b)\rangle = \langle a,f\rangle + \langle b,g\rangle \quad (a \in A, f \in A^*, b \in B, g \in B^*),$$

when we consider $A^* \times B^*$ under the norm

$$\|(f,g)\| = \|f\| + \|g\| \qquad (f \in A^*, g \in B^*).$$

For more details see Theorem 1.10.13 of [15]. Moreover, $(A \times_{T_{\mathfrak{A}}} B)^*$ is a $(A \times_{T_{\mathfrak{A}}} B)$-bimodule with the module operations given by

$$(f,g).(a,b) = \big(f.a + f.T(b), f \circ (L_aT) + g.b\big) \quad (a \in A, f \in A^*, b \in B, g \in B^*), \tag{3}$$

$$(a,b).(f,g) = \big(a.f + T(b).f, f \circ (R_aT) + b.g\big) \quad (a \in A, f \in A^*, b \in B, g \in B^*), \tag{4}$$

where $L_aT : B \to A$ and $R_aT : B \to A$ are defined by $L_aT(b) = aT(b)$ and $R_aT(b) = T(b)a \ \ (b \in B)$.

Theorem 3.1. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathtt{Hom}_{\mathfrak{A}}(B,A)$. Then the following statements are valid:*

(i) *If $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module amenable, then so are A and B. In the case that A has a bounded approximate identity the converse is also valid.*

(ii) *If $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module approximate amenable, then so are A and B.*

Proof. (i) Suppose that $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module amenable. Let $d : A \to X^*$ be an $\mathfrak{A}$-module derivation for a commutative Banach A-$\mathfrak{A}$-module X. Define $\tilde{d} : A \times \{0\} \to X^* \times \{0\}$. by $\tilde{d}(a,0) = (d(a),0)(a \in A)$ and define $P : A \times_{T_{\mathfrak{A}}} B \to A \times \{0\}$ by

$$P(a,b) = \big(a + T(b), 0\big) \qquad ((a,b) \in A \times_{T_{\mathfrak{A}}} B).$$

Then $D = \tilde{d} \circ P : A \times_{T_{\mathfrak{A}}} B \to X^* \times \{0\}$ is an $\mathfrak{A}$-module derivation on $A \times_{T_{\mathfrak{A}}} B$. From the $\mathfrak{A}$-module amenability of $A \times_{T_{\mathfrak{A}}} B$ it follows that there exists $(x^*,0)$ in $X^* \times \{0\}$ such that

$$D(a,b) = (a,b).(x^*,0) - (x^*,0).(a,b) \qquad ((a,b) \in A \times_{T_{\mathfrak{A}}} B).$$

Hence

$$d(a) = \tilde{d}(a,0) = D(a,0) = (a,0).(x^*,0) - (x^*,0).(a,0) = a.x^* - x^*.a \quad (a \in A).$$

This means that d is inner. Therefore A is $\mathfrak{A}$-module amenable. Similarly, by taking $P : A \times_{T_{\mathfrak{A}}} B \to A \times \{0\}$ as $P(a,b) = (0,b)$, we can show that B is $\mathfrak{A}$-module amenable.

Conversely, suppose that both A and B are $\mathfrak{A}$-module amenable. Since A is a closed $\mathfrak{A}$-invariant ideal in $A \times_{T_{\mathfrak{A}}} B$ with a bounded approximate identity and $(A \times_{T_{\mathfrak{A}}} B)/A \cong B$, then $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module amenable, by Corollary 2.2 of [2].

(ii) Suppose that $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module approximate amenable. Let $d : A \to X^*$ be an $\mathfrak{A}$-module derivation for some commutative Banach A-$\mathfrak{A}$-module X. As above there exists an $\mathfrak{A}$-module derivation $D = \tilde{d} \circ P$:

$A \times_{T_{\mathfrak{A}}} B \to X^* \times \{0\}$ on $A \times_{T_{\mathfrak{A}}} B$. Since $A \times_{T_{\mathfrak{A}}} B$ is $\mathfrak{A}$-module approximate amenable, there exists a bounded net $((x^*_\alpha, 0))$ in $X^* \times \{0\}$ such that

$$D(a,b) = \lim_\alpha \Big((a,b).(x^*_\alpha, 0) - (x^*_\alpha, 0).(a,b)\Big) \qquad ((a,b) \in A \times_{T_{\mathfrak{A}}} B).$$

In particular,

$$\begin{aligned} d(a) = \tilde{d}(a,0) &= D(a,0) \\ &= \lim_\alpha \Big((a,0).(x^*_\alpha,0) - (x^*_\alpha,0).(a,0)\Big) \\ &= \lim_\alpha \Big(a.x^*_\alpha - x^*_\alpha.a\Big) \ \ (a \in A). \end{aligned}$$

So d is approximate inner derivation. Therefore A is $\mathfrak{A}$-module approximate amenable. Also, by taking $P : A \times_{T_{\mathfrak{A}}} B \to A \times \{0\}$ as $P(a,b) = (0,b)$, one can show that B is $\mathfrak{A}$-module approximate amenable. □

The following example shows that the converse of the (ii) from the previous theorem is not valid in general case.

Example 3.2. Let l^1 denote the well-know space of complex sequences and let $K(l^1)$ be the space of all compact operators on l^1. It is well known that the Banach algebra $K(l^1)$ is amenable. Renorm $K(l^1)$ with the family of equivalent norm $\|.\|^k$ such that its bounded left approximate identity will be the constant 1 and its bounded right approximate identity will be $k+1$. Let $A = \oplus_{k=1}^\infty (K(l^1), \|.\|^k)$, then A has a bounded left approximate identity but no bounded right identity (see page 3931 of [9]). Now by choosing $T = 0$ and $\mathfrak{A} = \mathbb{C}$ we have $A \oplus A^{op} = A \times_{T_{\mathfrak{A}}} A^{op}$. A and A^{op}, the opposite algebra, are boundedly approximate amenable (see Theorem 3.1 of [9]) but since $A \times_{T_{\mathfrak{A}}} A^{op}$ has no bounded approximate identity, it is not $\mathfrak{A}$-module approximate amenable (see Theorem 4.1 of [9]).

Let A be a Banach $\mathfrak{A}$-bimodule and let $\phi \in \Delta(\mathfrak{A})$, where $\Delta(\mathfrak{A})$ denote the character space of $\mathfrak{A}$. Consider the linear map $\varphi : A \to \mathfrak{A}$ such that

$$\varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(\alpha.a) = \varphi(a.\alpha) = \phi(\alpha)\varphi(a) \quad (a,b \in A, \alpha \in \mathfrak{A}).$$

The set of all continuous such maps denoted by Ω_A. A bounded linear functional $m : A^* \to \mathbb{C}$ is called a (φ,ϕ)-module mean on A^* if $m(f.a) = \phi \circ \varphi(a) m(f)$, $m(f.\alpha) = \phi(\alpha) m(f)$ and $m(\phi \circ \varphi) = 1$ for each $f \in A^*, a \in A$ and $\alpha \in \mathfrak{A}$. A Banach $\mathfrak{A}$-bimodule A is called (φ,ϕ)-module amenable if there exists a (φ,ϕ)-module mean on A^*. Also A is called module character amenable if it is (φ,ϕ)-module amenable for each $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A}) \cup \{0\}$ (see [6]).

One should remember that if $\mathfrak{A} = \mathbb{C}$ and ϕ is the identity map then the module (φ,ϕ)-amenability coincides with φ-amenability [12].

For every $\varphi \in \Omega_A$ and $\psi \in \Omega_B$, we define $(0,\psi) : A \times_{T_{\mathfrak{A}}} B \to \mathfrak{A}$ and $(\varphi, \varphi \circ T) : A \times_{T_{\mathfrak{A}}} B \to \mathfrak{A}$ by

$$(0,\psi)(a,b) = \psi(b), \quad (\varphi, \varphi \circ T)(a,b) = \varphi(a) + \varphi \circ T(b) \qquad (a \in A, b \in B).$$

Then it is easy to see that $(0,\psi)$ and $(\varphi, \varphi \circ T) \in \Omega_{A \times_{T_{\mathfrak{A}}} B}$.

Theorem 3.3. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathtt{Hom}_{\mathfrak{A}}(B,A)$. Then the following statements are valid:*

(i) *$A \times_{T_{\mathfrak{A}}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module amenable for every $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$ if and only if A is module character amenable.*

(ii) *$A \times_{T_{\mathfrak{A}}} B$ is $\big((0,\psi), \phi\big)$-module amenable for every $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if B is module character amenable.*

Proof. (i) Let $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$. Since $A \times_{T_{\mathfrak{A}}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module amenable, there exists $m \in (A \times_{T_{\mathfrak{A}}} B)^{**}$ such that

$$m\big((f,g).(a,b)\big) = \big(\phi \circ (\varphi, \varphi \circ T)(a,b)\big)m\big((f,g)\big), \quad m\big((f,g).\alpha\big) = \phi(\alpha)m\big((f,g)\big),$$

and $m\big(\phi \circ (\varphi, \varphi \circ T)\big) = 1$, for each $(f,g) \in (A \times_{T_{\mathfrak{A}}} B)^* \simeq A^* \times B^*$ and $(a,b) \in A \times_{T_{\mathfrak{A}}} B$. Define $m_A \in A^{**}$ by $m_A(f) = m(f, f \circ T)$ $(f \in A^*)$. By (3), for every $a \in A$ and $f \in A^*$, we have

$$\begin{aligned} m_A(f.a) &= m\big(f.a, (f.a) \circ T\big) \\ &= m\big(f.a, f \circ L_a T\big) \\ &= m\big((f, f \circ T).(a, 0)\big) \\ &= \big(\phi \circ (\varphi, \varphi \circ T)(a, 0)\big)m\big((f, f \circ T)\big) \\ &= \phi \circ \varphi(a) m_A(f), \end{aligned}$$

and for every $\alpha \in \mathfrak{A}$,

$$m_A(f.\alpha) = m\big(f.\alpha, (f.\alpha) \circ T\big) = m\big((f, f \circ T).\alpha\big) = \phi(\alpha)m\big(f, f \circ T\big) = \phi(\alpha)m_A(f).$$

Also

$$m_A(\phi \circ \varphi) = m\big((\phi \circ \varphi, \phi \circ \varphi \circ T)\big) = m\big(\phi \circ (\varphi, \varphi \circ T)\big) = 1.$$

Then A is (φ, ϕ)-module amenable. Therefore A is module character amenable.

Conversely, suppose that A is module character amenable. Let $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$. Then there exists $m \in A^{**}$ such that

$$m(f.a) = \phi \circ \varphi(a)m(f), \quad m(f.\alpha) = \phi(\alpha)m(f), \quad m(\phi \circ \varphi) = 1 \quad (f \in A^*, a \in A, \alpha \in \mathfrak{A}).$$

Define $\overline{m} \in (A \times_{T_{\mathfrak{A}}} B)^{**}$ by $\overline{m}\big((f,g)\big) = m(f)$. By (3), for every $(f,g) \in (A \times_{T_{\mathfrak{A}}} B)^*$ and $(a,b) \in A \times_{T_{\mathfrak{A}}} B$, we have

$$\begin{aligned} \overline{m}\big((f,g).(a,b)\big) &= \overline{m}\big(f.a + f.T(b), f \circ (L_a T) + g.b\big) \\ &= m\big(f.a + f.T(b)\big) \\ &= m\big(f.a\big) + m\big(f.T(b)\big) \\ &= \phi \circ \varphi(a)m(f) + \phi \circ \varphi\big(T(b)\big)m(f) \\ &= \big(\phi \circ (\varphi, \varphi \circ T)(a,b)\big)\overline{m}(f,g), \end{aligned}$$

and for every $\alpha \in \mathfrak{A}$,

$$\overline{m}\big((f,g).\alpha\big) = \overline{m}\big((f.\alpha, g.\alpha)\big) = m(f.\alpha) = \phi(\alpha)m(f) = \phi(\alpha)\overline{m}\big((f,g)\big).$$

Moreover

$$\overline{m}\big(\phi \circ (\varphi, \varphi \circ T)\big) = \overline{m}\big(\phi \circ \varphi, \phi \circ \varphi \circ T\big) = m(\phi \circ T) = 1.$$

So $\overline{m}$ is a $\big((\varphi, \varphi \circ T), \phi\big)$-module mean on $A \times_{T_{\mathfrak{A}}} B$. Therefore $A \times_T B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module amenable.

(ii) Let $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$. From the $\big((0, \psi), \phi\big)$-module amenability of $A \times_{T_{\mathfrak{A}}} B$ it follows that there exists $m \in (A \times_{T_{\mathfrak{A}}} B)^{**}$ such that

$$m\big((f,g).(a,b)\big) = \big(\phi \circ (0, \psi)(a,b)\big)m\big((f,g)\big), \quad m\big((f,g).\alpha\big) = \phi(\alpha)m\big((f,g)\big),$$

and $m\big(\phi \circ (0, \psi)\big) = 1$, for each $(f,g) \in (A \times_{T_{\mathfrak{A}}} B)^*$ and $(a,b) \in A \times_{T_{\mathfrak{A}}} B$. Define $m_B \in B^{**}$ by $m_B(g) = m(0, g)$. As in (i) we can show that m_B is a (ψ, ϕ)-module mean on B^*. So B is (ψ, ϕ)-module amenable. Therefore B is module character amenable.

Conversely, suppose that B is module character amenable. Let $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$. Thus there exists $m \in B^{**}$ such that

$$m(g.b) = \phi \circ \psi(b)m(g), \quad m(g.\alpha) = \phi(\alpha)m(g), \quad m(\phi \circ \psi) = 1 \quad (g \in B^*, b \in B, \alpha \in \mathfrak{A}).$$

Define $\overline{m} \in (A \times_{T_{\mathfrak{A}}} B)^{**}$ by $\overline{m}\Big((f,g)\Big) = m(g) - m(f \circ T)$. We show that $\overline{m}$ is a $\Big((0,\psi),\phi\Big)$-module mean on $A \times_{T_{\mathfrak{A}}} B$.

By (3), for every $(f,g) \in (A \times_{T_{\mathfrak{A}}} B)^*$ and $(a,b) \in A \times_{T_{\mathfrak{A}}} B$, we have

$$\begin{aligned}
\overline{m}\Big((f,g).(a,b)\Big) &= \overline{m}\Big(f.a + f.T(b), f \circ (L_aT) + g.b\Big) \\
&= m\Big(f \circ (L_aT) + g.b\Big) - m\Big(\big(f.a + f.T(b)\big) \circ T\Big) \\
&= m(g.b) - m\Big(f.T(b) \circ T\Big) \\
&= m(g.b) - m\Big((f \circ T).b\Big) \\
&= \phi \circ \psi(b)m(g) - \phi \circ \psi(b)m(f \circ T) \\
&= \Big(\phi \circ (0,\psi)(a,b)\Big)\overline{m}\Big((f,g)\Big),
\end{aligned}$$

and for every $\alpha \in \mathfrak{A}$,

$$\overline{m}\Big((f,g).\alpha\Big) = \overline{m}\Big((f.\alpha, g.\alpha)\Big) = m(g.\alpha) - m\Big((f \circ T).\alpha\Big) = \phi(\alpha)\overline{m}\Big((f,g)\Big).$$

Also $\overline{m}\Big(\phi \circ (0,\psi)\Big) = \overline{m}\Big(0, \phi \circ \psi\Big) = m(\phi \circ \psi) = 1$. Thus $\overline{m}$ is a $\Big((0,\psi),\phi\Big)$-module mean on $A \times_{T_{\mathfrak{A}}} B$, and so $A \times_{T_{\mathfrak{A}}} B$ is $\Big((0,\psi),\phi\Big)$-module amenable. □

Corollary 3.4. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathrm{Hom}_{\mathfrak{A}}(B,A)$. Then $A \times_{T_{\mathfrak{A}}} B$ is $\Big((\varphi, \varphi \circ T),\phi\Big)$-module amenable and $\Big((0,\psi),\phi\Big)$-module amenable for every $\varphi \in \Omega_A$, $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if both A and B are module character amenable.*

Definition 3.5. *Let A be a Banach $\mathfrak{A}$-bimodule and let $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$. A net $\{m_i\} \subset A^{**}$ is called a (φ,ϕ)-module approximate mean, if $m_i(\phi \circ \varphi) \longrightarrow 1$, $\|a.m_i - \phi \circ \varphi(a)m_i\| \longrightarrow 0$ and $\|\alpha.m_i - \phi(\alpha)m_i\| \longrightarrow 0$ for all $a \in A$ and $\alpha \in \mathfrak{A}$. A Banach $\mathfrak{A}$-bimodule A is called (φ,ϕ)-module approximate amenable if there exists a (φ,ϕ)-module approximate mean on A^*. Also A is called module character approximate amenable if it is (φ,ϕ)-module approximate amenable for each $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A}) \cup \{0\}$.*

Note that by Theorem 5.2 of [7], and from the fact that

$$\|a.m_i - \phi \circ \varphi(a)m_i\| \leqslant \|a.m_i - \varphi(a).m_i\| + \|\varphi(a).m_i - \phi \circ \varphi(a)m_i\|,$$

and

$$\|a.m_i - \varphi(a).m_i\| \leqslant \|a.m_i - \phi \circ \varphi(a)m_i\| + \|\phi \circ \varphi(a)m_i - \varphi(a).m_i\|,$$

one can easily show that the concept of (φ,ϕ)-module approximate amenability in Definition 3.5 and the concept of module approximately (φ,ϕ)-amenability introduced in [7] are equivalent.

Theorem 3.6. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathrm{Hom}_{\mathfrak{A}}(B,A)$. Then the following statements are valid:*

(i) *$A \times_{T_{\mathfrak{A}}} B$ is $\Big((\varphi, \varphi \circ T),\phi\Big)$-module approximate amenable for every $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$ if and only if A is module character approximate amenable.*

(ii) *$A \times_{T_{\mathfrak{A}}} B$ is $\Big((0,\psi),\phi\Big)$-module approximate amenable for every $\psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if B is module character approximate amenable.*

Proof. (i) Let $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$. Since $A \times_{T_{\mathfrak{A}}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module approximate amenable, there exists a net $\{m_i\} \subset (A \times_T B)^{**}$ such that

$$\|(a,b).m_i - \Big(\phi \circ (\varphi, \varphi \circ T)(a,b)\Big)m_i\| \longrightarrow 0, \quad \|\alpha.m_i - \phi(\alpha)m_i\| \longrightarrow 0,$$

and $m_i\Big(\phi \circ (\varphi, \varphi \circ T)\Big) \longrightarrow 1$, for each $(a,b) \in A \times_T B$ and $\alpha \in \mathfrak{A}$. Let $\{m_i^A\} \subset A^{**}$ define by $m_i^A(f) = m_i(f, f \circ T)$ $(f \in A^*)$. By (3), for every $a \in A$, we have

$$\begin{aligned}
\|a.m_i^A - \phi \circ \varphi(a)m_i^A\| &= \sup_{f \in A^*, \|f\| \leq 1} |a.m_i^A(f) - \phi \circ \varphi(a)m_i^A(f)| \\
&= 2 \sup_{f \in A^*, \|f\| \leq 1} |m_i^A(\frac{1}{2}f.a) - \phi \circ \varphi(a)m_i^A(\frac{1}{2}f.a)| \\
&= 2 \sup_{f \in A^*, \|f\| \leq 1} |(a,0).m_i(\frac{1}{2}f, \frac{1}{2}f \circ T) - \Big(\phi \circ (\varphi, \varphi \circ T)(a,0)\Big)m_i(\frac{1}{2}f, \frac{1}{2}f \circ T)| \\
&\leq 2\|(a,0).m_i - \Big(\phi \circ (\varphi, \varphi \circ T)(a,0)\Big)m_i\| \longrightarrow 0,
\end{aligned}$$

and similarly one can prove that $\|\alpha.m_i^A - \phi(\alpha)m_i^A\| \longrightarrow 0$, for every $\alpha \in \mathfrak{A}$. Also

$$m_i^A(\phi \circ \varphi) = m_i\Big((\phi \circ \varphi, \phi \circ \varphi \circ T)\Big) = m_i\Big(\phi \circ (\varphi, \varphi \circ T)\Big) \longrightarrow 1.$$

Then $\{m_i^A\}$ is a (φ, ϕ)-module approximate mean on A^*. So A is (φ, ϕ)-module approximate amenable. Therefore A is module character approximate amenable.

Conversely, suppose that A is module character approximate amenable. Let $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A})$. Then there exists a net $\{m_i\} \subset A^{**}$ such that

$$\|a.m_i - \phi \circ \varphi(a)m_i\| \longrightarrow 0, \quad \|\alpha.m_i - \phi(\alpha)m_i\| \longrightarrow 0, m_i(\phi \circ \varphi) \longrightarrow 1 (a \in A, \alpha \in \mathfrak{A}).$$

Define the net $\{\overline{m}_i\} \subset (A \times_{T_{\mathfrak{A}}} B)^{**}$ by $\overline{m}_i\Big((f,g)\Big) = m_i(f)$. By (3), for every $(a,b) \in A \times_{T_{\mathfrak{A}}} B$, we have

$$\begin{aligned}
&\|(a,b).\overline{m}_i - \Big(\phi \circ (\varphi, \varphi \circ T)(a,b)\Big)\overline{m}_i\| \\
&= \sup_{(f,g) \in (A \times_T B)^*, \|(f,g)\| \leq 1} |(a,b).\overline{m}_i\Big((f,g)\Big) - \Big(\phi \circ (\varphi, \varphi \circ T)(a,b)\Big)\overline{m}_i\Big((f,g)\Big)| \\
&= \sup_{(f,g) \in (A \times_T B)^*, \|(f,g)\| \leq 1} |m_i\Big((f.a + f.T(b)\Big) - \Big(\phi \circ (\varphi, \varphi \circ T)(a,b)\Big)m_i(f)| \\
&\leq \sup_{(f,g) \in (A \times_T B)^*, \|(f,g)\| \leq 1} \Big(|a.m_i(f) - \phi \circ \varphi(a)m_i(f)| + |T(b).m_i(f) - \phi \circ \varphi\Big(T(b)\Big)m_i(f)|\Big) \\
&\leq \|a.m_i - \phi \circ \varphi(a)m_i\| + \|T(b).m_i - \phi \circ \varphi\Big(T(b)\Big)m_i\|| \longrightarrow 0.
\end{aligned}$$

It is easy to check that $\|\alpha.\overline{m}_i - \phi(\alpha)\overline{m}_i\| \longrightarrow 0$ $(\alpha \in \mathfrak{A})$. Moreover

$$\overline{m}_i\Big(\phi \circ (\varphi, \varphi \circ T)\Big) = \overline{m}_i\Big(\phi \circ \varphi, \phi \circ \varphi \circ T\Big) = m_i(\phi \circ T) \longrightarrow 1.$$

So $\{\overline{m}_i\}$ is a$\Big((\varphi, \varphi \circ T), \phi\Big)$-module approximate mean on $(A \times_{T_{\mathfrak{A}}} B)^*$. Therefore $A \times_{T_{\mathfrak{A}}} B$ is $\Big((\varphi, \varphi \circ T), \phi\Big)$-module approximate amenable.

Similarly, we can show that (ii) is also valid. □

Corollary 3.7. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathrm{Hom}_{\mathfrak{A}}(B, A)$. Then $A \times_{T_{\mathfrak{A}}} B$ is $\Big((\varphi, \varphi \circ T), \phi\Big)$-module approximate amenable and $\Big((0, \psi), \phi\Big)$-module approximate amenable for every $\varphi \in \Omega_A, \psi \in \Omega_B$ and $\phi \in \Delta(\mathfrak{A})$ if and only if both A and B are module character approximate amenable.*

4. Module character biprojectivity and module character biflatness of $A \times_{T_{\mathfrak{A}}} B$

We commence this section with the following definition from [3]:

Definition 4.1. *We say a Banach algebra $\mathfrak{A}$ acts trivially on A from the left (right) if for every $\alpha \in \mathfrak{A}$ and $a \in A$, $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$), where f is a continuous linear functional on $\mathfrak{A}$.*

We recall the following remark from [5] for the proof of the next results:

Remark 4.2. Let $I_{A\widehat{\otimes}A}$ and J_A be the closed ideals defined in (1) and (2), respectively. Suppose that A has a bounded approximate identity and $\mathfrak{A}$ acts on A trivially from the left. Then $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ is an A/J_A-bimodule with the actions given by

$$(a + J_A).(c \otimes b + I_{A\widehat{\otimes}A}) = ac \otimes b + I_{A\widehat{\otimes}A}, \tag{5}$$

and

$$(c \otimes b + I_{A\widehat{\otimes}A}).(a + J_A) = c \otimes ba + I_{A\widehat{\otimes}A}, \tag{6}$$

for $a, b, c \in A$ and $\alpha \in \mathfrak{A}$.

Throughout the next results, when we consider $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ as A/J_A-bimodule, we have suppose that A has a bounded approximate identity, and $\mathfrak{A}$ acts on A trivially from the left.

Recall that a Banach algebra A is called $\mathfrak{A}$-module biprojective if $\tilde{\omega}_A$ has a bounded right inverse which is an A/J_A-$\mathfrak{A}$-module map, and A is called $\mathfrak{A}$-module biflat if $\tilde{\omega}_A^*$ has a bounded left inverse which is an A/J_A-$\mathfrak{A}$-module map (see [5]).

Let $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$. Clearly $\varphi\Big((a.\alpha)b - a(\alpha.b)\Big) = 0$ $(\alpha \in \mathfrak{A}, a, b \in A)$, hence $\varphi = 0$ on J_A and $\tilde{\varphi} : A/J_A \longrightarrow \mathfrak{A}$ given by $\tilde{\varphi}(a + J_A) = \varphi(a)$ is well defined. Then $\tilde{\varphi} \in \Omega_{A/J_A}$.

Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\widehat{\phi} \in \Delta(A^{**})$ which is denote by $\widehat{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Definition 4.3. *Let $\varphi \in \Omega_A$. A Banach $\mathfrak{A}$-bimodule A is called φ-module biprojective (resp. φ-module approximate biprojective) if there exists A/J_A-$\mathfrak{A}$-module map $\tilde{\rho_A} : A/J_A \to A\widehat{\otimes}A/I_{A\widehat{\otimes}A}$ (resp. there exists a net of A/J_A-$\mathfrak{A}$-module maps $\tilde{\rho}_i : A/J_A \to A\widehat{\otimes}A/I_{A\widehat{\otimes}A}$) such that $\tilde{\varphi} \circ \tilde{\omega}_A \circ \tilde{\rho_A}(a + J_A) = \tilde{\varphi}(a + J_A)$(resp. $\tilde{\varphi} \circ \tilde{\omega}_A \circ \tilde{\rho}_i(a + J_A) \longrightarrow \tilde{\varphi}(a + J_A)$).*

Definition 4.4. *Let $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$. A Banach $\mathfrak{A}$-bimodule A is called (φ, ϕ)-module biflat (resp. (φ, ϕ)-module approximate biflat) if there exists A/J_A-$\mathfrak{A}$-module map $\tilde{\rho_A} : A/J_A \to \Big(A\widehat{\otimes}A/I_{A\widehat{\otimes}A}\Big)^{**}$ (resp. there exists a net of A/J_A-$\mathfrak{A}$-module maps $\tilde{\rho}_i : A/J_A \to \Big(A\widehat{\otimes}A/I_{A\widehat{\otimes}A}\Big)^{**}$) such that $\widehat{\phi \circ \tilde{\varphi}} \circ \tilde{\omega}_A^{**} \circ \tilde{\rho_A}(a + J_A) = \phi \circ \tilde{\varphi}(a + J_A)$ (resp. $\widehat{\phi \circ \tilde{\varphi}} \circ \tilde{\omega}_A^{**} \circ \tilde{\rho}_i(a + J_A) \longrightarrow \phi \circ \tilde{\varphi}(a + J_A)$). A is called module character biflat (resp. module character approximate biflat) if it is (φ, ϕ)-module biflat (resp. (φ, ϕ)-module approximate biflat) for each $\varphi \in \Omega_A$ and $\phi \in \Delta(\mathfrak{A}) \cup \{0\}$.*

Lemma 4.5. *Let A and B be two Banach $\mathfrak{A}$-bimodules and $T \in \mathrm{Hom}_{\mathfrak{A}}(B, A)$. Then $J_{A\times_{T_{\mathfrak{A}}}B} = J_A \times_{T_{\mathfrak{A}}} J_B$.*

Proof. For every $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}\Big((a_1, b_1).\alpha\Big)(a_2, b_2) - (a_1, b_1)\Big(\alpha.(a_2, b_2)\Big) &= \Big(\Big((a_1.\alpha)a_2 - a_1(\alpha.a_2)\Big) + \Big((a_1.\alpha)\varphi(b_2) - a_1(\alpha.\varphi(b_2))\Big)\\ &\quad + \Big((\varphi(b_1).\alpha)a_2 - \varphi(b_1)(\alpha.a_2)\Big), (b_1.\alpha)b_2 - b_1(\alpha.b_2)\Big).\end{aligned}$$

That is $J_{A\times_\varphi B} \subset J_A \times_\varphi J_B$. Also for every $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $\alpha, \alpha' \in \mathfrak{A}$, we have

$$\begin{aligned}\Big((a_1.\alpha)a_2 - a_1(\alpha.a_2), (b_1.\alpha')b_2 - b_1(\alpha'.b_2)\Big) &= \Big((a_1.\alpha)a_2 - a_1(\alpha.a_2), 0\Big) + \Big(0, (b_1.\alpha')b_2 - b_1(\alpha'.b_2)\Big)\\ &= \Big[\Big((a_1, 0).\alpha\Big)(a_2, 0) - (a_1, 0)\Big(\alpha.(a_2, 0)\Big)\Big]\\ &\quad + \Big[\Big((0, b_1).\alpha'\Big)(0, b_2) - (0, b_1)\Big(\alpha'.(0, b_2)\Big)\Big].\end{aligned}$$

So $J_A \times_\varphi J_B \subset J_{A\times_\varphi B}$. Therefore $J_{A\times_\varphi B} = J_A \times_\varphi J_B$. □

We note that $(A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}} B}$ is an A/J_A-bimodule through the actions given by

$$(a' + J_A).\Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big) = \Big((a',0) + J_{A\times_{T_{\mathfrak{A}}} B}\Big)\Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big), \tag{7}$$

and

$$\Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big).(a' + J_A) = \Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big)\Big((a',0) + J_{A\times_{T_{\mathfrak{A}}} B}\Big), \tag{8}$$

for all $a', a \in A$ and $b \in B$. By Lemma 4.5, one can easily check that these actions are well defined. Also $(A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}} B}$ can be made into a Banach B/J_B-bimodule in a similar fashion.

Lemma 4.6. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras with bounded approximate identity. Define the mappings $r_A : (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}} B} \to A/J_A$ and $q_A : A/J_A \to (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_T B}$ by*

$$r_A\Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big) = \Big(a + T(b)\Big) + J_A, \quad q_A(a + J_A) = (a,0) + J_{A\times_{T_{\mathfrak{A}}} B} \qquad (a \in A, b \in B).$$

Also define $p_B : (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}} B} \to B/J_B$ and $s_B : B/J_B \to (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}} B}$ by

$$p_B\Big((a,b) + J_{A\times_{T_{\mathfrak{A}}} B}\Big) = b + J_B, \quad s_B(b + J_B) = \Big(-T(b), b\Big) + J_{A\times_{T_{\mathfrak{A}}} B} \qquad (a \in A, b \in B).$$

Then r_A, q_A are Banach A/J_A-$\mathfrak{A}$-module maps and p_B, s_B are B/J_B-$\mathfrak{A}$-module maps.

Proof. By Lemma 4.5, it is easy to see that r_A, q_A, p_B and s_B are well defined. Let (e_i) (resp. (e'_j)) be a bounded approximate identity for A (resp. B) with bound $m > 0$ (resp. $m' > 0$). For every $a \in A$ and $b \in B$, we have

$$\begin{aligned}
\|q_A(a + J_A)\| &= \|(a,0) + J_{A\times_\varphi B}\| = \lim_i \|(a + J_A).\Big((e_i,0) + J_{A\times_\varphi B}\Big)\| \\
&\leq \lim_i k\|a + J_A\|\|(e_i,0) + J_{A\times_\varphi B}\| \\
&\leq \lim_i k\|a + J_A\|\|e_i\| \leq km\|a + J_A\|,
\end{aligned}$$

and

$$\begin{aligned}
\|s_B(b + J_B)\| &= \|(-\varphi(b), b) + J_{A\times_\varphi B}\| \\
&= \lim_i \lim_j \|(b + J_A).\Big((-e_i, e'_j) + J_{A\times_\varphi B}\Big)\| \\
&\leq \lim_i \lim_j k\|b + J_B\|\|(-e_i, e'_j) + J_{A\times_\varphi B}\| \\
&\leq \lim_i \lim_j k\|b + J_B\|\|e_i\|\|e'_j\| \leq kmm'\|b + J_B\|.
\end{aligned}$$

Thus q_A and s_B are bounded. By using Lemma 4.5, we obtain

$$\|a + J_A\| + \|b + J_B\| \leq \|(a,b) + J_{A\times_\varphi B}\| \qquad (a \in A, b \in B).$$

It follows that $\|p_B\Big((a,b) + J_{A\times_\varphi B}\Big)\| = \|b + J_B\| \leq \|(a,b) + J_{A\times_\varphi B}\|$, and since $\varphi(J_B) \subset J_A$,

$$\begin{aligned}
\|r_A\Big((a,b) + J_{A\times_\varphi B}\Big)\| &= \|\Big(a + \varphi(b)\Big) + J_A\| \\
&\leq \|a + J_A\| + \|\varphi(b) + \varphi(J_B)\| \\
&\leq \|(a,b) + J_{A\times_\varphi B}\|.
\end{aligned}$$

Therefore p_B and r_A is bounded. Also one can easily check that r_A, q_A are Banach A/J_A-bimodule and $\mathfrak{A}$-bimodule and p_B, s_B are B/J_B-bimodule and $\mathfrak{A}$-bimodule. Then r_A, q_A are Banach A/J_A-$\mathfrak{A}$-module maps and p_B, s_B are B/J_B-$\mathfrak{A}$-module maps. □

For the proof of the following result we refer to Proposition 2.6 of [22].

Proposition 4.7. *Let A be a Banach algebra with a bounded approximate identity and $\mathfrak{A}$ acts on A trivially from the left. Let $\Phi_A : (A\widehat{\otimes}A)/I_{A\widehat{\otimes}A} \to A/J_A\widehat{\otimes}A/J_A$ be defined by*

$$\Phi_A\Big((a_1 \otimes a_2) + I_{A\widehat{\otimes}A}\Big) = (a_1 + J_A) \otimes (a_2 + J_A) \quad (a_1, a_2 \in A).$$

Then Φ_A is a bijective A/J_A-$\mathfrak{A}$-module map.

Lemma 4.8. *Let A be a Banach algebra with a bounded approximate identity and $\mathfrak{A}$ acts on A trivially from the left. Let Φ_A be as in Proposition 4.7. Then the inverse of Φ_A that we denote by Φ_A^{-1} is a A/J_A-$\mathfrak{A}$-module map.*

Proof. Let (e_i) be a bounded approximate identity for A with bound $m > 0$. By (5) and (6), for every $a_1, a_2 \in A$, we have

$$\begin{aligned}
\|\Phi^{-1}\Big((a_1 + J_A) \otimes (a_2 + J_A)\Big)\| &= \|a_1 \otimes a_2 + I_{A\widehat{\otimes}A}\| = \lim_i \|a_1 e_i \otimes e_i a_2 + I_{A\widehat{\otimes}A}\| \\
&= \lim_i \|(a_1 + J_A).\Big((e_i \otimes e_i + I_{A\widehat{\otimes}A}).(a_2 + J_A)\Big)\| \\
&\leq k \lim_i \|e_i \otimes e_i + I_{A\widehat{\otimes}A}\|\|a_1 + J_A\|\|a_2 + J_A\| \\
&\leq k \lim_i \|e_i \otimes e_i\|\|(a_1 + J_A) \otimes (a_2 + J_A)\| \\
&\leq km^2 \|(a_1 + J_A) \otimes (a_2 + J_A)\|.
\end{aligned}$$

Therefore Φ_A^{-1} is a bounded and it is also easy to see that Φ_A^{-1} is an A/J_A-bimodule and $\mathfrak{A}$-bimodule map. So Φ_A^{-1} is a A/J_A-$\mathfrak{A}$-module map. □

Theorem 4.9. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $\varphi \in \Omega_A, \psi \in \Omega_B$ and $T \in \mathtt{Hom}_{\mathfrak{A}}(B, A)$. Then the following statements are valid:*

(i) *$A \times_{T_{\mathfrak{A}}} B$ is $(\varphi, \varphi \circ T)$-module biprojective (resp. $(\varphi, \varphi \circ T)$-module approximate biprojective) if and only if A is φ-module biprojective (resp. φ-module approximate biprojective).*

(ii) *$A \times_{T_{\mathfrak{A}}} B$ is $(0, \psi)$-module biprojective (resp. $(0, \psi)$-module approximate biprojective) if and only if B is ψ-module biprojective (resp. ψ-module approximate biprojective).*

Proof. Let r_A, q_A and p_B, s_B be as in Proposition 4.6, and Φ_A, Φ_B and $\Phi_{A\times_{T_{\mathfrak{A}}}B}$ be as in Proposition 4.7.

(i) Suppose that $A \times_{T_{\mathfrak{A}}} B$ is $(\varphi, \varphi \circ T)$-module biprojective. So there exists a $(A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B}$-$\mathfrak{A}$-module map

$$\eta : (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B} \to \Big((A \times_{T_{\mathfrak{A}}} B)\widehat{\otimes}(A \times_{T_{\mathfrak{A}}} B)\Big)/I_{(A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)},$$

such that $(\widetilde{\varphi, \varphi \circ T}) \circ \tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B} \circ \eta\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) = (\widetilde{\varphi, \varphi \circ T})\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big)$. A direct verification shows that the equalities

$$\tilde{\omega}_A \circ \Phi_A^{-1} \circ (r_A \otimes r_A) \circ \Phi_{A\times_{T_{\mathfrak{A}}}B} = r_A \circ \tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}, \quad \tilde{\varphi} \circ r_A = (\widetilde{\varphi, \varphi \circ T})$$

are valid. Define $\tilde{\rho}_A : A/J_A \to (A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ by

$$\tilde{\rho}_A = \Phi_A^{-1} \circ (r_A \otimes r_A) \circ \Phi_{A\times_{T_{\mathfrak{A}}}B} \circ \eta \circ q_A.$$

We claim that $\tilde{\rho}_A$ is an A/J_A-$\mathfrak{A}$-module map. By (7), for every $a', a \in A$, we have

$$\begin{aligned}
\tilde{\rho}_A\Big((a' + J_A).(a + J_A)\Big) &= \Big(\Phi_A^{-1} \circ (r_A \otimes r_A) \circ \Phi_{A\times_{T_{\mathfrak{A}}}B} \circ \eta \circ q_A\Big)\Big((a' + J_A).(a + J_A)\Big) \\
&= \Big(\Phi_A^{-1} \circ (r_A \otimes r_A) \circ \Phi_{A\times_{T_{\mathfrak{A}}}B} \circ \eta\Big)\Big((a' + J_A).q_A(a + J_A)\Big) \\
&= \Big(\Phi_A^{-1} \circ (r_A \otimes r_A) \circ \Phi_{A\times_{T_{\mathfrak{A}}}B} \circ \eta\Big)\Big((a', 0) + J_{A\times_{T_{\mathfrak{A}}}B}.(a, 0) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) \\
&= \Big(\Phi_A^{-1} \circ (r_A \otimes r_A)\Big)\Big[(a', 0) + J_{A\times_{T_{\mathfrak{A}}}B}.\Phi_{A\times_{T_{\mathfrak{A}}}B}\Big(\eta\Big((a, 0) + J_{A\times_{T_{\mathfrak{A}}}B}\Big)\Big)\Big].
\end{aligned}$$

Put $\Phi_{A\times_{T_{\mathfrak{A}}}B}\Big(\eta\big((a,0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\Big)=\big((a_1,b_1)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\otimes\big((a_2,b_2)+J_{A\times_{T_{\mathfrak{A}}}B}\big)$ $(a_1,a_2\in A, b_1,b_2\in B)$. Since

$$\begin{aligned}
&\big(\Phi_A^{-1}\circ(r_A\otimes r_A)\big)\Big[(a',0)+J_{A\times_{T_{\mathfrak{A}}}B}.\Phi_{A\times_{T_{\mathfrak{A}}}B}\Big(\eta\big((a,0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\Big)\Big]\\
&=\big(\Phi_A^{-1}\circ(r_A\otimes r_A)\big)\Big[(a',0)+J_{A\times_{T_{\mathfrak{A}}}B}.\big((a_1,b_1)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\otimes\big((a_2,b_2)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\Big]\\
&=\big(\Phi_A^{-1}\circ(r_A\otimes r_A)\big)\Big[\big((a'a_1+a'T(b_1),0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\otimes\big((a_2,b_2)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\Big]\\
&=\Phi_A^{-1}\Big(\big(a'a_1+a'T(b_1)+J_A\big)\otimes\big(a_2+T(b_2)+J_A\big)\Big)\\
&=(a'+J_A).\Phi_A^{-1}\Big(\big(a_1+T(b_1)+J_A\big)\otimes\big(a_2+T(b_2)+J_A\big)\Big)\\
&=(a'+J_A).\big(\Phi_A^{-1}\circ(r_A\otimes r_A)\circ\Phi_{A\times_{T_{\mathfrak{A}}}B}\circ\eta\circ q_A\big)\big((a+J_A)\big),
\end{aligned}$$

it follows that $\tilde{\rho}_A\big((a'+J_A).(a+J_A)\big)=(a'+J_A).\tilde{\rho}_A(a+J_A)$. By a similar argument one can show that $\tilde{\rho}_A\big((a+J_A).(a'+J_A)\big)=\tilde{\rho}_A(a+J_A).(a'+J_A)$. So $\tilde{\rho}_A$ is an A/J_A-bimodule. Also it is easy to see that $\tilde{\rho}_A$ is $\mathfrak{A}$-bimodule. Therefore $\tilde{\rho}_A$ is an A/J_A-$\mathfrak{A}$-module map. Moreover, for every $a\in A$, we have

$$\begin{aligned}
\tilde{\varphi}\circ\tilde{\omega}_A\circ\tilde{\rho}_A(a+J_A) &= \tilde{\varphi}\circ\tilde{\omega}_A\circ\big(\Phi_A^{-1}\circ(r_A\otimes r_A)\circ\Phi_{A\times_{T_{\mathfrak{A}}}B}\circ\eta\circ q_A\big)(a+J_A)\\
&= \tilde{\varphi}\circ r_A\circ\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\eta\big((a,0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\\
&= (\widetilde{\varphi,\varphi\circ T})\circ\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\eta\big((a,0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\\
&= (\widetilde{\varphi,\varphi\circ T})\big((a,0)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\\
&= \tilde{\varphi}(a+J_A).
\end{aligned}$$

Therefore A is φ-module biprojective.

Conversely, suppose that A is φ-module biprojective. Then there exists A/J_A-$\mathfrak{A}$-module map $\tilde{\rho}_A : A/J_A\to(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ such that $\tilde{\varphi}\circ\tilde{\omega}_A\circ\tilde{\rho}_A(a+J_A)=\tilde{\varphi}(a+J_A)$. It is easy to see that

$$\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\Phi^{-1}_{A\times_{T_{\mathfrak{A}}}B}\circ(q_A\otimes q_A)\circ\Phi_A=q_A\circ\tilde{\omega}_A,\quad (\widetilde{\varphi,\varphi\circ T})\circ q_A=\tilde{\varphi}.$$

Define $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} : (A\times_{T_{\mathfrak{A}}}B)/J_{A\times_{T_{\mathfrak{A}}}B}\to(A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)/I_{(A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)}$ by $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}=\Phi^{-1}_{A\times_{T_{\mathfrak{A}}}B}\circ(q_A\otimes q_A)\circ\Phi_A\circ\tilde{\rho}_A\circ r_A$. By similar argument as above show that $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}$ is $(A\times_{T_{\mathfrak{A}}}B)/J_{A\times_{T_{\mathfrak{A}}}B}$-$\mathfrak{A}$-module map. For every $(a,b)\in A\times_{T_{\mathfrak{A}}}B$, we have

$$\begin{aligned}
&(\widetilde{\varphi,\varphi\circ T})\circ\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\big((a,b)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\\
&=(\widetilde{\varphi,\varphi\circ T})\circ\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\Phi^{-1}_{A\times_{T_{\mathfrak{A}}}B}\circ(q_A\otimes q_A)\circ\Phi_A\circ\tilde{\rho}_A\circ r_A\big((a,b)+J_{A\times_{T_{\mathfrak{A}}}B}\big)\\
&=(\widetilde{\varphi,\varphi\circ T})\circ q_A\circ\tilde{\omega}_A\circ\tilde{\rho}_A\big((a+T(b))+J_A\big)\\
&=\tilde{\varphi}\circ\tilde{\omega}_A\circ\tilde{\rho}_A\big((a+T(b))+J_A\big)\\
&=\tilde{\varphi}\big((a+T(b))+J_A\big)\\
&=(\widetilde{\varphi,\varphi\circ T})\big((a,b)+J_{A\times_{T_{\mathfrak{A}}}B}\big).
\end{aligned}$$

So $A\times_{T_{\mathfrak{A}}}B$ is $(\varphi,\varphi\circ T)$-module biprojective.

(ii) Suppose that $A\times_{T_{\mathfrak{A}}}B$ is $(0,\psi)$-module biprojective. Then there exists $(A\times_{T_{\mathfrak{A}}}B)/J_{A\times_{T_{\mathfrak{A}}}B}$-$\mathfrak{A}$-module map

$$\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} : (A\times_{T_{\mathfrak{A}}}B)/J_{A\times_{T_{\mathfrak{A}}}B}\to\big((A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)\big)/I_{(A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)},$$

such that $(\widetilde{0,\psi})\circ\tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B}\circ\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\big((a,b)+J_{A\times_{T_{\mathfrak{A}}}B}\big)=(\widetilde{0,\psi})\big((a,b)+J_{A\times_{T_{\mathfrak{A}}}B}\big)$. Define $\tilde{\rho}_B : B/J_B\to(B\widehat{\otimes}B)/I_{B\widehat{\otimes}B}$ by $\tilde{\rho}_B=\Phi_B^{-1}\circ(p_B\otimes p_B)\circ\Phi_{A\times_{T_{\mathfrak{A}}}B}\circ\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\circ s_B$. By the same argument as in (i), one can show that $\tilde{\rho}_B$ is B/J_B-$\mathfrak{A}$-module map and for every $b\in B$, $\tilde{\psi}\circ\tilde{\omega}_B\circ\tilde{\rho}_B(b+J_B)=\tilde{\psi}(b+J_B)$. So B is ψ-module biprojective.

Let B be a ψ-module biprojective, then there exists $\tilde{\rho}_B : B/J_B \to (B\widehat{\otimes}B)/I_{B\widehat{\otimes}B}$ such that $\tilde{\psi} \circ \tilde{\omega}_B \circ \tilde{\rho}_B(b + J_B) = \tilde{\psi}(b + J_B)$. Now if we define $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} : (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B} \to (A \times_{T_{\mathfrak{A}}} B)\widehat{\otimes}(A \times_{T_{\mathfrak{A}}} B)/I_{(A\times_T B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)}$ by $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} = \Phi^{-1}_{A\times_{T_{\mathfrak{A}}}B} \circ (s_B \otimes s_B) \circ \Phi_B \circ \tilde{\rho}_B \circ p_B$, then by a similar argument as in (i) we can prove that $\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}$ is $(A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B}$-$\mathfrak{A}$-module map and for every $(a, b) \in A \times_{T_{\mathfrak{A}}} B$ we have

$$\widetilde{(0,\psi)} \circ \tilde{\omega}_{A\times_{T_{\mathfrak{A}}}B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) = \widetilde{(0,\psi)}\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big).$$

Therefore $A \times_{T_{\mathfrak{A}}} B$ is $(0, \psi)$-module biprojective.

The proof of module approximate biprojectivity is similar. □

Theorem 4.10. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T \in \mathtt{Hom}_{\mathfrak{A}}(B, A)$. Then the following statements are valid:*

(i) *$A \times_{T_{\mathfrak{A}}} B$ is $\big((\varphi, \varphi \circ T), \phi\big)$-module biflat (resp. $\big((\varphi, \varphi \circ T), \phi\big)$-module approximate biflat) for every $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$ if and only if A is module character biflat (resp. module character approximate biflat).*

(ii) *$A \times_{T_{\mathfrak{A}}} B$ is $\big((0, \psi), \phi\big)$-module biflat (resp. $\big((0, \psi), \phi\big)$-module approximate biflat) for every $\phi \in \Delta(\mathfrak{A})$ and $\psi \in \Omega_B$ if and only if B is module character biflat (resp. module character approximate biflat).*

Proof. Let r_A, q_A and p_B, s_B be as in Proposition 4.6, and Φ_A, Φ_B and $\Phi_{A\times_{T_{\mathfrak{A}}}B}$ be as in Proposition 4.7.

(i) Let $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$. From the $\big((\varphi, \varphi \circ T), \phi\big)$-module biflatness of $A \times_{T_{\mathfrak{A}}} B$ it follows that there exists a $(A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B}$-$\mathfrak{A}$-module map

$$\tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} : (A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B} \to \Big(\big((A \times_{T_{\mathfrak{A}}} B)\widehat{\otimes}(A \times_{T_{\mathfrak{A}}} B)\big)/I_{(A\times_{T_{\mathfrak{A}}}B)\widehat{\otimes}(A\times_{T_{\mathfrak{A}}}B)}\Big)^{**},$$

such that $\phi \circ \widehat{\widetilde{(\varphi, \varphi \circ T)}} \circ \tilde{\omega}^{**}_{A\times_{T_{\mathfrak{A}}}B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) = \phi \circ \widetilde{(\varphi, \varphi \circ T)}\Big((a, b) + J_{A\times_{T_{\mathfrak{A}}}B}\Big)$. Since $\tilde{\varphi} \circ r_A = \widetilde{(\varphi, \varphi \circ T)}$, for every $F \in \Big((A \times_{T_{\mathfrak{A}}} B)/J_{A\times_{T_{\mathfrak{A}}}B}\Big)^{**}$, it follows that

$$\begin{aligned}
\widehat{\phi \circ \tilde{\varphi}} \circ r^{**}_A(F) &= \widehat{\phi \circ \tilde{\varphi}}\Big(r^{**}_A(F)\Big) = r^{**}_A(F)(\phi \circ \tilde{\varphi}) \\
&= F\Big(\phi \circ \tilde{\varphi} \circ r\Big) = F\Big(\phi \circ \widetilde{(\varphi, \varphi \circ T)}\Big) \\
&= \phi \circ \widehat{\widetilde{(\varphi, \varphi \circ T)}}(F).
\end{aligned}$$

Thus $\widehat{\phi \circ \tilde{\varphi}} \circ r^{**}_A = \phi \circ \widehat{\widetilde{(\varphi, \varphi \circ T)}}$. A straightforward computation shows that

$$\tilde{\omega}^{**}_A \circ (\Phi^{-1}_A)^{**} \circ (r_A \otimes r_A)^{**} \circ \Phi^{**}_{A\times_{T_{\mathfrak{A}}}B} = r^{**}_A \circ \tilde{\omega}^{**}_{A\times_{T_{\mathfrak{A}}}B}.$$

Define $\tilde{\rho}_A : A/J_A \to \Big((A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}\Big)^{**}$ by

$$\tilde{\rho}_A = (\Phi^{-1}_A)^{**} \circ (r_A \otimes r_A)^{**} \circ \Phi^{**}_{A\times_\varphi B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} \circ q_A.$$

By the same argument as in the proof of the Theorem 4.9, one can show that $\tilde{\rho}_A$ is an A/J_A-$\mathfrak{A}$-module map. For every $a \in A$, we have

$$\begin{aligned}
\widehat{\phi \circ \tilde{\varphi}} \circ \tilde{\omega}^{**}_A \circ \tilde{\rho}_A(a + J_A) &= \widehat{\tilde{\varphi}} \circ \tilde{\omega}^{**}_A \circ (\Phi^{-1}_A)^{**} \circ (r_A \otimes r_A)^{**} \circ \Phi^{**}_{A\times_{T_{\mathfrak{A}}}B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B} \circ q_A(a + J_A) \\
&= \widehat{\phi \circ \tilde{\varphi}} \circ r^{**}_A \circ \tilde{\omega}^{**}_{A\times_{T_{\mathfrak{A}}}B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\Big((a, 0) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) \\
&= \phi \circ \widehat{\widetilde{(\varphi, \varphi \circ T)}} \circ \tilde{\omega}^{**}_{A\times_{T_{\mathfrak{A}}}B} \circ \tilde{\rho}_{A\times_{T_{\mathfrak{A}}}B}\Big((a, 0) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) \\
&= \phi \circ \widetilde{(\varphi, \varphi \circ T)}\Big((a, 0) + J_{A\times_{T_{\mathfrak{A}}}B}\Big) \\
&= \phi \circ \tilde{\varphi}(a + J_A).
\end{aligned}$$

So A is (φ, ϕ)-module biflat. Therefore A is module character biflat.

Conversely, suppose that A is module character biflat. Let $\phi \in \Delta(\mathfrak{A})$ and $\varphi \in \Omega_A$. Then there exists A/J_A-$\mathfrak{A}$-module map $\tilde{\rho}_A : A/J_A \to \Big((A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}\Big)^{**}$ such that $\widehat{\phi\circ\tilde{\varphi}} \circ \tilde{\omega}_A^{**} \circ \tilde{\rho}_A(a+J_A) = \phi\circ\tilde{\varphi}(a+J_A)$. A direct verification shows that

$$\tilde{\omega}^{**}_{A\times_{T_\mathfrak{A}}B} \circ (\Phi^{-1}_{A\times_{T_\mathfrak{A}}B})^{**} \circ (q_A\otimes q_A)^{**} \circ \Phi_A^{**} = q_A^{**}\circ\tilde{\omega}_A^{**}, \quad \phi\circ(\widetilde{\varphi,\varphi\circ T})\circ q_A^{**} = \widehat{\phi\circ\tilde{\varphi}}.$$

Define $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B} : (A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B} \to \Big((A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)/I_{(A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)}\Big)^{**}$ by $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B} = (\Phi^{-1}_{A\times_{T_\mathfrak{A}}B})^{**}\circ(q_A\otimes q_A)^{**}\circ \Phi_A^{**}\circ\tilde{\rho}_A\circ r_A$. By a similar argument as in the proof of the Theorem 4.9 for $\tilde{\rho}_A$, one can prove that $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}$ is $(A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B}$-$\mathfrak{A}$-module map. Now for every $(a,b)\in A\times_{T_\mathfrak{A}}B$, we have

$$\begin{aligned}
&\phi\circ(\widetilde{\varphi,\varphi\circ T})\circ\tilde{\omega}^{**}_{A\times_{T_\mathfrak{A}}B}\circ\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big)\\
&=\phi\circ(\widetilde{\varphi,\varphi\circ T})\circ\tilde{\omega}^{**}_{A\times_{T_\mathfrak{A}}B}\circ(\Phi^{-1}_{A\times_{T_\mathfrak{A}}B})^{**}\circ(q_A\otimes q_A)^{**}\circ\Phi_A^{**}\circ\tilde{\rho}_A\circ r_A\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big)\\
&=\phi\circ(\widetilde{\varphi,\varphi\circ T})\circ q_A^{**}\circ\tilde{\omega}_A^{**}\circ\tilde{\rho}_A\Big((a+T(b))+J_A\Big)\\
&=\widehat{\phi\circ\tilde{\varphi}}\circ\tilde{\omega}_A^{**}\circ\tilde{\rho}_A\Big((a+T(b))+J_A\Big)\\
&=\phi\circ\tilde{\varphi}\Big((a+T(b))+J_A\Big)\\
&=\phi\circ(\widetilde{\varphi,\varphi\circ T})\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big).
\end{aligned}$$

So $A\times_{T_\mathfrak{A}}B$ is $\Big((\varphi,\varphi\circ T),\phi\Big)$-module biflat.

(ii) Let $\phi\in\Delta(\mathfrak{A})$ and $\psi\in\Omega_B$. Since $A\times_{T_\mathfrak{A}}B$ is $\Big((0,\psi),\phi\Big)$-module biflat. Then there exists $(A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B}$-$\mathfrak{A}$-module map

$$\tilde{\rho}_{A\times_{T_\mathfrak{A}}B} : (A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B} \to \Big(\Big((A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)\Big)/I_{(A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)}\Big)^{**},$$

such that $\phi\circ\widehat{\widetilde{(0,\psi)}}\circ\tilde{\omega}^{**}_{A\times_{T_\mathfrak{A}}B}\circ\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big)=\phi\circ\widetilde{(0,\psi)}\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big)$. Define $\tilde{\rho}_B : B/J_B \to \Big((B\widehat{\otimes}B)/I_{B\widehat{\otimes}B}\Big)$ by

$$\tilde{\rho}_B = (\Phi_B^{-1})^{**}\circ(p_B\otimes p_B)^{**}\circ\Phi^{**}_{A\times_{T_\mathfrak{A}}B}\circ\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}\circ s_B.$$

As above we may show that $\tilde{\rho}_B$ is B/J_B-$\mathfrak{A}$-module map and for every $b\in B$,

$$\widehat{\phi\circ\tilde{\psi}}\circ\tilde{\omega}_B^{**}\circ\tilde{\rho}_B(b+J_B)=\phi\circ\tilde{\psi}(b+J_B).$$

Thus B is (ψ,ϕ)-module biflat, and so B is module character biflat.

Assume that B is module character biflat. Let $\phi\in\Delta(\mathfrak{A})$ and $\psi\in\Omega_B$. Then there exists $\tilde{\rho}_B : B/J_B \to \Big((B\widehat{\otimes}B)/I_{B\widehat{\otimes}B}\Big)^{**}$ such that $\widehat{\phi\circ\tilde{\psi}}\circ\tilde{\omega}_B^{**}\circ\tilde{\rho}_B(b+J_B)=\phi\circ\tilde{\psi}(b+J_B)$. Define $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B} : (A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B} \to \Big((A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)/I_{(A\times_{T_\mathfrak{A}}B)\widehat{\otimes}(A\times_{T_\mathfrak{A}}B)}\Big)^{**}$ by $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}=(\Phi^{-1}_{A\times_{T_\mathfrak{A}}B})^{**}\circ(s_B\otimes s_B)^{**}\circ\Phi_B^{**}\circ\tilde{\rho}_B\circ p_B$, then by the same argument as in (i), we can show that $\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}$ is $(A\times_{T_\mathfrak{A}}B)/J_{A\times_{T_\mathfrak{A}}B}$-$\mathfrak{A}$-module map and for every $(a,b)\in A\times_{T_\mathfrak{A}}B$, we have

$$\phi\circ\widehat{\widetilde{(0,\psi)}}\circ\tilde{\omega}^{**}_{A\times_{T_\mathfrak{A}}B}\circ\tilde{\rho}_{A\times_{T_\mathfrak{A}}B}\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big)=\phi\circ\widetilde{(0,\psi)}\Big((a,b)+J_{A\times_{T_\mathfrak{A}}B}\Big).$$

Therefore $A\times_{T_\mathfrak{A}}B$ is $\Big((0,\psi),\phi\Big)$-module biflat.

The proof of module approximate biflat is similar. □

Corollary 4.11. *Let A and B be two $\mathfrak{A}$-bimodule Banach algebras and let $T\in \mathtt{Hom}_\mathfrak{A}(B,A)$. Then $A\times_{T_\mathfrak{A}}B$ is $\Big((\varphi,\varphi\circ T),\phi\Big)$-module biflat and $\Big((0,\psi),\phi\Big)$-module biflat (resp. $\Big((\varphi,\varphi\circ T),\phi\Big)$-module approximate biflat and $\Big((0,\psi),\phi\Big)$-module approximate biflat) for every $\varphi\in\Omega_A$, $\psi\in\Omega_B$ and $\phi\in\Delta(\mathfrak{A})$ if and only if both A and B are module character biflat (resp. module character approximate biflat).*

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References

[1] F. Abtahi, A. Ghafarpanah and A. Rejali, Biprojectivity and biflatness of Lau product of Banach algebras defined by a Banach algebra morphism, Bull. Austral. Math. Soc 91 (2015) 134–144.
[2] M. Amini, Module amenability for semigroup algebras, Semigroup Forum 69 (2004) 243–254.
[3] M. Amini, A. Bodaghi and D. Ebrahimi Bagha, Module amenability of the second dual and module topological center of semigroup algebras, Semigroup Forum 80 (2010) 302–312.
[4] S. J. Bhatt and P. A. Dabhi, Arens regularity and amenability of lau product of Banach algebras defined by a Banach algebra morphism, Bull. Austral. Math. Soc 87 (2013) 197–206.
[5] A. Bodaghi and M. Amini, Module biprojective and module biflat Banach algebras, U.P.B. Sci. Bull, Series A 75 (2013) 25–37.
[6] A. Bodaghi and M. Amini, Module character amenability of Banach algebras, Arch. Math 99 (2012) 353–365.
[7] A. Bodaghi, H. Ebrahimi and M. Lashkarizaheh Bami, Generalized notions of module character amenability, Filomat 31 (2017) 1639–1654.
[8] A. Bodaghi, H. Ebrahimi, M. Lashkarizaheh Bami and M. Nemati, Module mean for Banach algebras, U.P.B. Sci. Bull, Series A 78 (2016) 21–30.
[9] F. Ghahramani and C. J. Read, Approximate identities in approximate amenability, J. Funct. Anal 262 (2012) 3929–3945.
[10] A. Ya. Helemskii, Flat Banach modules and amenable algebras, Trans. Moscow Math. Soc 47 (1985) 199–244.
[11] A. Ya. Helemskii, On a method for calculating and estimating the global homological dimentional of Banach algebras, Mat. Sb 87 (1972) 122–135.
[12] E. Kaniuth, A. Lau and J. Pym, On φ-amenability of Banach algebras, Math. Proc. Camp. Phil. Soc 144 (2008) 85–96.
[13] A. R. Khoddami and H. R. Ebrahimi vishki, Biflatness and biprojectivity of Lau product of Banach algebras, Bull. Iranian Math. Soc 39 (2013) 559–568.
[14] A. T. Lau, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups, Fund. Math 118 (1983) 161–175.
[15] R. E. Megginson, An introduction to Banach space theory, Springer, New York, 1998.
[16] M. S. Monfared, Character amenability of Banach algebras, Math. Proc. Camb. Phil. Soc 144 (2008) 697–706.
[17] M. S. Monfared, On certain products of Banach algebras with application to harmonic analysis, Studia Math 178 (2007) 277–294.
[18] H. Pourmahmood Aghababa and A. Bodaghi, module approximate amenability of Banach algebras, Bull. Iranian Math. Soc 39 (2013) 1137–1158.
[19] H. Pourmahmood Aghababa, L. Y. Shi and Y. J. Wu, Generalized notions of character amenability, Acta Math. Sinica, English Series 29 (2013) 1329–1350.
[20] M. A. Rieffel, Induced Banach representations of Banach algebras and locally compact groups, J. Funct. Anal 1 (1967) 443–491.
[21] H. Sadeghi and M. Lashkarizadeh Bami, Module character inner amenability of Banach algebras, Mathematical Sciences 11 (2017) 173–179.
[22] H. Sadeghi and M. Lashkarizadeh Bami, Module (ϕ, φ)-projectivity and module (ϕ, φ)-biflatness of Banach algebras, Proc. Rom. Academy, Series A, 19 (2018) 521–528.