Filomat 32:2 (2018), 553–573
https://doi.org/10.2298/FIL1802553A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Localization Theorems for Matrices and Bounds for the Zeros of Polynomials over Quaternion Division Algebra

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Abstract. In this paper, we derive Ostrowski and Brauer type theorems for the left and right eigenvalues of a quaternionic matrix. Generalizations of Gerschgorin type theorems are discussed for the left and the right eigenvalues of a quaternionic matrix. After that, a sufficient condition for the stability of a quaternionic matrix is given that generalizes the stability condition for a complex matrix. Finally, a characterization of bounds is derived for the zeros of quaternionic polynomials.

1. Introduction

Quaternions are extensively used in the programming of video games, computer graphics, quantum physics, flight dynamics, and control theory, etc. The solutions of linear differential equations with quaternion constant coefficients lead to quaternionic polynomials. So, the stability analysis of such differential equations can be studied through localization theorems of quaternionic matrices. In recent past, finding the zeros of quaternionic polynomials and finding the bounds of zeros of quaternionic polynomials have gained much attention in the literature. This paper attempts to study the localization theorems for matrices over a quaternion division algebra, which includes the Ostrowski, Brauer, and Gerschgorin type of theorems. Bounds for the zeros of quaternionic polynomials are also considered. Localization theorems for quaternionic matrices have received much attention in the literature due to their numerous applications in pure and applied sciences; see, e. g., [1, 2, 4, 6, 8, 13, 17–21, 27, 30, 31, 36–38] and the references therein. Unlike the case of matrices over the field of complex numbers [3, 5, 11, 25, 35], localization theorems for quaternionic matrices have been proposed for left and right eigenvalues separately in [16, 38, 39]. Ostrowski and Brauer type theorems for the right eigenvalues of a quaternionic matrix with all real diagonal entries have been introduced in [39]. A Brauer type theorem for the left eigenvalues of a quaternionic matrix has been considered in [16, Theorem 4] for the deleted absolute row sums which is not same for the deleted absolute column sums of a quaternionic matrix. Similar differences arise on the Gerschgorin and Ostrowski type theorems for a quaternionic matrix. Therefore, more research is required to understand the Ostrowski, Gershgorin, and Brauer type theorems for matrices over a quaternion division algebra. Furthermore, to investigating their applications in finding various bounds for the zeros of quaternionic polynomials and to

2010 *Mathematics Subject Classification*. 12E15; 34L15; 15A18; 15A66
Keywords. Skew field; quaternionic matrix; left and right eigenvalues; Gerschgorin type theorems; Brauer type theorem; quaternionic polynomials; quaternionic companion matrices; stable quaternionic matrix
Received: 27 January 2017; Accepted: 09 October 2017
Communicated by Dragana Cvetković Ilić
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analyze conditions for the stability of a quaternionic matrix, one has to do further research in this direction. Therefore we have developed a general framework using generalized Holder inequality of quaternions to enhance our theory.

In the first part of this paper, we provide a general framework for localization theorems for quaternionic matrices. Let $M_n(\mathbb{H})$ be the space of all $n\times n$ quaternionic matrices. Then, for any $A=(a_{ij})\in M_n(\mathbb{H})$, we prove a Ostrowski type theorem which states that all the left eigenvalues of A are located in the union of n balls $T_i(A):=\{z\in\mathbb{H}:|z-a_{ii}|\leq r_i(A)^{\gamma}c_i(A)^{1-\gamma}\}$, where $r_i(A):=\sum_{j=1,\,j\neq i}^{n}|a_{ij}|$ and $c_i(A):=\sum_{j=1,\,j\neq i}^{n}|a_{ji}|$, $\forall\ \gamma\in[0,1]$. From this result, we deduce a sufficient condition for invertibility of a quaternionic matrix. We find that the Brauer type theorem, proved in [16, Theorem 5] for the left eigenvalues in the case of deleted absolute column sums of a quaternionic matrix, is incorrect, and we prove a corrected version. In fact, in the case of the generalized Hölder inequality over the skew field of quaternions, we show that all the left eigenvalues of $A=(a_{ij})\in M_n(\mathbb{H})$ are contained in the union of n generalized balls: $B_i(A):=\{z\in\mathbb{H}:|z-a_{ii}|\leq(n-1)^{\frac{1-\gamma}{q}}r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma}\}$, where $\gamma\in[0,1]$, $n_i^{(p)}(A):=\left(\sum_{j=1,\,j\neq i}^{n}|a_{ij}|^{p}\right)^{\frac{1}{p}}$, for any $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Further, we prove that all the right eigenvalues of $A\in M_n(\mathbb{H})$ with all real diagonal entries are contained in the union of n generalized balls $B_i(A)$. In the sequel, we present localization theorems for the right eigenvalues of quaternionic matrices.

In the second part of this paper, we provide bounds for the zeros of quaternionic polynomials using the aforementioned localization theorems. Recall that quaternionic polynomials in general are expressed in the following forms

$$p_l(z) := q_m z^m + q_{m-1}z^{m-1} + \cdots + q_1 z + q_0, \tag{1}$$
$$p_r(z) := z^m q_m + z^{m-1}q_{m-1} + \cdots + z q_1 + q_0, \tag{2}$$

where $q_j,\ z\in\mathbb{H},\ (0\leq j\leq m)$. The polynomials (1) and (2) are called simple and monic if $q_m=1$. Some recent developments on the location and computation of zeros of quaternionic polynomials can be found in [7, 14, 15, 22–24, 28, 32]. As a consequence of the localization theorems for quaternionic matrices, we provide sharper bounds compared to the bound introduced by G. Opfer in [24] for the zeros of quaternionic polynomials. Finally, we provide bounds for the zeros of quaternionic polynomials in terms of powers of the companion matrices associated with the quaternionic polynomials (1) and (2). Some of our bounds are sharper than the bound from [24].

The paper is organized as follows: Section 2 reviews some existing results from [26, 37]. Section 3 discusses the Greshgorin type, Ostrowski type, and Brauer type theorems for the left and right eigenvalues of a quaternionic matrix. Section 4 explains bounds for the zeros of $p_l(z)$ and $p_r(z)$. Comparisons are made with the bound provided in [24]. A sufficient condition for the stability of a quaternionic matrix is also given. Section 5 introduces bounds for the zeros of the polynomials $p_l(z)$ and $p_r(z)$ in terms of powers of their companion matrices. Finally, Section 6 summarizes this work.

2. Preliminaries

Notation: Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set of real quaternions is defined by

$$\mathbb{H} := \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$. The conjugate of $q\in\mathbb{H}$ is $\overline{q}:=a_0-a_1\mathbf{i}-a_2\mathbf{j}-a_3\mathbf{k}$ and the modulus of q is $|q|:=\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}$. $\Im(a)$ denotes the imaginary part of $a\in\mathbb{C}$. The real part of a quaternion $q=a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ is defined as $\Re(q)=a_0$. The collection of all n-column vectors with elements in $\mathbb{H}$ is denoted by $\mathbb{H}^n$. For $x\in\mathcal{K}^n$, where $\mathcal{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, the transpose of x is x^T. If $x=[x_1,\ldots,x_n]^T$, the conjugate of x is defined as $\overline{x}=[\overline{x_1},\ldots,\overline{x_n}]^T$ and the conjugate transpose of x is defined as $x^H=[\overline{x_1},\ldots,\overline{x_n}]$. For $x,y\in\mathbb{H}^n$, the inner product is defined as $\langle x,y\rangle:=y^Hx$ and the norm of x is defined as $\|x\|:=\sqrt{\langle x,x\rangle}$. The sets of $m\times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively.

When $m = n$, these sets are denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $A \in M_{m\times n}(\mathcal{K})$, the conjugate, transpose, and conjugate transpose of A are defined as $\overline{A} = (\overline{a_{ij}})$, $A^T = (a_{ji}) \in M_{n\times m}(\mathbb{H})$, and $A^H = (\overline{A})^T \in M_{n\times m}(\mathbb{H})$, respectively. For $z \in \mathbb{H}^n$, the vector p-norm on $\mathbb{H}^n$ is defined by $\|z\|_p := (\sum_{i=1}^n |z_i|^p)^{1/p}$, where $1 \le p < \infty$ and $\|z\|_\infty := \max\limits_{1\le i\le n}\{|z_i|\}$. Define $\mathbb{R}^+ := \{\alpha : \alpha \in \mathbb{R}, \alpha > 0\}$. The set

$$[q] := \{r \in \mathbb{H} : r = \rho^{-1}\, q\, \rho \;\; for\ all\ 0 \neq \rho \in \mathbb{H}\}$$

is called an equivalence class of $q \in \mathbb{H}$.

Let $x \in \mathbb{H}^n$. Then x can be uniquely expressed as $x = x_1 + x_2\mathbf{j}$, where $x_1, x_2 \in \mathbb{C}^n$. Define the function $\psi : \mathbb{H}^n \to \mathbb{C}^{2n}$ by

$$\psi_x := \begin{bmatrix} x_1 \\ -\overline{x_2} \end{bmatrix}.$$

This function ψ is an injective linear transformation from $\mathbb{H}^n$ to $\mathbb{C}^{2n}$.

Definition 2.1. *Let $A \in M_n(\mathbb{H})$. Then A can be uniquely expressed as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$. Define the function $\Psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by*

$$\Psi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}.$$

The matrix Ψ_A is called the complex adjoint matrix of A.

Definition 2.2. *Let $A \in M_n(\mathbb{H})$. Then the left, right, and the standard eigenvalues, respectively, are given by*

$$\begin{aligned} \Lambda_l(A) &:= \{\lambda \in \mathbb{H} : Ax = \lambda x \text{ for some nonzero } x \in \mathbb{H}^n\}, \\ \Lambda_r(A) &:= \{\lambda \in \mathbb{H} : Ax = x\lambda \text{ for some nonzero } x \in \mathbb{H}^n\} \text{ and} \\ \Lambda_s(A) &:= \{\lambda \in \mathbb{C} : Ax = x\lambda \text{ for some nonzero } x \in \mathbb{H}^n,\ \Im(\lambda) \ge 0\}. \end{aligned}$$

Definition 2.3. *Let $A \in M_n(\mathbb{H})$. Then the matrix A is said to be stable if and only if $\Lambda_r(A) \subset \mathbb{H}^- := \{q \in \mathbb{H} : \Re(q) < 0\}$.*

Definition 2.4. *Let $A \in M_n(\mathbb{H})$. Then A is said to be η-Hermitian if $A = (A^\eta)^H$, where $A^\eta = \eta^H A \eta$ and $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.*

Definition 2.5. *A matrix $A \in M_n(\mathbb{H})$ is said to be invertible if there exists $B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.*

We next recall the following result necessary for the development of our theory.

Theorem 2.6. *[37, Theorem 4.3]. Let $A \in M_n(\mathbb{H})$. Then the following statements are equivalent:*
(a) A is invertible, (b) $Ax = 0$ has the unique solution, (c) $\det(\Psi_A) \neq 0$, (d) Ψ_A is invertible, (e) A has no zero eigenvalue.

Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and define the absolute row and column sums of A as

$$r_i'(A) := r_i(A) + |a_{ii}| \text{ and } c_i'(A) := c_i(A) + |a_{ii}| \quad (1 \le i \le n).$$

3. Distribution of the left and right eigenvalues of quaternionic matrices

It is known from [29, Corollary 3.2] that a quaternionic matrix A and its conjugate transpose A^H have the same right eigenvalues. However, A and A^H may not have the same left eigenvalues, take for example $A = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix}$ and $A^H = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & -\mathbf{j} \end{bmatrix}$. We now present the following lemma for left eigenvalues of A and A^H.

Lemma 3.1. *Let $A \in M_n(\mathbb{H})$ and let $\lambda \in \mathbb{H}$. Then λ is a left eigenvalue of A if and only if $\overline{\lambda}$ is a left eigenvalue of A^H.*

Proof. Let λ be a left eigenvalue of A. Then there exists $x(\neq 0) \in \mathbb{H}^n$ such that $(A - \lambda I_n)x = 0$. This can be written as $\Psi_{(A-\lambda I_n)}\psi_x = 0$. Hence it follows that λ is a left eigenvalue of A if and only if $\det\left[\Psi_{(A-\lambda I_n)}\right] = 0$ $\Leftrightarrow \det\left[\Psi^H_{(A-\lambda I_n)}\right] = 0 \Leftrightarrow \det\left[\Psi_{(A-\lambda I_n)^H}\right] = 0 \Leftrightarrow \det\left[\Psi_{(A^H-\overline{\lambda} I_n)}\right] = 0$. Thus, $\overline{\lambda}$ is a left eigenvalue of A^H. ∎

The Gerschgorin type theorem is proved in [38] for the left eigenvalues using deleted absolute row sums of a matrix $A \in M_n(\mathbb{H})$. However, the Gerschgorin type theorem for the left eigenvalues using deleted absolute column sums of A has not yet been established. We now state and prove the theorem.

Theorem 3.2. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are located in the union of n Gerschgorin balls $\Omega_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq c_i(A)\}, 1 \leq i \leq n$, that is,*

$$\Lambda_l(A) \subseteq \Omega(A) := \cup_{i=1}^n \Omega_i(A).$$

Proof. Let λ be a left eigenvalue of A. Then from Lemma 3.1, $\overline{\lambda}$ is a left eigenvalue of A^H. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $A^H x = \overline{\lambda} x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i|, 1 \leq i \leq n$. Then, $|x_t| > 0$. From the t-th equation of $A^H x = \overline{\lambda} x$, we have

$$\sum_{j=1}^{n} \overline{a_{jt}} x_j = \overline{\lambda} x_t.$$

This shows

$$|\lambda - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{jt}| := c_t(A). \quad ∎$$

We now have the following localization theorem for the deleted absolute row and column sums of a matrix $A \in M_n(\mathbb{H})$ which is known as *Ostrowski type theorem*.

Theorem 3.3. (*Ostrowski type theorem for the left eigenvalues*) *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0,1]$. Then all the left eigenvalues of A are located in the union of n balls $T_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)^\gamma c_i(A)^{1-\gamma}\}, 1 \leq i \leq n$, that is,*

$$\Lambda_l(A) \subseteq T(A) := \cup_{i=1}^n T_i(A).$$

Proof. Let $\gamma \in (0,1)$; as the cases $\gamma = 0$ and $\gamma = 1$ (Gerschgorin type theorems for column and row sums, respectively) can be obtained by taking limits. We may assume that all $r_i(A) > 0$, because we may perturb A by inserting a small nonzero entry $\epsilon > 0$ into any row in which $r_i(A) = 0$; the resulting matrix has a ball that is larger the ball for A, and the result follows in the limit as the perturbation goes to zero.

Let λ be a left eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = \lambda x$. Let $x = [x_1, \ldots, x_n]^T \in \mathbb{H}^n$. Then for each $i = 1, 2, \ldots, n$, we have

$$|\lambda - a_{ii}||x_i| = |\sum_{j=1, j\neq i}^{n} a_{ij} x_j| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}||x_j| = \sum_{j=1, j\neq i}^{n} |a_{ij}|^\gamma (|a_{ij}|^{1-\gamma}|x_j|).$$

Applying the generalized Holder inequality with $p = \frac{1}{\gamma}$ and $q = \frac{1}{1-\gamma}$, we obtain

$$\begin{aligned} |\lambda - a_{ii}||x_i| &\leq \left(\sum_{j=1, j\neq i}^{n} (|a_{ij}|^\gamma)^{1/\gamma}\right)^\gamma \left(\sum_{j=1, j\neq i}^{n} (|a_{ij}|^{1-\gamma}|x_j|)^{\frac{1}{1-\gamma}}\right)^{1-\gamma} \\ &= \left(\sum_{j=1, j\neq i}^{n} |a_{ij}|\right)^\gamma \left(\sum_{j=1, j\neq i}^{n} |a_{ij}||x_j|^{\frac{1}{1-\gamma}}\right)^{1-\gamma} = r_i(A)^\gamma \left(\sum_{j=1, j\neq i}^{n} |a_{ij}||x_j|^{\frac{1}{1-\gamma}}\right)^{1-\gamma}. \end{aligned} \tag{3}$$

Since $r_i(A) > 0$, then from (3) we have

$$\left(\frac{|\lambda - a_{ii}|}{r_i(A)^\gamma}\right)^{\frac{1}{1-\gamma}} |x_i|^{\frac{1}{1-\gamma}} \leq \sum_{j=1, j\neq i}^{n} |a_{ij}||x_j|^{\frac{1}{1-\gamma}}.$$

Summing over all i, one obtains

$$\sum_{i=1}^{n}\left(\frac{|\lambda - a_{ii}|}{r_i(A)^\gamma}\right)^{\frac{1}{1-\gamma}} |x_i|^{\frac{1}{1-\gamma}} \leq \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} |a_{ij}||x_j|^{\frac{1}{1-\gamma}} = \sum_{j=1}^{n} c_j(A)|x_j|^{\frac{1}{1-\gamma}}. \tag{4}$$

If

$$\left(\frac{|\lambda - a_{ii}|}{r_i(A)^\gamma}\right)^{\frac{1}{1-\gamma}} > c_i(A)$$

for each i such that $x \neq 0$, then (4) could not hold. Hence, we can conclude that at least one i exists such as

$$\left(\frac{|\lambda - a_{ii}|}{r_i(A)^\gamma}\right)^{\frac{1}{1-\gamma}} \leq c_i(A)$$

that is $|\lambda - a_{ii}| \leq r_i(A)^\gamma c_i(A)^{1-\gamma}$. Thus, all the left eigenvalues of A are located in the union of n balls $T_i(A)$. ∎

Corollary 3.4. *For any $A := (a_{ij}) \in M_n(\mathbb{H})$, $n \geq 2$ and for any $\gamma \in [0, 1]$. Let us assume that*

$$|a_{ii}| > r_i(A)^\gamma\ r_i(A)^{1-\gamma}, \quad 1 \leq i \leq n. \tag{5}$$

Then A is invertible.

Proof. On the contrary, suppose A is not invertible. Then by Theorem 2.6, there is a left eigenvalue $\lambda = 0$ of A. Now from Theorem 3.3, we obtain $|a_{ii}| \leq r_i(A)^\gamma c_i(A)^{1-\gamma}$. This contradicts our assumption (5). Hence A is invertible. ∎

The Brauer type theorem is proved in [16] for the left eigenvalues in the case of deleted absolute column sums of a matrix $A \in M_n(\mathbb{H})$. That is, if $\lambda \in \Lambda_l(A)$, then its conjugate $\overline{\lambda}$ lies in the union of $\frac{n(n-1)}{2}$ ovals of Cassini. However, this is incorrect as the following example suggests:

Example 3.5. Let $A = \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ 0 & \mathbf{j} \end{bmatrix}$. Then by [16, Theorem 5], oval of Cassini is given by $\{z \in \mathbb{H} : |z - \mathbf{i}|\ |z - \mathbf{j}| \leq 0\}$. Here, $\mathbf{i}$ is a left eigenvalue of A and its conjugate $-\mathbf{i}$ is not contained in the above oval of Cassini.

According to [16, Theorem 5], if $\lambda \in \Lambda_l(A)$, then $\overline{\lambda} \in \cup_{\substack{i,j=1,\\ i\neq j}}^{n} F_{ij}(A)$, where

$$F_{ij}(A) := \left\{z \in \mathbb{H} : |z - a_{ii}|\ |z - a_{jj}| \leq c_i(A)c_j(A)\right\}, \quad 1 \leq i, j \leq n, \quad i \neq j.$$

However, this result is not necessarily true as

$$|\overline{\lambda} - a_{ii}|\ |\overline{\lambda} - a_{jj}| > c_i(A)c_j(A), \quad 1 \leq i, j \leq n, \quad i \neq j,$$

which follows from Example 3.5. Now, we derive a corrected version of [16, Theorem 5] as follows:

Theorem 3.6. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ ovals of Cassini*

$$F_{ij}(A) := \left\{z \in \mathbb{H} : |z - a_{ii}|\ |z - a_{jj}| \leq c_i(A)c_j(A)\right\}, \quad 1 \leq i, j \leq n, \quad i \neq j,$$

that is, $\Lambda_l(A) \subseteq F(A) := \cup_{\substack{i,j=1,\\ i\neq j}}^{n} F_{ij}(A)$.

Proof. Let λ be a left eigenvalue of A. Then by Lemma 3.1, $\overline{\lambda}$ is a left eigenvalue of A^H, so that there exists some nonzero $x \in \mathbb{H}^n$ such that $A^H x = \overline{\lambda} x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_s be an element of x such that $|x_s| \geq |x_i|$, $1 \leq i \leq n$. Then, $|x_s| > 0$. Clearly, if all the other elements of x are zero, then the required result holds.

Let x_s and x_t be two nonzero elements of x such that $|x_s| \geq |x_t| \geq |x_i|, 1 \leq i \leq n, i \neq s$. From the s-th equation of $A^H x = \overline{\lambda} x$, we have $\sum_{j=1}^{n} \overline{a_{js}} x_j = \overline{\lambda} x_s$, which implies $(\overline{\lambda} - \overline{a_{ss}}) x_s = \sum_{j=1,\, j \neq s}^{n} \overline{a_{js}} x_j$. Thus

$$|\lambda - a_{ss}| \leq \left(\frac{|x_t|}{|x_s|} \right) c_s(A). \tag{6}$$

Similarly, from $A^H x = \overline{\lambda} x$, we obtain

$$|\lambda - a_{tt}| \leq \left(\frac{|x_s|}{|x_t|} \right) c_t(A). \tag{7}$$

Combining (6) and (7), we have

$$|\lambda - a_{ss}|\, |\lambda - a_{tt}| \leq c_s(A) c_t(A).$$

Hence, all the left eigenvalues of A are located in the union of $\frac{n(n-1)}{2}$ ovals of Cassini $F_{ij}(A)$, $\quad 1 \leq i, j \leq n, \quad i \neq j$. ■

For $A := (a_{ij}) \in M_n(\mathbb{H})$, define

$$n_i^{(p)}(A) := \left(\sum_{j=1,\, j \neq i}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}, \quad 1 \leq i \leq n, \quad p \in (1, \infty).$$

We are now ready to derive the following localization theorem for left eigenvalues of a quaternionic matrix.

Theorem 3.7. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. Then all the left eigenvalues of A are contained in the union of n generalized balls*

$$B_i(A) := \left\{ z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma} \right\}, \quad 1 \leq i \leq n,$$

that is,

$$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A),$$

for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let μ be a left eigenvalue of A. Then there exists some nonzero $x \in \mathbb{H}^n$ such that $Ax = \mu x$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i|$, $1 \leq i \leq n$. Then from $Ax = \mu x$, we have

$$a_{tt} x_t + \sum_{j=1,\, j \neq t}^{n} a_{tj} x_j = \mu x_t.$$

This implies

$$|\mu - a_{tt}| |x_t| = \left| \sum_{j=1,\, j \neq t}^{n} a_{tj} x_j \right| \leq \sum_{j=1,\, j \neq t}^{n} |a_{tj}|\, |x_j|. \tag{8}$$

Applying the generalized Hölder inequality to (8), we have

$$|\mu - a_{tt}| |x_t| \leq \left(\sum_{j=1,\, j \neq t}^{n} |a_{tj}|^p \right)^{\frac{1}{p}} \left(\sum_{j=1,\, j \neq t}^{n} |x_j|^q \right)^{\frac{1}{q}}.$$

Since $|x_t| \geq |x_i|$ for all $1 \leq i \leq n$, we have $|\mu - a_{tt}||x_t| \leq n_t^{(p)}(A)\,((n-1)|x_t|^q)^{\frac{1}{q}}$ that is,

$$|\mu - a_{tt}| \leq n_t^{(p)}(A)\,(n-1)^{\frac{1}{q}}\,. \tag{9}$$

Similarly, using $|x_t| \geq |x_i|\ \forall\ i\ (1 \leq i \leq n)$ in (8), we get

$$|\mu - a_{tt}| \leq \sum_{j=1,\, j\neq t}^{n} |a_{tj}| = r_t(A). \tag{10}$$

Combining (9) and (10) for $\gamma \in [0, 1]$, we have

$$|\mu - a_{tt}|^{1-\gamma} \leq (n_t^{(p)}(A))^{1-\gamma}(n-1)^{\frac{1-\gamma}{q}} \text{ and } |\mu - a_{tt}|^{\gamma} \leq r_t(A)^{\gamma}, \tag{11}$$

that is,

$$|\mu - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}}(n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \ \blacksquare$$

Let us relate Theorem 3.7 to some existing results:

- Setting $p = q = 2$ and $\gamma = 1$ implies that the left eigenvalues of $A := (a_{ij}) \in M_n(\mathbb{H})$ are contained in the union of n Greschgorin balls $B_i(A) := \{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)\}, 1 \leq i \leq n$, that is,

  $$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A).$$

  This result can be found in [38, Theorem 6].
- Setting $p = q = 2$ and $\gamma = 0$ implies that the left eigenvalues of $A := (a_{ij}) \in M_n(\mathbb{H})$ are contained in the union of n balls $B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1}{2}} n_i^{(2)}(A)\right\}, 1 \leq i \leq n$, that is,

  $$\Lambda_l(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A).$$

  This result can be found in [36, Theorem 1].

We now present a generalization of [38, Theorem 7] and [39, Theorem 3.1] by applying the generalized Hölder inequality over the skew field of quaternions. For a general matrix $A := (a_{ij}) \in M_n(\mathbb{H})$, all the right eigenvalues may not lie in the union of n generalized balls $B_i(A), 1 \leq i \leq n$. On the other hand, we show that every connected region of the generalized balls $B_i(A), 1 \leq i \leq n$ contains some right eigenvalues of A.

Theorem 3.8. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. For every right eigenvalue μ of A there exists a nonzero quaternion β such that $\beta^{-1}\mu\beta$ (which is also a right eigenvalue) is contained in the union of n generalized balls*

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma}\right\}, \quad 1 \leq i \leq n,$$

that is, $\left\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\right\} \cap \cup_{i=1}^{n} B_i(A) \neq \emptyset$, where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let μ be a right eigenvalue of A. Then there exists some nonzero vector $x \in \mathbb{H}^n$ such that $Ax = x\mu$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and choose x_t from x as given in Theorem 3.7. Consider $\rho \in \mathbb{H}$ such that $x_t\mu = \rho x_t$. Then we have

$$|\rho - a_{tt}||x_t| = \left|\sum_{j=1,\, j\neq t}^{n} a_{tj}x_j\right| \leq \sum_{j=1,\, j\neq t}^{n} |a_{tj}|\,|x_j|. \tag{12}$$

Using the method from the proof of Theorem 3.7, we have

$$|\rho - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}}(n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \ \blacksquare$$

Let us relate Theorem 3.8 to some existing results:

- Substituting $p = q = 2$ and $\gamma = 1$, we obtain

$$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^{n}\{z \in \mathbb{H} : |z - a_{ii}| \leq r_i(A)\} \neq \emptyset.$$

  This result can be found in [38, Theorem 7].

- Substituting $p = q = 2$ and $\gamma = 0$, we get

$$\{z^{-1}\mu z : 0 \neq z \in \mathbb{H}\} \cap \cup_{i=1}^{n}\left\{z \in \mathbb{H} : |z - a_{ii}| \leq \sqrt{n-1}\, n_i^{(2)}(A)\right\} \neq \emptyset.$$

  This result can be found in [39, Theorem 3.1].

We next present a sufficient condition for the stability of a matrix $A \in M_n(\mathbb{H})$.

Proposition 3.9. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ and let $\gamma \in [0, 1]$. Assume that*

$$\Re(a_{ii}) + (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma} < 0, \quad 1 \leq i \leq n, \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q \in (1, \infty)$. Then the matrix A is stable.

Proof. Let $\lambda \in \Lambda_r(A)$. From Theorem 3.8 there exists $0 \neq \rho \in \mathbb{H}$ such that $\rho^{-1}\lambda\rho \in \cup_{i=1}^{n} B_i(A)$. Without loss of generality, we assume $\rho^{-1}\lambda\rho \in B_l(A)$, that is,

$$|\rho^{-1}\lambda\rho - a_{ll}| \leq (n-1)^{\frac{1-\gamma}{q}} r_l(A)^{\gamma}(n_l^{(p)}(A))^{1-\gamma}.$$

Consider $\lambda := \lambda_1 + \lambda_2\mathbf{i} + \lambda_3\mathbf{j} + \lambda_4\mathbf{k}$ and $a_{ll} = a_l + b_l\mathbf{i} + c_l\mathbf{j} + d_l\mathbf{k}$. Then from (13), we obtain

$$|(\lambda_1 - a_l) + (\rho^{-1}\lambda_2\mathbf{i}\rho - b_l\mathbf{i}) + (\rho^{-1}\lambda_3\mathbf{j}\rho - c_l\mathbf{j}) + (\rho^{-1}\lambda_4\mathbf{k}\rho - d_l\mathbf{k})| < -\Re(a_{ll}) = -a_l. \tag{14}$$

The equality (14) is possible when $\lambda_1 < 0$, that is, $\Re(\lambda) < 0$, hence $\lambda \in \mathbb{H}^-$. This shows that the matrix A is stable. ∎

When all the diagonal entries of a matrix $A \in M_n(\mathbb{H})$ are real, we have the following theorem.

Theorem 3.10. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0, 1]$. Then all the right eigenvalues of A are contained in the union of n generalized balls*

$$B_i(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma}(n_i^{(p)}(A))^{1-\gamma}\right\}, \quad 1 \leq i \leq n,$$

that is, $\Lambda_r(A) \subseteq B(A) := \cup_{i=1}^{n} B_i(A)$, where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let λ be a right eigenvalue of A. Then there exists some nonzero vector $x \in \mathbb{H}^n$ such that $Ax = x\lambda$. Let $x := [x_1, \ldots, x_n]^T \in \mathbb{H}^n$ and let x_t be an element of x such that $|x_t| \geq |x_i|, 1 \leq i \leq n$. Then $|x_t| > 0$. Thus from $Ax = x\lambda$, we have

$$a_{tt}x_t + \sum_{j=1,\, j\neq t}^{n} a_{tj}x_j = x_t\lambda,$$

since $a_{tt} \in \mathbb{R}$, so $a_{tt}x_t = x_t a_{tt}$. Then from the proof method of Theorem 3.7, we have

$$|\lambda - a_{tt}| \leq (n-1)^{\frac{1-\gamma}{q}} (n_t^{(p)}(A))^{1-\gamma} r_t(A)^{\gamma}. \ \blacksquare$$

The above result has great significance as Hermitian, and η-Hermitian matrices have all real diagonal entries. In general, η-Hermitian matrices arise widely in applications [12, 33, 34]. To that end, we state the following proposition when all diagonal entries of $A \in M_n(\mathbb{H})$ are real. In particular, this result gives a sufficient condition for the stability of a matrix $A \in M_n(\mathbb{H})$.

Proposition 3.11. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$ with $a_{ii} \in \mathbb{R}$ and let $\gamma \in [0, 1]$. Assume that*

$$a_{ii} + (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma} < 0, \quad 1 \le i \le n,$$

where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the matrix A is stable.

From Theorem 3.10, all the complex right eigenvalues of a matrix $A = (a_{ij}) \in M_n(\mathbb{H})$ with all real diagonal entries lie in the union of n-discs $D_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \le (n-1)^{\frac{1-\gamma}{q}} r_i(A)^{\gamma} (n_i^{(p)}(A))^{1-\gamma}\}, 1 \le i \le n$, that is,

$$\Lambda_c(A) \subseteq D(A) := \cup_{i=1}^n D_i(A). \tag{15}$$

However, if diagonal entries are from $\mathbb{C} \setminus \mathbb{R}$, then it is not necessary that all the complex right eigenvalues of A are contained in the union of n-discs $D_i(A), 1 \le i \le n$ as the following examples suggest.

Example 3.12. Let $A := \begin{bmatrix} 1-2\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2\mathbf{i} & -\mathbf{i} \\ 0 & \mathbf{k} & 3+\mathbf{i} \end{bmatrix}$. The set of complex right eigenvalues of A is

$$\Lambda_c(A) := \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\},$$

where $\lambda_1 = -0.0164 + 2.0083\mathbf{i}$, $\lambda_2 = -0.0164 - 2.0083\mathbf{i}$, $\lambda_3 = 1 + 2\mathbf{i}$, $\lambda_4 = 1 - 2\mathbf{i}$, $\lambda_5 = 3.0164 + 1.0324\mathbf{i}$, and $\lambda_6 = 3.0164 + 1.0324\mathbf{i}$. For $\gamma = 1$ in (15), the discs $D_1(A), D_2(A)$, and $D_3(A)$ are as follows:

$$D_1(A) := \{z \in \mathbb{C} : |z - 1 + 2\mathbf{i}| \le 2\}, \;\; D_2(A) := \{z \in \mathbb{C} : |z + 2\mathbf{i}| \le 1\}, \;\text{and}$$

$$D_3(A) := \{z \in \mathbb{C} : |z - 3 - \mathbf{i}| \le 1\}.$$

From Figure 1, it is clear that λ_1, λ_3, and λ_6 lie outside the discs $D_1(A), D_2(A)$, and $D_3(A)$.

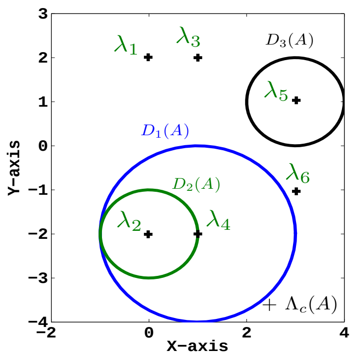

Figure 1: Location of the complex right eigenvalues of A from Example 3.12.

Example 3.13. Let $A = \begin{bmatrix} -4 & 1+\mathbf{j}+\sqrt{2}\mathbf{k} & \mathbf{j} \\ \mathbf{i}+\mathbf{j} & -10 & 2\mathbf{j}-\mathbf{k} \\ \mathbf{i}-2\mathbf{j}+2\mathbf{k} & \sqrt{3}+2\mathbf{j}-3\mathbf{k} & -8 \end{bmatrix}$. In this example, there are six complex right eigenvalues λ_j $(1 \le j \le 6)$ which are shown in Figure 2. Substituting $\gamma = 1$ in (15), then all the complex right eigenvalues of the matrix A are contained in the union of three discs $D_1(A), D_2(A)$, and $D_3(A)$, where

$$D_1(A) := \{z \in \mathbb{C} : |z + 4| \le 3\}, \;\; D_2(A) := \{z \in \mathbb{C} : |z + 10| \le \sqrt{2} + \sqrt{5}\}, \;\text{and}$$

$$D_3(A) := \{z \in \mathbb{C} : |z + 8| \le 7\}.$$

From Figure 2, the standard right eigenvalues of A are λ_1, λ_3, and λ_5. Then

$$\Lambda_r(A) = [\lambda_1] \cup [\lambda_3] \cup [\lambda_5].$$

Also, from Figure 2, we observe that $\Re(\lambda_i) \in \mathbb{H}^-$ $(i = 1, 3, 5)$. Hence

$$\Re(\lambda_1) = \Re(\rho^{-1}\lambda_1\rho),\ \Re(\lambda_2) = \Re(\tau^{-1}\lambda_2\tau), \text{and } \Re(\lambda_3) = \Re(\nu^{-1}\lambda_3\nu)\quad \forall\, \rho, \tau, \nu \in \mathbb{H}$$

Thus the matrix A is stable.

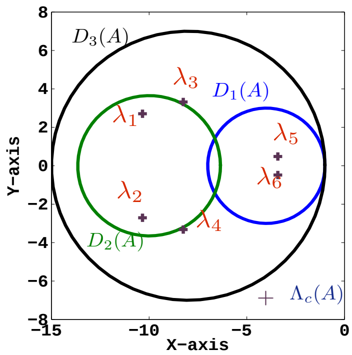

Figure 2: Location of the complex right eigenvalues of A from Example 3.13.

In general, similar quaternionic matrices may not have the same left eigenvalues, see, [38, Example 3.3]. However, the following result is true.

Proposition 3.14. *Let $A \in M_n(\mathbb{H})$ and let W be any invertible real matrix. Then A and WAW^{-1} have the same left eigenvalues.*

Proof. Let λ be a left eigenvalue of A. Then there exists some nonzero vector $x \in \mathbb{H}^n$ such that $Ax = \lambda x$. Let W be an invertible real matrix. Then

$$WAx = W\lambda x = \lambda Wx.$$

Now, $WAW^{-1}Wx = \lambda Wx$. Setting $Wx = y$ implies $WAW^{-1}y = \lambda y$. ■

Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Suppose $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$ with $w_i \in \mathbb{R}^+, 1 \le i \le n$. Then

$$W^{-1}AW = \left(\frac{a_{ij}w_j}{w_i}\right) \text{ and } \Lambda_l(A) = \Lambda_l(W^{-1}AW).$$

Define

$$r_i^W(A) := \sum_{j=1,\, j\neq i}^{n} \frac{|a_{ij}|w_j}{w_i} \text{ and } c_i^W(A) := \sum_{j=1,\, j\neq i}^{n} \frac{|a_{ji}|w_i}{w_j}, \quad 1 \le i \le n.$$

Applying Theorem 3.3 to $W^{-1}AW$, we get the following theorem which may be sharper than Theorem 3.3 depending upon the choice of W.

Theorem 3.15. *Let $A := (a_{ij}) \in M_n(\mathbb{H})$. Then all the left eigenvalues of A are contained in the union of n balls*

$$T_i^W(A) := \{z \in \mathbb{H} : |z - a_{ii}| \le (r_i^W(A))^{\gamma}\, (c_i^W(A))^{1-\gamma}\}, \quad 1 \le i \le n,$$

that is,

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq T^W(A) := \cup_{i=1}^{n} T_i^W(A).$$

Since the above theorem holds for every $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$, where $w_i \in \mathbb{R}^+$, we have

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} T^W(A) =: T^S(A),$$

where $M_n(S)$ is a set of real diagonal matrices with non-negative entries. $T^S(A)$ is called the minimal Ostrowski type set for the matrix A.

Substituting $\gamma = 1$ in Theorem 3.15, we obtain

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \eta^W(A) := \cup_{i=1}^{n} \eta_i^W(A), \tag{16}$$

where $\eta_i^W(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq r_i^W(A)\right\}$. Therefore,

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} \eta^W(A) =: \eta^S(A),$$

where $\eta^S(A)$ is called the first minimal Gerschgorin type set for the matrix A.

For $\gamma = 0$ in Theorem 3.15, we have

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \Omega^W(A) := \cup_{i=1}^{n} \Omega_i^W(A), \tag{17}$$

where $\Omega_i^W(A) := \left\{z \in \mathbb{H} : |z - a_{ii}| \leq c_i^W(A)\right\}$. Then

$$\Lambda_l(A) = \Lambda_l(W^{-1}AW) \subseteq \bigcap_{W \in M_n(S)} \Omega^W(A) =: \Omega^S(A),$$

where $\Omega^S(A)$ is called the second minimal Gerschgorin type set for the matrix A.

4. Bounds for the zeros of quaternionic polynomials

In this section, we derive bounds for the zeros of quaternionic polynomials by applying the localization theorems for the left eigenvalues of a quaternionic matrix. Due to noncommutivity of quaternions, we first define some basic facts on multiplication of quaternions. For $p, q \in \mathbb{H}$, define $p \times q := pq$. For $0 \neq p \in \mathbb{H}$ and $q \in \mathbb{H}$, define

$$\frac{1}{p} \times q := p^{-1} \times q := p^{-1}q, \; q \times \frac{1}{p} := q \times p^{-1} := qp^{-1}.$$

Recall the quaternionic polynomials $p_l(z)$ and $p_r(z)$ from (1) and (2). Then the corresponding companion matrices of the simple monic polynomials $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} := \left[\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -q_0 & -q_1 & \ldots & -q_{m-1} \end{array}\right] := \begin{array}{c} \\ m-1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & m-1 \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m, 2:m) \end{array}\right] \end{array} \text{and } C_{p_r} := C_{p_l}^T,$$

respectively. Let $q_0 \neq 0$, and define simple monic reversal polynomials of $p_l(z)$ and $p_r(z)$ as follows:

$$q_l(z) := \frac{1}{q_0} \times p_l\left(\frac{1}{z}\right) \times z^m = z^m + q_0^{-1}q_1 z^{m-1} + \cdots + q_0^{-1}q_{m-1}z + q_0^{-1},$$

$$q_r(z) := z^m \times p_r\left(\frac{1}{z}\right) \times \frac{1}{q_0} = z^m + z^{m-1}q_1 q_0^{-1} + \cdots + z q_{m-1} q_0^{-1} + q_0^{-1},$$

respectively. The corresponding companion matrices of the simple monic reversal polynomials $q_l(z)$ and $q_r(z)$ are denoted by C_{q_l} and C_{q_r}, respectively. We observe that the zeros of $q_l(z)$ and $q_r(z)$ are the reciprocal of zeros of $p_l(z)$ and $p_r(z)$, respectively.

Now, we need the following result:

Proposition 4.1. *[32, Proposition 1]. Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_l(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{p_l}.*

In general, a right eigenvalue of C_{p_l} is not necessarily a zero of the simple monic polynomial $p_l(z)$. For example, let a simple monic polynomial $p_l(z) = z^2 + \mathbf{j}z + 2$. Then its companion matrix is given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 \\ -2 & -\mathbf{j} \end{bmatrix}.$$

Here $\mathbf{i}$ is a right eigenvalue of C_{p_l}. However, $\mathbf{i}$ is not a zero of $p_l(z)$.

Analogous to Proposition 4.1, the following result is presented for $p_r(z)$.

Proposition 4.2. *Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_r(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{p_r}.*

We now present bounds for the zeros of $p_l(z)$ as follows.

Theorem 4.3. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequality:*

$$\left(\max_{1\leq i\leq m}\left(r_i'(C_{q_l})^{\gamma}\, c_i'(C_{q_l})^{1-\gamma}\right)\right)^{-1} \leq |\tilde{z}| \leq \max_{1\leq i\leq m}\left(r_i'(C_{p_l})^{\gamma}\, c_i'(C_{p_l})^{1-\gamma}\right),$$

for every $\gamma \in [0, 1]$.

Proof. From Proposition 4.1, zeros of $p_l(z)$ and left eigenvalues of C_{p_l} are same. Thus, if $\tilde{z}$ is a zero of $p_l(z)$, then $\tilde{z}$ is a left eigenvalue of C_{p_l}. By applying Theorem 3.3 (Ostrowski type theorem) to C_{p_l}, we obtain

$$|\tilde{z}| \leq \max_{1\leq i\leq m}\left(r_i'(C_{p_l})^{\gamma}\, c_i'(C_{p_l})^{1-\gamma}\right).$$

We use the respective upper bounds for the zeros of the simple monic reversal polynomial $q_l(z)$ for the desired lower bounds for the zeros of $p_l(z)$. ■

Corollary 4.4. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

1. $\dfrac{|q_0|}{\max\limits_{1\leq i\leq (m-1)}\{1, |q_0| + |q_i|\}} \leq |\tilde{z}| \leq \max\limits_{1\leq i\leq (m-1)}\{|q_0|, 1 + |q_i|\}.$
2. $\dfrac{|q_0|}{\max\left\{|q_0|, 1 + \sum_{i=1}^{m-1}|q_i|\right\}} \leq |\tilde{z}| \leq \max\left\{1, \sum_{i=0}^{m-1}|q_i|\right\}.$

Proof. Substituting $\gamma = 0, 1$ in Theorem 4.3, we obtain the desired results. ■

Next, we derive the following lemma which gives a better bound than Opfer's bound [24, Theorem 4.2] for $|q_0| \geq 1$.

Lemma 4.5. *Assume that $|q_0| \geq 1$. Then $\alpha \leq \mathcal{T}$, where $\alpha := \max\limits_{1\leq i\leq m-1}\{|q_0|, 1 + |q_i|\}$ and $\mathcal{T} := \max\left\{1, \sum_{i=0}^{m-1}|q_i|\right\}$.*

Proof. **Case 1:** If $|q_0| = 1$, then $\alpha = \max\limits_{1\leq i\leq m-1}\{|q_0|, 1 + |q_i|\} = \max\limits_{1\leq i\leq m-1}\{1 + |q_i|\}$ and $\mathcal{T} := \max\left\{1, \sum_{i=0}^{m-1}|q_i|\right\} = \max\left\{1, |q_0| + \sum_{i=1}^{m-1}|q_i|\right\} = 1 + \sum_{i=1}^{m-1}|q_i|$.

Case 2: If $|q_0| > 1$, then $\alpha = \max\limits_{1\leq i\leq (m-1)}\{|q_0|, 1 + |q_i|\} = |q_0|$ or $\max_{1\leq i\leq (m-1)}\{1 + |q_i|\}$ and $\mathcal{T} := \max\{1, \sum_{i=0}^{m-1}|q_i|\} = \max\left\{1, |q_0| + \sum_{i=1}^{m-1}|q_i|\right\} = |q_0| + \sum_{i=1}^{m-1}|q_i|$. Thus $\alpha \leq \mathcal{T}$. This completes the proof. ■

On the other hand, if $|q_0| < 1$ in Lemma 4.5 then $\alpha \leq \mathcal{T}$ or $\alpha > \mathcal{T}$. For example, for a simple monic polynomial $p_l'(z) := z^3 + (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})z^2 - 2\mathbf{k}z + 0.5\mathbf{k}$, we have $\alpha = 4$ and $\mathcal{T} = 5.5$. Hence $\alpha < \mathcal{T}$. Further, if we consider $p_l''(z) = z^3 + 0.5\mathbf{j}z^2 + (0.2\mathbf{i} + 0.3\mathbf{j})z + 0.5\mathbf{i}$, then $\alpha = 1.5$ and $\mathcal{T} = 1.36$. Hence $\alpha > \mathcal{T}$.

Next, by applying Theorem 3.3 to $WC_{p_l}W^{-1}$ and $WC_{q_l}W^{-1}$ (W is an invertible real diagonal matrix), we obtain different and potentially sharper bounds.

Theorem 4.6. *Let $w_i \in \mathbb{R}^+$, $1 \le i \le m$. Then every zero $\tilde{z}$ of the simple monic polynomial $p_l(z)$ satisfies the following inequality:*

$$\left[\max_{1\le i\le m}\left\{r_i'(WC_{q_l}W^{-1})^{\gamma}\, c_i'(WC_{q_l}W^{-1})^{1-\gamma}\right\}\right]^{-1} \le |\tilde{z}| \le \max_{1\le i\le m}\left\{r_i'(WC_{p_l}W^{-1})^{\gamma}\, c_i'(WC_{p_l}W^{-1})^{1-\gamma}\right\},$$

where $W := \operatorname{diag}(w_1, w_2, \ldots, w_m)$ and $\gamma \in [0,1]$.

Proof. The companion matrix of $p_l(z)$ is given by

$$C_{p_l} = \begin{array}{c} \\ m-1 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 1 & m-1 \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline -q_0 & [-q_1 \ldots - q_{m-1}] \end{array}\right] \end{array}.$$

Then

$$WC_{p_l}W^{-1} = \begin{array}{c} \\ m-1 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 1 & m-1 \end{array} \\ \left[\begin{array}{c|c} 0 & \operatorname{diag}\left(\frac{w_1}{w_2}, \ldots, \frac{w_{m-1}}{w_m}\right) \\ \hline -\frac{w_m}{w_1}q_0 & -\frac{w_m}{w_2}q_1 \ldots - q_{m-1} \end{array}\right] \end{array}.$$

By Proposition 3.14, C_{p_l} and $WC_{p_l}W^{-1}$ have the same left eigenvalues. Rest of the proof follows from the proof method of Theorem 4.3. ■

Corollary 4.7. *Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies the following inequalities:*

1. $\left[\max\limits_{0\le j\le m-1}\left\{\dfrac{(|q_0|w_j + w_m|q_{m-j}|)}{|q_0|d_{j+1}}\right\}\right]^{-1} \le |\tilde{z}| \le \max\limits_{0\le j\le m-1}\left\{\dfrac{w_j + w_m|q_j|}{w_{j+1}}\right\}$, *where* $w_0 = 0$.
2. $\left[\max\limits_{1\le j\le m-1}\left\{\dfrac{w_j}{w_{j+1}}, \sum\limits_{i=0}^{m-1}\dfrac{w_m|q_i|}{|q_0|w_{i+1}}\right\}\right]^{-1} \le |\tilde{z}| \le \max\limits_{1\le j\le m-1}\left\{\dfrac{w_j}{w_{j+1}}, \sum\limits_{i=0}^{m-1}\dfrac{w_m|q_i|}{w_{i+1}}\right\}$.

Proof. Substituting $\gamma = 0, 1$ in Theorem 4.6, we get the desired results. ■

Let $w_j = w_m|q_j|, 1 \le j \le m-1$, in the part (1) of Corollary 4.7. Then we obtain

$$|\tilde{z}| \le \max_{1\le j\le m-1}\left\{\left|\frac{q_0}{q_1}\right|, 2\left|\frac{q_j}{q_{j+1}}\right|\right\}.$$

This is called the Kojima type bound for the zeros of the simple monic polynomial $p_l(z)$.

For computation of bounds of the zeros of $p_r(z)$, we define the following polynomial:

$$\tilde{p}_l(z) := \overline{p_r(\overline{z})} := \sum_{j=0}^{m} \overline{q_j} z^j,\ q_j \in \mathbb{H}.$$

Now, we discuss the following theorem which shows relation between the zeros of $p_r(z)$ and $\tilde{p}_l(z)$.

Theorem 4.8. *Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_r(z)$ if and only if $\overline{\lambda}$ is a zero of the simple monic polynomial $\tilde{p}_l(z)$.*

Proof. The corresponding companion matrices of $p_r(z)$ and $\tilde{p}_l(z)$ are given by

$$C_{p_r} := C_{p_l}^T \text{ and } C_{\tilde{p}_l} := C_{p_r}^H,$$

respectively. By Lemma 3.1, if λ is a left eigenvalue of C_{p_r}, then $\overline{\lambda}$ is a left eigenvalue of $C_{p_r}^H = C_{\tilde{p}_l}$. By Propositions 4.1 and 4.2, the left eigenvalues of C_{p_r} and $C_{\tilde{p}_l}$ imply the zeros of $p_r(z)$ and $\tilde{p}_l(z)$, respectively. Hence if λ is a zero of $p_r(z)$, then $\overline{\lambda}$ is also a zero of $\tilde{p}_l(z)$. ■

Remark 4.9. *Similar results can be obtained for the quaternionic polynomial $p_r(z)$ as well.*

5. Bounds for the zeros of quaternionic polynomials by using the powers of companion matrices

We present some preliminaries results for the powers of companion matrices C_{p_l} and C_{p_r}. In general, if λ is a left eigenvalue of a quaternionic matrix A, then λ^2 is not necessarily a left eigenvalue of A^2. For example, for a quaternionic matrix $A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}$, we have $\Lambda_l(A) := \left\{\mu : \mu = \alpha + \beta\mathbf{j} + \gamma\mathbf{k}, \alpha^2 + \beta^2 + \gamma^2 = 1\right\}$ and $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So $\Lambda_l(A^2) := \{1\}$. Here $\mathbf{j}$ is a left eigenvalue of A but $\mathbf{j}^2$ is not a left eigenvalue of A^2.

Proposition 5.1. *If λ is a left eigenvalue of C_{p_l} with respect to the eigenvector $x \in \mathbb{H}^n$, then λ^t is a left eigenvalue of $C^t_{p_l}$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.*

Proof. **Case** (*a*): Let t be a positive integer and let λ be a left eigenvalue of C_{p_l}. Then, there exists $0 \neq x := \left[1, \lambda, \lambda^2, \dots, \lambda^{m-1}\right]^T \in \mathbb{H}^n$ such that $C_{p_l}x = \lambda x$. Therefore,

$$\begin{aligned} C^2_{p_l}x &= C_{p_l}(C_{p_l}x) = C_{p_l}x\lambda = x\lambda^2 \\ &\vdots \\ C^t_{p_l}x &= C^{t-1}_{p_l}(C_{p_l}x) = C^{t-1}_{p_l}x\lambda = \cdots = x\lambda^t = \lambda^t x. \end{aligned}$$

Thus, λ^t is a left eigenvalue of matrix $C^t_{p_l}$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.

Case (*b*): Let t be a negative integer. From **Case (a)**, we have $C_{p_l}x = x\lambda$. This implies $C^{-1}_{p_l}x = x\lambda^{-1}$. Therefore,

$$\begin{aligned} C^{-2}_{p_l}x &= C^{-1}_{p_l}(C^{-1}_{p_l}x) = C^{-1}_{p_l}x\lambda^{-1} = x\lambda^{-2} \\ &\vdots \\ C^t_{p_l}x &= C^{(t+1)}_{p_l}(C^{-1}_{p_l}x) = C^{(t+1)}_{p_l}x\lambda^{-1} = \cdots = x\lambda^t = \lambda^t x. \end{aligned}$$

Thus, λ^t is a left eigenvalue of $C^t_{p_l}$ with respect to the same eigenvector $x \in \mathbb{H}^n$. ■

Next, we state the following result for left eigenvalues of C_{p_r} and $C^t_{p_r}$ (t is a nonzero integer).

Proposition 5.2. *If λ is a left eigenvalue of C_{p_r} with respect to the eigenvector $x \in \mathbb{H}^n$, then λ^t (t is a nonzero integer) is a left eigenvalue of $C^t_{p_r}$ corresponding to the same eigenvector $x \in \mathbb{H}^n$.*

Proof. **Case** (*a*): Let t be a positive integer and let λ be a left eigenvalue of C_{p_r}. Now from Lemma 3.1, $\overline{\lambda}$ is a left eigenvalue of $C^H_{p_r}$. Then there exists $0 \neq x := \left[1, \overline{\lambda}, (\overline{\lambda})^2, \dots, (\overline{\lambda})^{m-1}\right] \in \mathbb{H}^n$ such that $C^H_{p_r}x = \overline{\lambda}x = x\overline{\lambda}$. This gives

$$\begin{aligned} \left(C^H_{p_r}\right)^2 x &= C^H_{p_r}(C^H_{p_r}x) = C^H_{p_r}x\overline{\lambda} = x(\overline{\lambda})^2 \\ &\vdots \\ \left(C^H_{p_r}\right)^t x &= \left(C^H_{p_r}\right)^{t-1}(C^H_{p_r}x) = \left(C^H_{p_r}\right)^{t-1} x\overline{\lambda} = \cdots = x(\overline{\lambda})^t = (\overline{\lambda})^t x. \end{aligned}$$

Thus, $(\overline{\lambda})^t$ is a left eigenvalue of $\left(C^H_{p_r}\right)^t$. Then by Lemma 3.1, λ^t is a left eigenvalue of $C^t_{p_r}$.

Case (*b*): Let t be a negative integer. From **Case (a)**, we have $C^H_{p_r}x = \overline{\lambda}x = x\overline{\lambda}$. This implies $(C^H_{p_r})^{-1}x = x(\overline{\lambda})^{-1}$. Thus

$$\begin{aligned} (C^H_{p_r})^{-2}x &= (C^H_{p_r})^{-1}\{(C^H_{p_r})^{-1}x\} = (C^H_{p_r})^{-1}x(\overline{\lambda})^{-1} = x(\overline{\lambda})^{-2} \\ &\vdots \\ (C^H_{p_r})^t x &= (C^H_{p_r})^{(t+1)}\{(C^H_{p_r})^{-1}x\} = (C^H_{p_r})^{(t+1)}x(\overline{\lambda})^{-1} = \cdots = x(\overline{\lambda})^t = (\overline{\lambda})^t x. \end{aligned}$$

Thus, $(\overline{\lambda})^t$ is a left eigenvalue of $\left(C_{p_r}^H\right)^t$. Then by Lemma 3.1, λ^t is a left eigenvalue of $C_{p_r}^t$. ∎

Further, we present a framework to find the powers of the companion matrix C_{p_l} which can be derived in a simple procedure as follows, keeping in view that quaternions do not commute.

Theorem 5.3. *Consider* $C_{p_l} = \begin{array}{c|c} & \begin{array}{cc} 1 & m-1 \end{array} \\ \begin{array}{c} m-1 \\ 1 \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m,2:m) \end{array}\right] \end{array}$.
(*a*) *If $t < m$ is a positive integer, then*

$$C_{p_l}^t = \begin{array}{cc} & \begin{array}{cc} t & m-t \end{array} \\ \begin{array}{c} m-t \\ t \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \tag{18}$$

(*b*) *if $t \geq m$, then*

$$C_{p_l}^t = \begin{bmatrix} C_{p_l}^{t-(m-1)}(m,1:m) \\ C_{p_l}^{t-(m-2)}(m,1:m) \\ \vdots \\ C_{p_l}^{t-1}(m,1:m) \\ C_{p_l}^{t}(m,1:m) \end{bmatrix}_{m\times m}, \tag{19}$$

where

$$\begin{aligned} C_{p_l}^t(m,1) &:= C_{p_l}^{t-1}(m,m)C_{p_l}(m,1), \\ C_{p_l}^t(m,2:m) &:= C_{p_l}^{t-1}(m,1:m-1) + C_{p_l}^{t-1}(m,m)C_{p_l}(m,2:m), \end{aligned}$$

$$C := \begin{bmatrix} C_{p_l}(m,1:t) \\ C_{p_l}^2(m,1:t) \\ \vdots \\ C_{p_l}^t(m,1:t) \end{bmatrix}_{t\times t}, \text{ and } D := \begin{bmatrix} C_{p_l}(m,t+1:m) \\ C_{p_l}^2(m,t+1:m) \\ \vdots \\ C_{p_l}^t(m,t+1:m) \end{bmatrix}_{t\times(m-t)}.$$

Note that $C_{p_l}(k,1:m)$ denotes the k-th row of the matrix C_{p_l}.

Proof. Assuming $t = 1$, (18) becomes $C_{p_l} = \begin{array}{cc} & \begin{array}{cc} 1 & m-1 \end{array} \\ \begin{array}{c} m-1 \\ 1 \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(m,1) & C_{p_l}(m,2:m) \end{array}\right] \end{array}$, where $C_{p_l}(m,1) := -q_0, C_{p_l}(m,2:m) := [-q_1 \ldots -q_{m-1}]$. Thus the theorem is true for $t = 1$. Now, let us consider C_{p_l} as

$$C_{p_l} = \begin{array}{cc} & \begin{array}{cc} m-k & k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array}, \text{ where}$$

$A' := C_{p_l}(1:k, 1:m-k), B' := C_{p_l}(k+1:m, m-k+1:m), C' := C_{p_l}(k+1:m, 1:m-k), D' := C_{p_l}(k+1:m, m-k+1:m)$. For $t = k = 3$, we get

$$C_{p_l}^3 = \begin{array}{cc} & \begin{array}{cc} 2 & m-2 \end{array} \\ \begin{array}{c} m-2 \\ 2 \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array} \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} 2 \\ m-2 \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} m-2 \\ 2 \end{array} & \left[\begin{array}{c|c} C' & D' \\ \hline CA' + DC' & CB' + DD' \end{array}\right] \end{array}.$$

Note that in each step, size of the identity matrix I reduces by order 1 and the size of matrix C increases by order 1. Similarly, the matrix D increases by 1 row and decreases by 1 column. Finally, after rearranging and separating 0 and I matrices we get

$$\begin{array}{c|c} & \begin{array}{cc} 2+1 & m-2-1 \end{array} \\ \begin{array}{c} m-2-1 \\ 2+1 \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right], \end{array}$$

where C and D are of size 3×3 and $3 \times (m-3)$, respectively. Assuming that the theorem is true for $t = k$, we have

$$C_{p_l}^{k+1} = C_{p_l}^{k} C_{p_l} = \begin{array}{c} m-k \\ k \end{array} \overset{\begin{array}{cc} m-k & k \end{array}}{\left[\begin{array}{c|c} C' & D' \\ \hline CA' + DC' & CB' + DD' \end{array}\right]} = \begin{array}{c} m-k-1 \\ k+1 \end{array} \overset{\begin{array}{cc} k+1 & m-k-1 \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right]},$$

where the corresponding C and D matrices are given in the statement of the theorem.
The proof for $t \geq m$ is similar. ■

In the case of quaternionic matrix, $C_{p_l} = C_{p_r}^T$ but $C_{p_r}^t \neq (C_{p_l}^t)^T$ for $t \geq 2$. This is illustrated by the following example.

Example 5.4. Consider the following simple monic polynomials over $\mathbb{H}$:

$$p_l(z) = z^3 - \mathbf{k}z^2 + (\mathbf{k} - \mathbf{j})z + (\mathbf{i} + \mathbf{j}) \text{ and } p_r(z) = z^3 - z^2\mathbf{k} + z(\mathbf{k} - \mathbf{j}) + (\mathbf{i} + \mathbf{j}).$$

The corresponding companion matrices of $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} = \begin{array}{c} 2 \\ 1 \end{array} \overset{\begin{array}{cc} 1 & 2 \end{array}}{\left[\begin{array}{c|c} 0 & I \\ \hline C_{p_l}(3,1) & C_{p_l}(3,2:3) \end{array}\right]} \text{ and } C_{p_r} = C_{p_l}^T,$$

respectively, where $C_{p_l}(3,1) = -\mathbf{i} - \mathbf{j}$ and $C_{p_l}(3,2:3) := [\mathbf{j} - \mathbf{k}, \mathbf{k}]$. Then

$$C_{p_l}^2 = \begin{bmatrix} 0 & 0 & 1 \\ -\mathbf{i} - \mathbf{j} & \mathbf{j} - \mathbf{k} & \mathbf{k} \\ \mathbf{i} - \mathbf{j} & \mathbf{1} - 2\mathbf{i} - \mathbf{j} & \mathbf{j} - \mathbf{k} - \mathbf{1} \end{bmatrix} \text{ and } C_{p_r}^2 = \begin{bmatrix} 0 & -\mathbf{i} - \mathbf{j} & \mathbf{j} - \mathbf{i} \\ 0 & \mathbf{j} - \mathbf{k} & \mathbf{1} - \mathbf{j} \\ 1 & \mathbf{k} & \mathbf{j} - \mathbf{k} - \mathbf{1} \end{bmatrix}.$$

This shows that $C_{p_r}^2 \neq (C_{p_l}^2)^T$.

Hence, we can derive results analogous to Theorem 5.3 for the case of $C_{p_r}^t$, $t \geq 2$.

Theorem 5.5. *Consider* $C_{p_r} = \begin{array}{c} 1 \\ m-1 \end{array} \overset{\begin{array}{cc} m-1 & 1 \end{array}}{\left[\begin{array}{c|c} 0 & C_{p_r}(1,m) \\ \hline I & C_{p_r}(2:m,m) \end{array}\right]}$.

(*a*) *If $t < m$ is a positive integer, then*

$$C_{p_r}^t = \begin{array}{c} t \\ m-t \end{array} \overset{\begin{array}{cc} m-t & t \end{array}}{\left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right]}, \tag{20}$$

(*b*) *if $t \geq m$, then*

$$C_{p_r}^t = \left[\; C_{p_r}^{t-(m-1)}(1:m,m) \quad C_{p_r}^{t-(m-2)}(1:m,m) \quad \ldots \quad C_{p_r}^{t-1}(1:m,m) \quad C_{p_r}^{t}(1:m,m) \;\right]_{m\times m},$$

where

$$\begin{aligned} C &:= \left[C_{p_r}(1:t,m) \quad C_{p_r}^2(1:t,m) \quad \ldots \quad C_{p_r}^t(1:t,m)\right], \\ D &:= \left[C_{p_r}(t+1:m,m) \quad C_{p_r}^2(t+1:m,m) \quad \ldots \quad C_{p_r}^t(t+1:m,m)\right], \\ C_{p_r}^t(1,m) &:= C_{p_r}(1,m)\, C_{p_r}^{t-1}(m,m), \text{ and} \\ C_{p_r}^t(2:m,m) &:= C_{p_r}^{t-1}(1:m-1,m) + C_{p_r}(2:m,m)\, C_{p_r}^{t-1}(m,m). \end{aligned}$$

Proof. The proof follows from the proof method of Theorem 5.3. ■

Polynomials from Example 5.4 satisfy

$$\tilde{p}_l(z) := \overline{p_r(\overline{z})} = z^3 + \mathbf{k}z^2 + (\mathbf{j}-\mathbf{k})z + (-\mathbf{i}-\mathbf{j}), \text{ and } \tilde{p}_r(z) := \overline{p_l(\overline{z})} = z^3 + z^2\mathbf{k} + z(\mathbf{j}-\mathbf{k}) - (\mathbf{i}+\mathbf{j}).$$

Thus the companion matrices corresponding to $\tilde{p}_l(z)$ and $\tilde{p}_r(z)$ are given by $C_{\tilde{p}_l} = \overline{C_{p_l}}$ and $C_{\tilde{p}_r} = \overline{C_{p_r}}$, respectively. Next,

$$C^2_{\tilde{p}_l} = \begin{bmatrix} 0 & 0 & 1 \\ \mathbf{i}+\mathbf{j} & -\mathbf{j}+\mathbf{k} & -\mathbf{k} \\ \mathbf{i}-\mathbf{j} & \mathbf{1}+\mathbf{j} & \mathbf{k}-\mathbf{j}-\mathbf{1} \end{bmatrix} \text{ and } C^2_{\tilde{p}_r} = \begin{bmatrix} 0 & \mathbf{i}+\mathbf{j} & \mathbf{j}-\mathbf{i} \\ 0 & -\mathbf{j}+\mathbf{k} & \mathbf{1}+\mathbf{2i}+\mathbf{j} \\ 1 & -\mathbf{k} & -\mathbf{1}-\mathbf{j}+\mathbf{k} \end{bmatrix}.$$

Then

$$\max_{1\le i\le 3}\left[(r'_i(C^2_{p_l}))^{1/2}\right] = 2.3655 \text{ and } \max_{1\le i\le 3}\left[(r'_i(C^2_{\tilde{p}_r}))^{1/2}\right] = 1.9656,$$

$$\max_{1\le i\le 3}\left[\left(r'_i\left(C^2_{p_r}\right)\right)^{1/2}\right] = 1.9319 \text{ and } \max_{1\le i\le 3}\left[(r'_i(C^2_{\tilde{p}_l}))^{1/2}\right] = 2.1355.$$

Now, we have

$$\max_{1\le i\le 3}\left[(r'_i(C^2_{p_l}))^{1/2}\right] \neq \max_{1\le i\le 3}\left[(r'_i(C^2_{\tilde{p}_r}))^{1/2}\right] \text{ and } \max_{1\le i\le 3}\left[\left(r'_i\left(C^2_{p_r}\right)\right)^{1/2}\right] \neq \max_{1\le i\le 3}\left[(r'_i(C^2_{\tilde{p}_l}))^{1/2}\right].$$

Further, we have the following bounds for the zeros of $p_l(z)$ and $p_r(z)$ for $\gamma \in [0, 1]$.

Theorem 5.6. *Let $p_l(z)$ and $p_r(z)$ be the simple monic polynomials over $\mathbb{H}$ of degree m and let $C^t_{p_l}$ and $C^t_{p_r}$ $(t \ge 2)$ be the t-th power of the companion matrices C_{p_l} and C_{p_r}, corresponding to $p_l(z)$ and $p_r(z)$, respectively. Then, for $\gamma \in [0,1]$ bounds for every zero $\tilde{z}$ of $p_l(z)$ satisfy the following inequalities:*

$$\left(\max_{1\le i\le m}\left[\left(r'_i\left(C^t_{q_l}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{q_l}\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \max_{1\le i\le m}\left[\left(r'_i\left(C^t_{p_l}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{p_l}\right)\right)^{(1-\gamma)/t}\right], \tag{21}$$

$$\left(\max_{1\le i\le m}\left[\left(r'_i\left(C^t_{\tilde{q}_r}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{\tilde{q}_r}\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \max_{1\le i\le m}\left[\left(r'_i\left(C^t_{\tilde{p}_r}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{\tilde{p}_r}\right)\right)^{(1-\gamma)/t}\right], \tag{22}$$

and bounds for every zero $\tilde{z}$ of $p_r(z)$ satisfy the following inequalities:

$$\left(\max_{1\le i\le m}\left[\left(r'_i\left(C^t_{q_r}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{q_r}\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \max_{1\le i\le m}\left[\left(r'_i\left(C^t_{p_r}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{p_r}\right)\right)^{(1-\gamma)/t}\right], \tag{23}$$

$$\left(\max_{1\le i\le m}\left[\left(r'_i\left(C^t_{\tilde{q}_l}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{\tilde{q}_l}\right)\right)^{(1-\gamma)/t}\right]\right)^{-1} \le |\tilde{z}| \le \max_{1\le i\le m}\left[\left(r'_i\left(C^t_{\tilde{p}_l}\right)\right)^{\gamma/t}\left(c'_i\left(C^t_{\tilde{p}_l}\right)\right)^{(1-\gamma)/t}\right]. \tag{24}$$

Proof. Let λ be a left eigenvalue of C_{p_l}. Then by Proposition 5.1, λ^t ($t \ge 2$ is positive integer) is a left eigenvalue of $C^t_{p_l}$. Hence by applying Theorem 3.3, we get (21).

By Lemma 3.1, $\overline{\lambda}$ is a left eigenvalue of $C_{\tilde{p}_r}$ and by Proposition 5.2, $(\overline{\lambda})^t$ is a left eigenvalue of $(C_{\tilde{p}_r})^t$. Then from Theorem 3.3, (22) follows. The proof of (23) and (24) are similar. ■

Substituting $t = 2$ and $\gamma = 1$ in Theorem 5.6, we have the following corollary.

Corollary 5.7. *Let $p_l(z)$ and $p_r(z)$ be the simple monic polynomials over $\mathbb{H}$ of degree m. Then bounds for every zero $\tilde{z}$ of $p_l(z)$ satisfy the following inequalities:*

$$\frac{1}{\beta_1} \le |\tilde{z}| \le \alpha_1 \text{ and } \frac{1}{\beta_2} \le |\tilde{z}| \le \alpha_2,$$

where

$$
\begin{aligned}
\alpha_1 &= \max\left\{1, \left(\sum_{j=0}^{m-1} |q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |q_{m-1}q_j - q_{j-1}|\right)^{1/2}\right\},\\
\alpha_2 &= \max_{2\le j\le m-1}\left\{(|q_0| + |\overline{q_0}\ \overline{q_{m-1}}|)^{1/2}, (|q_1| + |\overline{q_1}\ \overline{q_{m-1}} - \overline{q_0}|)^{1/2}, \left(1 + |q_j| + |\overline{q_j}\ \overline{q_{m-1}} - \overline{q_{j-1}}|\right)^{1/2}\right\},\\
\beta_1 &= \max\left\{1, \left(\sum_{j=1}^{m-1} |q_0^{-1}q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |q_0^{-1}q_1q_0^{-1}q_{m-j} - q_0^{-1}q_{m-j+1}|\right)^{1/2}\right\},\\
\beta_2 &= \max_{2\le j\le m-1}\left\{\left(|q_0^{-1}| + |\overline{q_0^{-1}}\ \overline{q_1q_0^{-1}}|\right)^{1/2}, \left(|q_{m-1}q_0^{-1}| + |\overline{q_{m-1}q_0^{-1}}\ \overline{q_1q_0^{-1}} - \overline{q_0^{-1}}|\right)^{1/2},\right.\\
&\left.\left(1 + |q_{m-j}q_0^{-1}| + |\overline{q_{m-j}q_0^{-1}}\ \overline{q_1q_0^{-1}} - \overline{q_{m-j+1}q_0^{-1}}|\right)^{1/2}\right\},
\end{aligned}
$$

and bounds for every zero $\tilde{z}$ of $p_r(z)$ satisfy the following inequalities:

$$\frac{1}{\beta_3} \le |\tilde{z}| \le \alpha_3, \text{ and } \frac{1}{\beta_4} \le |\tilde{z}| \le \alpha_4,$$

where

$$
\begin{aligned}
\alpha_3 &= \max_{2\le j\le m-1}\left\{(|q_0| + |q_0\ q_{m-1}|)^{1/2}, (|q_1| + |q_1\ q_{m-1} - q_0|)^{1/2}, \left(1 + |q_j| + |q_j\ q_{m-1} - q_{j-1}|\right)^{1/2}\right\},\\
\alpha_4 &= \max\left\{1, \left(\sum_{j=0}^{m-1} |q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |\overline{q_{m-1}}\ \overline{q_j} - \overline{q_{j-1}}|\right)^{1/2}\right\},\\
\beta_3 &= \max_{2\le j\le m-1}\left\{\left(|q_0^{-1}| + |q_0^{-1}\ q_1q_0^{-1}|\right)^{1/2}, \left(|q_{m-1}q_0^{-1}| + |q_{m-1}q_0^{-1}\ q_1q_0^{-1} - q_0^{-1}|\right)^{1/2},\right.\\
&\left.\left(1 + |q_{m-j}q_0^{-1}| + |q_{m-j}q_0^{-1}\ q_1q_0^{-1} - q_{m-j+1}q_0^{-1}|\right)^{1/2}\right\},\\
\beta_4 &= \max\left\{1, \left(\sum_{j=1}^{m-1} |q_0^{-1}q_j|\right)^{1/2}, \left(\sum_{j=0}^{m-1} |\overline{q_0^{-1}q_1}\ \overline{q_0^{-1}q_{m-j}} - \overline{q_0^{-1}q_{m-j+1}}|\right)^{1/2}\right\}, q_{-1} = 0 = q_{m+1}, q_m = 1.
\end{aligned}
$$

Proof. The proof follows from Theorem 5.6 and Appendix A. ∎

Example 5.8. Consider the following polynomials $p_l(z)$ and $p_r(z)$ over $\mathbb{H}$:

$$
\begin{aligned}
p_l(z) &= z^6 + (\mathbf{i} + 3\mathbf{k})z^5 + (3 + \mathbf{j})z^4 + (5\mathbf{i} + 15\mathbf{k})z^3 + (-4 + 5\mathbf{j})z^2 + (6\mathbf{i} + 18\mathbf{k})z + (6\mathbf{j} - 12),\\
p_r(z) &= z^6 + z^5(\mathbf{i} + 3\mathbf{k}) + z^4(3 + \mathbf{j}) + z^3(5\mathbf{i} + 15\mathbf{k}) + z^2(-4 + 5\mathbf{j}) + z(6\mathbf{i} + 18\mathbf{k}) + (6\mathbf{j} - 12).
\end{aligned}
$$

The zeros of $p_l(z)$ are given in [32]. Moreover, we find the zeros of $p_r(z)$ by Niven's algorithm [23]. Thus, the zeros and bounds for the zeros of $p_l(z)$ and $p_r(z)$. are given in the following tables.

6. Conclusion

In this paper, we have derived Ostrowski type theorem for left eigenvalues of a quaternionic matrix that generalizes Ostrowski type theorem for right eigenvalues of a quaternionic matrix when all the diagonal entries of a quaternionic matrix are real. We have derived a corrected version of the Brauer type theorem for left eigenvalues for the deleted absolute column sums of a quaternionic matrix. Moreover, we have extended localization theorems by applying the generalized Hölder inequality for left as well as right eigenvalues of a quaternionic matrix. Bounds for the zeros of quaternionic polynomials have derived. As

z_1	$\lvert z_1\rvert$	z_2	$\lvert z_2\rvert$
$-\mathbf{i}-2\mathbf{k}$	2.2361	$-0.4\mathbf{i}-2.2\mathbf{k}$	2.2361
$[\mathbf{i}\sqrt{3}]$	1.7321	$[\mathbf{i}\sqrt{3}]$	1.7321
$[\mathbf{i}\sqrt{2}]$	1.4142	$[\mathbf{i}\sqrt{2}]$	1.4142
$-0.6\mathbf{i}-0.8\mathbf{k}$	1	$-\mathbf{k}$	1

Table 1: Zeros of $p_l(z)$ and $p_r(z)$ and their absolute values, where z_1 and z_2 are the set of zeros of $p_l(z)$, and $p_r(z)$, respectively.

Example 5.8	lower bound	upper bound
Corollary 4.4 (1)	0.4142	19.9737
Corollary 4.4 (2)	0.2766	60.9291
Theorem 4.3, $\gamma = 1/4$	0.3744	8.1415

Table 2: Lower and upper bounds for the zeros of $p_l(z)$ and $p_r(z)$.

Example 5.4	lower bound	lower bound
Corollary 5.7 1(a)	0.6156	2.3655
Corollary 5.7 1(b)	0.6078	1.9656
Corollary 5.7 2(a)	0.6078	1.9319
Corollary 5.7 2(b)	0.6436	2.1355

Table 3: Lower and upper bounds for the zeros of $p_l(z)$ and $p_r(z)$.

a consequence, we have shown that some of our bounds are sharper than the bound given in [24]. Further, we have derived bounds via the powers of companion matrices which are always sharper than the bound given in [24].

Appendix A.

In this appendix, we state formulas for the squares of quaternionic companion matrices. For $t = 2$, Theorem 5.3 implies

$$C_{p_l}^2 = \begin{array}{c} \\ {\scriptstyle m-2} \\ {\scriptstyle 2} \end{array}\!\!\begin{array}{c} \begin{array}{cc} {\scriptstyle 2} & {\scriptstyle m-2} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C := \begin{bmatrix} C_{p_l}(m,1:2) \\ C_{p_l}^2(m,1:2) \end{bmatrix} = \begin{bmatrix} -q_0 & -q_1 \\ q_{m-1}q_0 & q_{m-1}q_1 - q_0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} C_{p_l}(m,3:m) \\ C_{p_l}^2(m,3:m) \end{bmatrix} = \begin{bmatrix} -q_2 & -q_3 & \dots & -q_{m-1} \\ q_{m-1}q_2 - q_1 & q_{m-1}q_3 - q_1 & \dots & (q_{m-1})^2 - q_{m-2} \end{bmatrix},$$

$$C_{\tilde{p}_l}^2 = \begin{array}{c} \\ {\scriptstyle m-2} \\ {\scriptstyle 2} \end{array}\!\!\begin{array}{c} \begin{array}{cc} {\scriptstyle 2} & {\scriptstyle m-2} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} C_{\tilde{p}_l}(m,1:2) \\ C_{\tilde{p}_l}^2(m,1:2) \end{bmatrix} = \begin{bmatrix} -\overline{q_0} & -\overline{q_1} \\ \overline{q_{m-1}}\,\overline{q_0} & \overline{q_{m-1}}\,\overline{q_1} - \overline{q_0} \end{bmatrix}$$

and

$$D = \begin{bmatrix} C_{\tilde{p}_l}(m,3:m) \\ C_{\tilde{p}_l}^2(m,3:m) \end{bmatrix} = \begin{bmatrix} -\overline{q_2} & -\overline{q_3} & \dots & -\overline{q_{m-1}} \\ \overline{q_{m-1}}\,\overline{q_2} - \overline{q_1} & \overline{q_{m-1}}\,\overline{q_3} - \overline{q_1} & \dots & (\overline{q_{m-1}})^2 - \overline{q_{m-2}} \end{bmatrix},$$

$$C_{q_l}^2 = \begin{array}{c} \\ {\scriptstyle m-2} \\ {\scriptstyle 2} \end{array}\!\!\begin{array}{c} \begin{array}{cc} {\scriptstyle 2} & {\scriptstyle m-2} \end{array} \\ \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -q_0^{-1} & -q_0^{-1}q_{m-1} \\ q_0^{-1}q_1q_0^{-1} & q_0^{-1}q_1q_0^{-1}q_{m-1} - q_0^{-1} \end{bmatrix}$$

and

$$D = \begin{bmatrix} -q_0^{-1}q_{m-2} & \cdots & -q_0^{-1}q_1 \\ q_0^{-1}q_1q_0^{-1}q_{m-2} - q_0^{-1}q_{m-1} & \cdots & (q_0^{-1}q_1)^2 - q_0^{-1}q_2 \end{bmatrix},$$

$$C^2_{\tilde{q}_l} = \begin{array}{cc} & \begin{array}{cc} 2 & m-2 \end{array} \\ \begin{array}{c} m-2 \\ 2 \end{array} & \left[\begin{array}{c|c} 0 & I \\ \hline C & D \end{array}\right] \end{array}, \text{ where } C = \begin{bmatrix} -\overline{q_0^{-1}} & -\overline{q_0^{-1}q_{m-1}} \\ \overline{q_0^{-1}q_1}\,\overline{q_0^{-1}} & \overline{q_0^{-1}q_1}\,\overline{q_0^{-1}q_{m-1}} - \overline{q_0^{-1}} \end{bmatrix}$$

and

$$D = \begin{bmatrix} -\overline{q_0^{-1}q_{m-2}} & \cdots & -\overline{q_0^{-1}q_1} \\ \overline{q_0^{-1}q_1}\,\overline{q_0^{-1}q_{m-2}} - \overline{q_0^{-1}q_{m-1}} & \cdots & \left(\overline{q_0^{-1}q_1}\right)^2 - \overline{q_0^{-1}q_2} \end{bmatrix}.$$

For $t = 2$, Theorem 5.5 implies

$$C^2_{p_r} = \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} 2 \\ m-2 \end{array} & \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{where } C = \begin{bmatrix} C_{p_r}(1:2,m) & C^2_{p_r}(1:2,m) \end{bmatrix} = \begin{bmatrix} -q_0 & q_0q_{m-1} \\ -q_1 & q_1q_{m-1} - q_0 \end{bmatrix},$$

and

$$D = \begin{bmatrix} C_{p_r}(3:m,m) & C^2_{p_r}(3:m,m) \end{bmatrix} = \begin{bmatrix} -q_2 & q_2q_{m-1} - q_1 \\ -q_3 & q_3q_{m-1} - q_2 \\ \vdots & \vdots \\ -q_{m-1} & (q_{m-1})^2 - q_{m-2} \end{bmatrix},$$

$$C^2_{\tilde{p}_r} = \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} 2 \\ m-2 \end{array} & \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{where } C = \begin{bmatrix} -\overline{q_0} & \overline{q_0}\,\overline{q_{m-1}} \\ -\overline{q_1} & \overline{q_1}\,\overline{q_{m-1}} - \overline{q_0} \end{bmatrix} \text{ and } D = \begin{bmatrix} -\overline{q_2} & \overline{q_2}\,\overline{q_{m-1}} - \overline{q_1} \\ -\overline{q_3} & \overline{q_3}\,\overline{q_{m-1}} - \overline{q_2} \\ \vdots & \vdots \\ -\overline{q_{m-1}} & (\overline{q_{m-1}})^2 - \overline{q_{m-2}} \end{bmatrix},$$

$$C^2_{q_r} = \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} 2 \\ m-2 \end{array} & \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{where}$$

$$C = \begin{bmatrix} -q_0^{-1} & q_0^{-1}\,q_1q_0^{-1} \\ -q_{m-1}q_0^{-1} & q_{m-1}q_0^{-1}q_1q_0^{-1} - q_0^{-1} \end{bmatrix} \text{ and } D = \begin{bmatrix} -q_{m-2}q_0^{-1} & q_{m-2}q_0^{-1}q_1q_0^{-1} - q_{m-1}q_0^{-1} \\ \vdots & \vdots \\ -q_1q_0^{-1} & (q_1q_0^{-1})^2 - q_2q_0^{-1} \end{bmatrix},$$

$$C^2_{\tilde{q}_r} = \begin{array}{cc} & \begin{array}{cc} m-2 & 2 \end{array} \\ \begin{array}{c} 2 \\ m-2 \end{array} & \left[\begin{array}{c|c} 0 & C \\ \hline I & D \end{array}\right] \end{array}, \text{where}$$

$$C = \begin{bmatrix} -\overline{q_0^{-1}} & \overline{q_0^{-1}}\,\overline{q_1q_0^{-1}} \\ -\overline{q_{m-1}q_0^{-1}} & \overline{q_{m-1}q_0^{-1}}\,\overline{q_1q_0^{-1}} - \overline{q_0^{-1}} \end{bmatrix} \text{ and } D = \begin{bmatrix} -\overline{q_{m-2}q_0^{-1}} & \overline{q_{m-2}q_0^{-1}}\,\overline{q_1q_0^{-1}} - \overline{q_{m-1}q_0^{-1}} \\ \vdots & \vdots \\ -\overline{q_1q_0^{-1}} & \left(\overline{q_1q_0^{-1}}\right)^2 - \overline{q_2q_0^{-1}} \end{bmatrix}.$$

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