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A Multiplicity Results for a Singular Problem Involving a Riemann-Liouville Fractional Derivative

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Abstract. In this work, we investigate the following nonlinear singular problem with Riemann-Liouville Fractional Derivative

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_1^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}\ {}_0D_t^\alpha u(t)\right) = \frac{g(t)}{u^\gamma(t)} + \lambda f(t,u(t))\ t\in(0,T); \\ \\ u(0)=u(T)=0, \end{cases}$$

where λ is a positive parameter, $p>1$, $\frac{1}{2}<\alpha\leq 1$, $0<\gamma<1$, $g\in C([0,T])$ and $f\in C([0,T]\times\mathbb{R},\mathbb{R})$. Under appropriate assumptions on the function f, we employ the method of the Nehari manifold combined with the fibering maps in order to show the existence of λ_0 such that for all $\lambda\in(0,\lambda_0)$ the problem (P_λ) has at least two positive solutions. Finally, some examples are given to illustrate our results.

1. Introduction

The purpose of this work is to study the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_1^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}\ {}_0D_t^\alpha u(t)\right) = \frac{g(t)}{u^\gamma(t)} + \lambda f(t,u(t)),\ \ t\in(0,T); \\ \\ u(0)=u(T)=0, \end{cases}$$

where λ is a positive parameter, $\frac{1}{2}<\alpha\leq 1$, $0<\gamma<1$ and $g\in C([0,1])$. While $f\in C\left([0,1]\times\mathbb{R},\mathbb{R}\right)$ is positively homogeneous of degree $r-1$ that is

$$f(x,tu) = t^{r-1}f(x,u) \text{ holds for all } (x,u)\in[0,T]\times\mathbb{R},$$

where r is such that $1<p<r$. Put $F(x,s) := \int_0^s f(x,t)dt$ and assume that F satisfies suitable growth conditions. Precisely, we assume the following:

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($\mathbf{H}_1$) $F:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is homogeneous of degree r that is

$$F(x,tu)=t^rF(x,u)\ (t>0)\ \text{ for all }\ x\in[0,T],\ u\in\mathbb{R}.$$

($\mathbf{H}_2$) $F^{\pm}(x,u)=max(\pm F(x,u),0)\neq 0$ for all $u\neq 0$.

Note that, from **($\mathbf{H}_1$)**, f leads to the so-called Euler identity

$$uf(t,u)=rF(t,u)$$

and

$$|F(t,u)|\leq K|u|^r \quad \text{for some constant } K>0. \tag{1}$$

The theory of fractional calculus may be used to the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [7, 8] and references therein.

By means of the critical point theory, Agrawal [1] discussed the existence of solutions for the following fractional boundary value problem

$$\begin{cases} -{}_tD_1^{\alpha}\,{}_0D_t^{\alpha}u(t)=\nabla F(t,u(t)),\ t\in(0,T)\\ u(0)=u(T)=0,\end{cases} \tag{2}$$

and obtained the existence of at least one nontrivial solution. We note that it is not easy to use the critical point theory to study (2), since it is often very difficult to establish a suitable space and variational functional for the fractional boundary value problem.

Based on the fixed point index theory, Xu-Yang [9] study the singular fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t)=h(t)f(t,u(t)),\ t\in(0,1)\\ u(0)=u(1)=u'(0)=u'(1)=0,\end{cases} \tag{3}$$

where $3<\alpha\leq 4$ is a real number, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f\in C([0,1]\times[0,\infty),[0,\infty))$, and $h\in C(0,1)\cap L(0,1)$ is nonnegative and may be singular at $t=0$ and/or $t=1$. The authors established two existence results of twin positive solutions for (3). It should be remarked our main results, even in the case of h being non-singular, essentially extend and improve the corresponding ones in the literature.

In this paper we want to contribute with the development of this new area on fractional differential equations theory. More precisely, our main results is the following.

Theorem 1.1. *Let $\frac{1}{2}<\alpha<1$, $1<p<r$, $0<\gamma<1$ and assume that f satisfies the conditions **($\mathbf{H}_1$)-($\mathbf{H}_2$)**. Then there exists a parameter $\lambda_0>0$, such that for all $\lambda\in(0,\lambda_0)$, problem (P_λ) has at least two nontrivial solutions.*

This paper is organized as follows. In Section 2, some preliminaries on the fractional calculus are presented. In Section 3, we set up the variational framework of problem (P_λ) and give some necessary lemmas. Finally, Section 4 presents the main result and its proof.

2. Preliminaries

In this section, we give some background theory on the concept of fractional calculus, in particular the Riemann-Liouville operators and results which will used throughout this paper. Let us start by introduce the definition of the fractional integral in the sense of Riemann-Liouville.

Definition 2.1. *Let $\alpha > 0$ and u be a function defined a.e. on $(a,b) \subset \mathbb{R}$ with values in $\mathbb{R}$. The Left (resp. right) fractional integral in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by*

$$ {}_aI_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad t \in (a,b] $$

respectively

$$ {}_tI_b^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}u(s)ds, \quad t \in [a,b), $$

provided the right side is point-wise defined on $[a,b]$, where Γ denotes Euler's Gamma function. If $u \in L^1(a,b)$, then ${}_aI_t^{\alpha}u$ and ${}_tI_b^{\alpha}u$ are defined a.e. on (a,b).

Now, we define the fractional derivative in the sense of Riemann-Liouville as follows.

Definition 2.2. *Let $0 < \alpha < 1$. Then, the Left (resp. right) fractional derivative in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by*

$$ {}_aD_t^{\alpha}u(t) = \frac{d}{dt}\left({}_aI_t^{1-\alpha}u\right)(t), \ \forall\ t \in (a,b], $$

respectively

$$ {}_tD_b^{\alpha}u(t) = \frac{d}{dt}\left({}_tI_b^{1-\alpha}u\right)(t), \ \forall\ t \in [a,b], $$

provided that the right-hand side is point-wise defined.

Remark 2.3. *From [7], if u is an absolutely continuous function in $[a,b]$. Then ${}_aD_t^{\alpha}u$ and ${}_tD_b^{\alpha}u$ are defined a.e. on (a,b) and satisfy*

$$ {}_aD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t) + \frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)} \tag{4} $$

and

$$ {}_tD_b^{\alpha}u(t) = -{}_tI_b^{1-\alpha}u'(t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)}. \tag{5} $$

Moreover, if $u(a) = u(b) = 0$, then ${}_aD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t)$ and ${}_tD_b^{\alpha}u(t) = -{}_tI_b^{1-\alpha}u'(t)$. So in this case we have the equality of Riemann-Liouville fractional derivative and Caputo derivative defined by

$$ {}_a^cD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t) $$

and

$$ {}_t^cD_b^{\alpha}u(t) = -\,{}_tI_b^{1-\alpha}u'(t). $$

Consequently, one gets

$$ {}_aD_t^{\alpha}u(t) = {}_a^cD_t^{\alpha}u(t) + \frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)}, $$

and

$$ {}_tD_b^{\alpha}u(t) = {}_t^cD_b^{\alpha}u(t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)}. $$

Next, we provide some properties concerning the left fractional operators of Riemann-Liouville. For more details we refer the reader to [3].

Proposition 2.4. *for any $\alpha,\ \beta > 0$ and any $u \in L^1(a,b)$, the following equality holds*

$$ {}_aI_t^{\alpha} \circ {}_aI_t^{\beta} u = {}_aI_t^{\alpha+\beta}. $$

From Property 2.4 and the equations (4) and (5), it is simple to deduce the following results concerning the composition between fractional integral and fractional derivative. That is, for any $0 < \alpha < 1$, if $u \in L^1(a,b)$ we have

$$ {}_aD_t^{\alpha} \circ {}_aI_t^{\alpha} u = u, $$

and if u is absolutely continuous such that $u(a) = 0$. Then, one has

$$ {}_aI_t^{\alpha} \circ {}_aD_t^{\alpha} u = u. $$

Now, we presented an important result on the boundness of the left fractional integral from $L^p(a,b)$ to $L^p(a,b)$:

Proposition 2.5. *for any $\alpha > 0$ and any $p \geq 1$, ${}_aI_t^{\alpha}$ is linear and continuous from $L^p(a,b)$ to $L^p(a,b)$. Moreover for all $u \in L^p(a,b)$, we have*

$$ \|{}_aI_t^{\alpha} u\|_p \leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} \|u\|_p. $$

In the same way, we give another classical result on the boundness of the left fractional integral from $L^p(a,b)$ to $C_a(a,b)$ which completes Property 2.5 in the case $0 < \frac{1}{p} < \alpha < 1$:

Proposition 2.6. *Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then, for any $u \in L^p(a,b)$, ${}_aI_t^{\alpha} u$ is Hölder continuous on $(a,b]$ with exponent $\alpha - \frac{1}{p} > 0$, moreover, ${}_aI_t^{\alpha} u(t) = 0$. Consequently, ${}_aI_t^{\alpha} u$ can be continuously extended by 0 in $t = a$. Finally, ${}_aI_t^{\alpha} u \in C_a(a,b)$, and*

$$ \|{}_aI_t^{\alpha} u\|_{\infty} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((a-1)q+1\right)^{\frac{1}{q}}} \|u\|_p. \tag{6} $$

Also, we will need the following formula for integration by parts:

Proposition 2.7. *Let $0 < \alpha < 1$ and p, q are such that*

$$ p \geq 1,\ q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \alpha \quad \text{or} \quad p \neq 1,\ q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \alpha. $$

Then, for all $u \in L^p(a,b)$ and all $v \in L^q(a,b)$, one has

$$ \int_a^b v(t)\ {}_aI_t^{\alpha} u(t) dt = \int_a^b u(t)\ {}_aI_t^{\alpha} v(t) dt, \tag{7} $$

and

$$ \int_a^b u(t)\ {}_a^cD_t^{\alpha} v(t) dt = v(t)\,{}_tI_b^{1-\alpha} u(t)\Big|_{t=a}^{t=b} + \int_a^b v(t)\ {}_aD_t^{\alpha} u(t) dt. \tag{8} $$

Moreover, if $v(a) = v(b) = 0$, then, one gets

$$ \int_a^b u(t)\ {}_aD_t^{\alpha} v(t) dt = \int_a^b v(t)\ {}_a^cD_t^{\alpha} u(t) dt. \tag{9} $$

3. A Variational setting and main results

To show the existence of solutions to the problem (P_λ) we will use critical point theory. For this purpose we introduce some basic notations and results, which we use to proof our main results.
The set of all functions $u \in C^\infty([0,T],\mathbb{R})$ with $u(0) = u(T) = 0$ is denoted by $C_0^\infty([0,T],\mathbb{R})$. For, $\alpha > 0$ we define the fractional derivative space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0,T],\mathbb{R})$ under the norm

$$\|u\|_{\alpha,p} = \left(\|u\|_p^p + \|{}_0^cD_t^\alpha u\|_p^p\right)^{\frac{1}{p}}. \tag{10}$$

Remark 3.1. *(i) It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T])$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^p([0,T])$ and $u(0) = u(T) = 0$.*

(ii) For any $u \in E_0^{\alpha,p}$, noting the fact $u(0) = 0$, we have

$${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t), \ t \in [0,T].$$

This means that the left and right Riemann-Liouville fractional derivatives of order α are equivalent to the left and right Caputo fractional derivatives of order α.

(iii) The fractional space $E_0^{\alpha,p}$ is reflexive and a separable Banach space.

Lemma 3.2. *Let $0 < \alpha \le 1$, and $1 < p < \infty$. Then, for all $u \in E_0^{\alpha,p}$, one has*

$$\|u\|_p \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_p. \tag{11}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$, we have

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\tilde{p}}}}\|{}_0D_t^\alpha u\|_p. \tag{12}$$

According to (11), we can consider $E_0^{\alpha,p}$ with respect to the equivalent norm

$$\|u\| = \|{}_0D_t^\alpha u\|_p.$$

Lemma 3.3. *Let $0 < \alpha \le 1$, and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_n\} \rightharpoonup u$ weakly in $E_0^{\alpha,p}$. Then, $\{u_n\} \to u$ in $C([0,T])$, that is*

$$\|u_n - u\|_\infty \to 0 \quad as\ n \to \infty.$$

Associated to the problem (P_λ) we define the functional

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{r}\int_0^T F(t,|u|)dt - \frac{1}{1-\gamma}\int_0^T g(t)|u|^{1-\gamma}dt, \ u \in E_0^{\alpha,p}. \tag{13}$$

We say that $u \in E_0^{\alpha,p}$ is a weak solution of problem (P_λ) if for every $v \in E_0^{\alpha,p}$ we have:

$$\int_0^T |{}_0D_t^\alpha u(t)|^{p-2}\, {}_0D_t^\alpha u(t)\, {}_0D_t^\alpha v(t)dt = \int_0^T g(t)u(t)^{-\gamma}v(t)dt + \lambda\int_0^T f(t,u(t))v(t)dt.$$

Note that u is a positive solution of problem (P_λ) , if u is positive and

$$\frac{1}{p}\|u\|^p - \frac{1}{1-\gamma}\int_0^T g(t)u(t)^{1-\gamma}dt - \frac{\lambda}{r}\int_0^T F(t,u(t))dt = 0.$$

It is easy to see that the energy functional J_λ is not bounded below on the space $E_0^{\alpha,p}$, but it is bounded below on a suitable subset of $E_0^{\alpha,p}$. In order to investigate the problem (P_λ), we define the constraint set

$$\mathcal{N}_\lambda := \left\{u \in E_0^{\alpha,p} \setminus \{0\} : t(u)u = 0\right\},$$

where $t(u)$ is the zero of the map $\Phi_u : (0,\infty) \to \mathbb{R}$ defined as

$$\Phi_u(t) = J_\lambda(tu).$$

Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [4].
For $u \in E_0^{\alpha,p}$, we have

$$\begin{aligned}
\Phi_u(t) &= \frac{t^p}{p}\|u\|^p - \lambda\frac{t^r}{r}\int_0^T F(s,u(s))ds - \lambda\frac{t^{1-\gamma}}{1-\gamma}\int_\Omega g(s)|u(s)|^{1-\gamma}ds,\\
\Phi_u'(t) &= t^{p-1}\|u\|^p - \lambda t^{r-1}\int_0^T F(s,u(s))ds - \frac{\lambda}{t^\gamma}\int_0^T g(s)|u(s)|^{1-\gamma}ds,\\
\Phi_u''(t) &= (p-1)t^{p-2}\|u\|^p - \lambda(r-1)t^{r-2}\int_0^T F(s,u(s))ds + \frac{\gamma}{t^{\gamma+1}}\int_0^T g(s)|u(s)|^{1-\gamma}ds.
\end{aligned}$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (P_λ), and $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\|^p - \lambda\int_0^T F(t,u(t))dt - \int_0^T g(t)|u(t)|^{1-\gamma}dt = 0. \tag{14}$$

To obtain the existence of solutions, we split $\mathcal{N}_\lambda$ into three parts: corresponding to local minima, local maxima and points of inflection, are measurable sets defined as follows:

$$\mathcal{N}_\lambda^+ = \left\{u \in \mathcal{N}_\lambda : (p+\gamma-1)\|u\|^p - \lambda(r+\gamma-1)\int_0^T F(t,u(t))dt > 0\right\},$$

$$\mathcal{N}_\lambda^- = \left\{u \in \mathcal{N}_\lambda : (p+\gamma-1)\|u\|^p - \lambda(r+\gamma-1)\int_0^T F(t,u(t))dt < 0\right\},$$

$$\mathcal{N}_\lambda^0 = \left\{u \in \mathcal{N}_\lambda : (p+\gamma-1)\|u\|^p - \lambda(r+\gamma-1)\int_0^T F(t,u(t))dt = 0\right\}.$$

Next, we present some important properties of $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$. Firstly, fix $\tilde{p}$ be such that $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$, and put

$$\lambda_0 = \frac{p+\gamma-1}{K(r-p)}\left(\frac{(r-p)\beta^p}{(r+\gamma-1)T^{\alpha p}\|g\|_\infty^{\frac{r-p}{r+\gamma-1}}}\right)^{\frac{r+\gamma-1}{p+\gamma-1}}. \tag{15}$$

Then, we have the following crucial results.

Lemma 3.4. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then, using equations (1) and (12), we obtain

$$\int_0^T F(t,u(t))dt \le K\int_0^T |u|^r dt \le \frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|u\|^r, \tag{16}$$

and

$$\int_0^T g(t)|u|^{1-\gamma}dt \le \|g\|_\infty\int_0^T |u|^{1-\gamma}dt \le \|g\|_\infty\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}{\beta^{1-\gamma}}\|u\|^{1-\gamma}. \tag{17}$$

Consequently, from (16) and (17), we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p - \frac{1}{1-\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - \frac{\lambda}{r}\int_0^T F(t,u(t))dt \\ &= \frac{r-p}{rp}\|u\|^p - \frac{r+\gamma-1}{r(1-\gamma)}\int_0^T g(t)|u|^{1-\gamma}dt \\ &\geq \frac{r-p}{rp}\|u\|^p - \frac{r+\gamma-1}{r(1-\gamma)}\|g\|_\infty \frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}{\beta^{1-\gamma}}\|u\|^{1-\gamma}. \end{aligned}$$

Since $0<\gamma<1$ and $1-\gamma<p<r$, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. The proof of the Lemma 3.4 is now completed. □

Lemma 3.5. *Let $\lambda \in (0,\lambda_0)$. Then, there exist t_0^+ and t_0^- such that*

$$\Phi_u(t_0^+) = \lambda \int_0^T F(t,u)\,\mathrm{d}x = \Phi_u(t_0^-)$$

and

$$\Phi_u'(t_0^+) < 0 < \Phi_u'(t_0^-);$$

that is, $t_0^+u \in \mathcal{N}_\lambda^+$ and $t_0^-u \in \mathcal{N}_\lambda^-$.

Proof. Firstly, for $\Phi_u'(t)=0$ it is simple to verify that Φ_u attains it's maximum at

$$t_{\max} = \left(\frac{(r+\gamma-1)\int_0^T g(t)u^{1-\gamma}dt}{(r-p)\|u\|^p}\right)^{\frac{1}{p+\gamma-1}}.$$

Moreover, $\Phi_u'(t)>0$ for all $0<t<t_{\max}$ and $\Phi_u'(t)<0$ for all $t>t_{\max}$. On the other hand,

$$\Phi(t_{\max}) = \frac{p+\gamma-1}{r-p}\left(\frac{r-p}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\|u\|^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{\left(\int_0^T g(t)u^{1-\gamma}dt\right)^{\frac{r-p}{p+\gamma-1}}}.$$

Combining equations (16) and (17), we obtain

$$\begin{aligned} &\Phi_u(t_{\max}) - \lambda\int_0^T F(t,u(x))dt \\ &\geq \frac{p+\gamma-1}{r-p}\left(\frac{r-p}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\|u\|^{\frac{p(r+\gamma-1)}{p+\gamma-1}}\beta^{\frac{(1-\gamma)(r-p)}{p+\gamma-1}}}{\|g\|_\infty^{\frac{r-p}{p+\gamma-1}}\|u\|^{\frac{(1-\gamma)(r-p)}{p+\gamma-1}}T^{\frac{(1+(1-\gamma)(\alpha-\frac{1}{p}))}{p+\gamma-1}}} - \lambda\frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|u\|^r, \\ &= \frac{p+\gamma-1}{r-p}\left(\frac{r-p}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\beta^{\frac{(1-\gamma)(r-p)}{p+\gamma-1}}\|u\|^r}{\|g\|_\infty^{\frac{r-p}{p+\gamma-1}}T^{\frac{(1+(1-\gamma)(\alpha-\frac{1}{p}))}{p+\gamma-1}}} - \lambda\frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|u\|^r \\ &= \frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}(\lambda_0-\lambda)\|u\|^r > 0, \end{aligned} \tag{18}$$

for all $\lambda\in(0,\lambda_0)$. Therefore, there exist $0<t_0^+<t_{\max}<t_0^-$ such that

$$\Phi_u(t_0^+) = \lambda \int_0^T F(t,u)\,\mathrm{d}x = \Phi_u(t_0^-)$$

and

$$\Phi_u'(t_0^+) < 0 < \Phi_u'(t_0^-);$$

that is, $t_0^+ u \in \mathcal{N}_\lambda^+$ and $t_0^- u \in \mathcal{N}_\lambda^-$. This completes the proof of the Lemma 3.5. □

Now, we prove the following crucial Lemma:

Lemma 3.6. *Suppose $\lambda \in (0, \lambda_0)$, then $\mathcal{N}_\lambda^\pm \neq \emptyset$ and $\mathcal{N}_\lambda^0 = \emptyset$. Moreover, $\mathcal{N}_\lambda^-$ is a closed set in $E_0^{\alpha,p}$−topology.*

Proof. Firstly, using Lemma 3.5, we conclude that $\mathcal{N}^\pm$ are non-empty for $\lambda \in (0, \lambda_0)$. Now, we proceed by contradiction to prove that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda_0)$. Let us suppose that there exists $u_0 \in \mathcal{N}_\lambda^0$. Then, it follows that

$$(p+\gamma-1)\|u_0\|^p - \lambda(r+\gamma-1)\int_0^T F(t, u_0(x))dt = 0,$$

which implies that

$$\begin{aligned} 0 &= \|u_0\|^p - \int_0^T g(t) u_0^{1-\gamma} dt - \lambda \int_0^T F(t, u_0) dt \\ &= \frac{r-p}{r+\gamma-1}\|u_0\|^p - \int_0^T g(t) u_0^{1-\gamma} dt. \end{aligned}$$

Using, (18) we obtain that

$$\begin{aligned} & 0 < \phi_{u_0}(t_{\max}) - \lambda \int_0^T F(t, u_0) dt \\ &= \frac{p+\gamma-1}{r-p}\left(\frac{r-p}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\|u_0\|^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{\left(\int_0^T a(x) u_0^{1-\gamma} dt\right)^{\frac{r-p}{p+\gamma-1}}} - \lambda \int_0^T F(t, u_0) dt \\ &\le \frac{p+\gamma-1}{r-p}\left(\frac{r-p}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\|u_0\|^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{\left(\frac{(r-p)\|u_0\|^p}{r+\gamma-1}\right)^{\frac{r-p}{p+\gamma-1}}} - \frac{p+\gamma-1}{r+\gamma-1}\|u_0\|^p = 0 \end{aligned} \tag{19}$$

which is impossible. Thus, $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda_0)$. Finally, to prove that $\mathcal{N}_\lambda^-$ is closed for all $\lambda \in (0, \lambda_0)$, we introduce the sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that $u_n \to u$ in E. Since $\{u_n\} \subset \mathcal{N}_\lambda^-$, then we have

$$\|u_n\|^p - \int_0^T a(x) u_n^{1-\gamma} dt - \lambda \int_0^T F(t, u_n) dt = 0,$$

and

$$(p+\gamma-1)\|u_n\|^p - \lambda(\gamma+r-1)\int_0^T F(t, u_n) dt < 0. \tag{20}$$

That is,

$$\|u\|^p - \int_0^T g(t) u^{1-\gamma} dt - \lambda \int_0^T F(t, u) dt = 0,$$

and

$$(p+\gamma-1)\|u\|^p - \lambda(\gamma+r-1)\int_0^T F(t, u) dt \le 0,$$

then $u \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^- = \mathcal{N}_\lambda^-$. Thus $u \in \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \lambda_0)$. Therefore the proof of the Lemma 3.6 is now completed. □

Lemma 3.7. *Given $u \in \mathcal{N}_\lambda^-$ (respectively $\mathcal{N}_\lambda^+$) with $u \geq 0$, for all $v \in E$ with $v \geq 0$, there exist $\varepsilon > 0$ and a continuous function ω such that for all $k \in \mathbb{R}$ with $|k| < \varepsilon$ we have*

$$\omega(0) = 1 \quad \textit{and} \quad \omega(k)(u + kv) \in \mathcal{N}_\lambda^- \ (\textit{respectively} \ \ \mathcal{N}_\lambda^+).$$

Proof. We introduce the function $\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ define by:

$$\psi(t,k) = t^{p+\gamma-1}\|u + kv\|^p - \int_0^T g(s)(u + kv)^{1-\gamma} ds - \lambda t^{r+\gamma-1} \int_0^T F(s, u + kv)ds.$$

Hence,

$$\psi_t(t,k) = (p + \gamma - 1)t^{p+\gamma-2}\|u + kv\|^p - \lambda(r + \gamma - 1)t^{r+\gamma-2} \int_0^T F(s, u + kv)ds,$$

is continuous on $\mathbb{R} \times \mathbb{R}$. Since $u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we have $\psi(1,0) = 0$, and

$$\psi_t(1,0) = (p + \gamma - 1)\|u\|^p - \lambda(r + \gamma - 1) \int_0^T F(t,u)dt < 0.$$

Therefore, applying the implicit function theorem to the function ψ at the point $(1,0)$. So, we obtain the existence of a parameter $\delta > 0$ and a positive continuous function ω satisfying

$$\omega(0) = 1, \ \omega(k)(u + kv) \in \mathcal{N}_\lambda, \ \forall \ k \in \mathbb{R}, \ |k| < \delta.$$

Hence, taking $\varepsilon > 0$ possibly smaller enough, we get

$$\omega(k)(u + kv) \in \mathcal{N}_\lambda^-, \ \forall \ k \in \mathbb{R}, \ |k| < \varepsilon.$$

The case $u \in \mathcal{N}_\lambda^+$. may be obtained in the same way. This completes the proof of the Lemma 3.7. □

4. Solutions of (P_λ) for all $\lambda \in (0, \lambda_0)$

Since $J_\lambda(u) = J_\lambda(|u|)$, we can assume that all the price elements in $\mathcal{N}_\lambda$ are nonnegative. On the other hand, according to Lemma 3.4 and Lemma 3.6, for all $\lambda \in (0, \lambda_0)$

$$m^+ := \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) \ \text{ and } \ m^- := \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u)$$

are well defined. Moreover, for all $u \in \mathcal{N}_\lambda^+$, it follows that

$$(p + \gamma - 1)\|u\|^p - \lambda(\gamma + r - 1) \int_0^T F(t, u(t))dt > 0,$$

and consequently, since $0 < \gamma < 1$, $p < r$ and $u \neq 0$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p - \frac{1}{1-\gamma} \int_0^T g(t)u(t)^{1-\gamma}dt - \frac{\lambda}{r} \int_0^T F(t, u(t))dt \\ &= (\frac{1}{p} - \frac{1}{1-\gamma})\|u\|^p + \lambda(\frac{1}{1-\gamma} - \frac{1}{r}) \int_0^T F(t, u(t))dt \\ &< \frac{1-\gamma-p}{p(1-\gamma)}\|u\|^p + \frac{p+\gamma-1}{r(1-\gamma)}\|u\|^p \\ &= -\frac{(r-p)(p+\gamma-1)}{pr(1-\gamma)}\|u\|^p < 0. \end{aligned}$$

Thus,

$$m^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) < 0 \tag{21}$$

for all $\lambda \in (0, \lambda_0)$.

Proof. [Proof of Theorem 1.1] The proof is done in two steps:

Step 1: (P_λ) have a positive solution in $\mathcal{N}_\lambda^+$.
Let us consider the sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ and applying Ekeland's variational principle (see [2] for the detail), we obtain

(i) $J_\lambda(u_n) < m^+ + \frac{1}{n}$,

(ii) $J_\lambda(u) \geq J_\lambda(u_n) - \frac{1}{n}\|u - u_n\|$, for all $u \in \mathcal{N}^+$.

Since $J_\lambda(u) = J_\lambda(|u|)$, we can assume that $u_n(x) \geq 0$. Consequently, as J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_n\}$ is a bounded sequence in $E_0^{\alpha,p}$, going to a sub-sequence denoted by $\{u_n\}$, and $u_0 \geq 0$ such that $u_n \rightharpoonup u_0$, weakly in $E_0^{\alpha,p}$, $u_n \to u_0$, strongly in $L^q(\Omega)$, for $1 \leq q < p^*$, and $u_n(x) \to u_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (21) and using the weak lower semi-continuity of norm $J_\lambda(u_0) \leq \liminf J_\lambda(u_n) = \inf_{\mathcal{N}^+} J_\lambda$, we see that $u_0 \not\equiv 0$ in Ω.

Claim 1. $u_0(x) > 0$ a.e. in Ω.
Firstly, we start by observing that, since $u_n \in \mathcal{N}_\lambda^+$, one has

$$(p + \gamma - 1)\|u_n\|^p - \lambda(\gamma + r - 1)\int_0^T F(t, u_n)dt > 0 \tag{22}$$

equivalent to

$$(p + \gamma - 1)\int_0^T g(t)u_n^{1-\gamma}dt - \lambda(r - p)\int_0^T F(t, u_n)dt > C_1. \tag{23}$$

Now, using Lemma 3.3, we get that, as $n \to \infty$,

$$\begin{aligned}\int_0^T u_n^{1-\gamma}\mathrm{d}t &\leq \int_0^T u_0^{1-\gamma}\mathrm{d}t + \int_0^T |\, u_n - u_0 \,|^{1-\gamma}\ \mathrm{d}t \\ &\leq \int_0^T u_0^{1-\gamma}\mathrm{d}x + T \,\|\, u_n - u_0 \,\|_\infty^{1-\gamma} \\ &= \int_0^T u_0^{1-\gamma}\mathrm{d}t + o(1).\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^T u_0^{1-\gamma}\mathrm{d}t &\leq \int_0^T u_n^{1-\gamma}\mathrm{d}t + \int_0^T |\, u_n - u_0 \,|^{1-\gamma}\ \mathrm{d}t \\ &\leq \int_0^T u_n^{1-\gamma}\mathrm{d}t + T \,\|\, u_n - u_0 \,\|_\infty^{1-\gamma} \\ &= \int_0^T u_n^{1-\gamma}\mathrm{d}t + o(1).\end{aligned}$$

Thus,

$$\int_0^T u_n^{1-\gamma}\mathrm{d}t = \int_0^T u_0^{1-\gamma}\mathrm{d}x + o(1). \tag{24}$$

On the other hand, using Vitali's convergence Theorem, we have

$$\lim_{n\to\infty}\int_0^T F(t, u_n)dt = \int_0^T F(t, u_0)dt. \tag{25}$$

Therefore, from (24) and (25), it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left((p+\gamma-1)\int_0^T g(t)u_n^{1-\gamma}dt-\lambda(r-p)\int_0^T F(t,u_n)dt\right)\\
&=(p+\gamma-1)\int_0^T g(t)u_0^{1-\gamma}dt-\lambda(r-p)\int_0^T F(t,u_0)dt\geq 0.
\end{aligned}$$

Now, we assume that

$$(p+\gamma-1)\int_0^T g(t)u_0^{1-\gamma}dt-\lambda(r-p)\int_0^T F(t,u_0)dt=0. \tag{26}$$

Consequently, Combining (24)-(25) and the weakly lower semi-continuity of the norm, we obtain

$$\begin{aligned}
0&\geq \|u_0\|^p-\int_0^T g(t)u_0^{1-\gamma}dt-\lambda\int_0^T F(t,u_0)dt\\
&=\|u_0\|^p-\frac{p+\gamma-1}{r-p}\int_0^T g(t)u_0^{1-\gamma}dt\\
&=\|u_0\|^p-\lambda\frac{r-p}{p+\gamma-1}\int_0^T F(t,u_0)dt
\end{aligned}\tag{27}$$

and consequently, from (19) one has a contradiction. That is

$$(p+\gamma-1)\int_0^T g(t)u_0^{1-\gamma}dt-\lambda(r-p)\int_0^T F(t,u_0)dt>0. \tag{28}$$

Now, let us consider the function $\varphi\in E_0^{\alpha,p}$, with $\varphi\geq 0$. From Lemma 3.7 with $u=u_n$, there exits a sequence of continuous functions $h_n=h_n(t)$ such that $h_n(t)(u_n+t\varphi)\in\Lambda^+$ and $h_n^p(0)=1$. That is,

$$[h_n(t)]^p\|u_n+t\varphi\|^p-[h_n(t)]^{1-\gamma}\int_0^T g(s)(u_n+t\varphi)^{1-\gamma}ds-\lambda[h_n(t)]^r\int_0^T F(s,u_n+t\varphi)ds=0.$$

Since

$$\|u_n\|^p-\int_0^T g(t)u_n^{1-\gamma}dt-\lambda\int_0^T F(t,u_n)dt=0, \tag{29}$$

it follows that, for t small enough

$$\begin{aligned}
0&=(h_n(t)]^p-1)\|u_n+t\varphi\|^p+(\|u_n+t\varphi\|^p-\|u_n\|^p)\\
&-(h_n(t)^{1-\gamma}-1)\int_0^T g(s)(u_n+t\varphi)^{1-\gamma}ds-\int_0^T g(s)\left((u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}\right)ds\\
&-\lambda(h_n(t)^r-1)\int_0^T F(s,u_n+t\varphi)ds-\lambda\int_0^T F(s,u_n+t\varphi)-F(s,u_n)ds\\
&\leq (h_n(t)^p-1)\|u_n+t\varphi\|^p+(\|u_n+t\varphi\|^p-\|u_n\|^p)-(h_n(t)^{1-\gamma}-1)\int_0^T g(s)(u_n+t\varphi)^{1-\gamma}ds\\
&-\lambda(h_n(t)^r-1)\int_0^T F(s,u_n+t\varphi)ds-\lambda\int_0^T F(s,u_n+t\varphi)-F(s,u_n)ds,
\end{aligned}$$

dividing the above inequality by $t > 0$, and passing to the limit for $t \to 0$, we obtain

$$\begin{aligned}
&0 \leq ph_n'(0)\|u_n\|^p - h_n'(0)(1-\gamma)\int_0^T u_n^{1-\gamma}\,\mathrm{d}s - \lambda r h_n'(0)\int_0^T F(s,u_n)\mathrm{d}s \\
&+p\int_0^T |\,{}_0D_s^\alpha u_n(s)|^{p-2}\,{}_0D_s^\alpha u_n(s)\,{}_0D_s^\alpha \varphi(s)ds \\
&= h_n'(0)\left((p-r)\|u_n\|^p + (r+\gamma-1)\int_0^T u_n^{1-\gamma}\,\mathrm{d}s\right) \\
&+p\int_0^T |\,{}_0D_s^\alpha u_n(s)|^{p-2}\,{}_0D_s^\alpha u_n(s)\,{}_0D_s^\alpha \varphi(s)ds.
\end{aligned}$$

where $h_n'(0) \in [-\infty,\infty]$ denotes the right derivate of $g_n(t)$ at zero and since $u_n \in \mathcal{N}^+$, $h_n'(0) \neq -\infty$. For simplicity, we assume that the right derivate of h_n at $t = 0$ exists. Moreover, from (27) $h_n'(0)$ is uniformly bounded from below. Now, using the condition (ii) ,

$$\begin{aligned}
&|h_n(t)-1|\frac{\|u_n\|}{n} + th_n(t)\frac{\|\varphi\|}{n} \geq J_\lambda(u_n) - J_\lambda\left(h_n(t)(u_n+t\varphi)\right) \\
&= -\frac{p+\gamma-1}{p(1-\gamma)}\|u_n\|^p + \lambda\frac{r+\gamma-1}{r(1-\gamma)}\int_0^T F(s,u_n)ds \\
&+\frac{p+\gamma-1}{p(1-\gamma)}h_n(t)^p\|u_n+t\varphi\|^p - \lambda\frac{r+\gamma-1}{r(1-\gamma)}h_n(t)^r\int_0^T F(s,u_n+t\varphi)ds \\
&= \frac{p+\gamma-1}{p(1-\gamma)}\left[\|u_n+t\varphi\|^p - \|u_n\|^p + (h_n(t)^p-1)\|u_n+t\varphi\|^p\right] \\
&-\lambda\frac{r+\gamma-1}{r(1-\gamma)}\left[\int_0^T F(s,u_n+t\varphi) - F(s,u_n)ds + (h_n(t)^r-1)\int_0^T F(s,u_n+t\varphi)ds\right].
\end{aligned}$$

Then, dividing the above inequality by $t > 0$, and passing to the limit $t \to 0$, we obtain

$$\begin{aligned}
\frac{1}{n}\left(|h_n'(0)|\|u_n\| + th_n(t)\|\varphi\|\right) \quad &\geq \frac{h_n'(0)}{1-\gamma}\left[\frac{r+\gamma-1}{(1-\gamma)}\int_0^T g(s)u_n^{1-\gamma}ds - \frac{p+\gamma-1}{(1-\gamma)}\|u_n\|^p\right] \\
&+\frac{p+\gamma-1}{(1-\gamma)}p\int_0^T |\,{}_0D_s^\alpha u_n(s)|^{p-2}\,{}_0D_s^\alpha u_n(s)\,{}_0D_s^\alpha\varphi(s)ds \\
&-\lambda\left(\frac{r+\gamma-1}{(1-\gamma)}\right)\int_0^T F(s,u_n)\varphi ds.
\end{aligned} \tag{30}$$

Then from (28), there exists a positive constant C such that

$$-\frac{1}{1-\gamma}\left((r-p)\|u_n\|^p - (r+\gamma-1)\int_0^T u_n^{1-\gamma}\,\mathrm{d}s\right) - \frac{\|u_n\|}{n} \geq C > 0. \tag{31}$$

Thus, according to (30) and (31), $h_n'(0)$ is uniformly bounded from above. Consequently,

$$h_n'(0) \quad \text{is uniformly bounded for } n \text{ large enough.} \tag{32}$$

Thus from condition (ii) it follows that for $t > 0$ small enough,

$$J_\lambda(u_n) \leq J_\lambda(h_n(t)(u_n+t\varphi)) + \frac{1}{n}\|h_n(t)(u_n+t\varphi) - u_n\|. \tag{33}$$

That is,

$$\begin{aligned}
&\frac{1}{n}\left(|h_n(t)-1|\|u_n\|+th_n(t)\|\varphi\|\right) \geq \frac{1}{n}\left\|h_n(t)(u_n+t\varphi)-u_n\right\| \\
&\geq J_\lambda(u_n)-J_\lambda\left(h_n(t)(u_n+t\varphi)\right) \\
&= -\frac{h_n^p(t)-1}{p}\|u_n\|^p+\frac{h_n^{1-\gamma}(t)-1}{1-\gamma}\int_0^T g(s)u_n^{1-\gamma}ds \\
&+\frac{h_n^{1-\gamma}(t)}{1-\gamma}\int_0^T g(s)\left((u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}\right)ds+\frac{h_n^p(t)}{p}\left(\|u_n\|^p-\|u_n+t\varphi\|^p\right) \\
&+\frac{\lambda}{r}h_n^r(t)\int_0^T F(s,u_n+t\varphi)-F(s,u_n)ds+\frac{\lambda}{r}(h_n^r(t)-1)\int_0^T F(s,u_n)ds.
\end{aligned}$$

Then dividing by $t>0$, and passing to the limit $t\to 0$, we obtain

$$\begin{aligned}
\tfrac{1}{n}\left(|h_n'(0)|\|u_n\|+\|\varphi\|\right) \quad &\geq -h_n'(0)\left[\|u_n\|^p+\int_0^T g(s)u_n^{1-\gamma}ds+\lambda\int_0^T F(s,u_n)ds\right]+\lambda\int_0^T F(s,u_n)\varphi ds \\
&-\int_0^T |\,{}_0D_s^\alpha u_n(s)|^{p-2}\,{}_0D_s^\alpha u_n(s)\,{}_0D_s^\alpha\varphi(s)ds \\
&+\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_0^T\left(\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}s\right) \\
&= -\int_0^T |\,{}_0D_s^\alpha u_n(s)|^{p-2}\,{}_0D_s^\alpha u_n(s)\,{}_0D_s^\alpha\varphi(s)ds \\
&+\lambda\int_0^T F(s,u_n)\varphi ds+\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_0^T |\,{}_0D_t^\alpha u_n(t)|^{p-2}\,{}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha\varphi(t)dt
\end{aligned}$$

From the above inequality we deduce that

$$\begin{aligned}
&\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_0^T\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}t \\
&\leq \int_0^T |\,{}_0D_t^\alpha u_n(t)|^{p-2}\,{}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha\varphi(t)dt \\
&-\lambda\int_0^T F(s,u_n)\varphi ds+\frac{1}{n}\left(|h_n'(0)|\|u_n\|+\|\varphi\|\right).
\end{aligned}\tag{34}$$

Since

$$g(s)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]\geq 0,\quad \forall\, s\in[0,T],\ \forall\, t>0,$$

using Fatou's Lemma we get

$$\int_0^T g(s)u_n^{-\gamma}\varphi\,\mathrm{d}s\leq\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_0^T g(s)\left(\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}s\right).$$

Hence, using (34), it follows that

$$\int_0^T u_n^{-\gamma}\varphi\,\mathrm{d}s\leq\int_0^T |\,{}_0D_t^\alpha u_n(t)|^{p-2}\,{}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha\varphi(t)dt-\lambda\int_0^T F(s,u_n)\varphi\,\mathrm{d}s+h_n'(0)\frac{\|u_n\|+\|\varphi\|}{n}$$

for n large enough. Therefore, from (32) and applying Fatou's Lemma again, to conclude that $u_0(t)>0$ a.e. in $[0,T]$ and

$$\int_0^T |\,{}_0D_t^\alpha u_n(t)|^{p-2}\,{}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha\varphi(t)dt-\int_0^T g(t)u_0^{-\gamma}\varphi\,\mathrm{d}t-\lambda\int_0^T F(t,u_0)\varphi\,\mathrm{d}t\geq 0,\tag{35}$$

for all $\varphi \in E$, with $\varphi \geq 0$. Now, we prove that $u_0 \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \lambda_0)$. Then, choosing $\varphi = u_0$ in (35), we get

$$\|u_0\|^p \geq \lambda \int_0^T F(s, u_0) ds + \int_0^T g(s)(u_0)^{1-\gamma} ds.$$

On the other hand, from (27) it follows that,

$$\|u_0\|^p \leq \lambda \int_0^T F(s, u_0) ds + \int_0^T g(s)(u_0)^{1-\gamma} ds.$$

Thus

$$\|u_0\|^p = \lambda \int_0^T F(s, u_0) ds + \int_0^T g(s)(u_0)^{1-\gamma} ds, \tag{36}$$

this implies that $u_0^+ \in \mathcal{N}_\lambda$. Moreover from (29), ones gets

$$\lim_{n\to\infty} \|u_n\|^p = \lambda \int_0^T F(t, u_0^+) dt + \int_0^T g(t)(u_0^+)^{1-\gamma} dt.$$

Hence according to (36), we have $u_n \to u_0$ in E as $n \to \infty$. In particular, combining (28) with (36), we obtain

$$(1+\gamma)\|u_0\|^p - \lambda(\gamma + r) \int_0^T F(t, u_0) dt > 0,$$

and therefore $u_0 \in \mathcal{N}_\lambda^+$.

Claim 2 : u_0 is a solution of problem (P_λ). Our proof is inspired by Ghanmi-Saoudi [5, 6]. Let $\phi \in E_0^{\alpha,p}$ and $\epsilon > 0$. We define $\Psi \in E_0^{\alpha,p}$ by $\Psi := (u_0 + \epsilon\phi)^+$ where $(u_0 + \epsilon\phi)^+ = \max\{u_0 + \epsilon\phi, 0\}$. Replace φ with Ψ in (35) and combining with (36) we obtain

$$\begin{aligned}
0 &\leq \int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha \Psi(t) dt - \int_0^T g(t) u_0^{-\gamma} \Psi \,\mathrm{d}t - \lambda \int_0^T f(t, u_0)\Psi \,\mathrm{d}t \\
&= \int_{\{t|u_0+\epsilon\phi>0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha (u_0+\epsilon\phi)(t) dt \\
&- \int_{\{t|u_0+\epsilon\phi>0\}} \Big(g(t)u_0^{-\gamma}(u_0+\epsilon\phi) + \lambda F(t,u_0)(u_0+\epsilon\phi)\Big)\,\mathrm{d}t \\
&= \int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha (u_0+\epsilon\phi)(t) dt \\
&- \int_0^T \Big(g(t)u_0^{-\gamma}(u_0+\epsilon\phi) + \lambda f(t,u_0)(u_0+\epsilon\phi)\Big)\,\mathrm{d}t \\
&- \int_{\{t|u_0+\epsilon\phi\leq 0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha (u_0+\epsilon\phi)(t) dt \\
&+ \int_{\{t|u_0+\epsilon\phi\leq 0\}} \Big(g(t)u_0^{-\gamma}(u_0+\epsilon\phi) + \lambda f(t,u_0)(u_0+\epsilon\phi)\Big)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \|u_0\|^p - \int_0^T u_0^{1-\gamma}\,\mathrm{d}t - \lambda\int_0^T F(t,u_0)\,\mathrm{d}t - \int_0^T \left(u_0^{-\gamma}\phi + \lambda f(t,u_0)\phi\right)\mathrm{d}t \\
&+\epsilon\int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha\phi(t)dt \\
&-\int_{\{t|u_0+\epsilon\phi\le 0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha(u_0+\epsilon\phi)(t)dt \\
&+\int_{\{t|u_0+\epsilon\phi\le 0\}} \left(g(t)u_0^{-\gamma}(u_0+\epsilon\phi) + \lambda f(t,u_0)(u_0+\epsilon\phi)\right)\mathrm{d}t \\
&= \epsilon\int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha\phi(t)dt - \epsilon\int_0^T\left(u_0^{-\gamma}\phi + \lambda f(t,u_0)\phi\right)\mathrm{d}t \\
&-\int_{\{t|u_0+\epsilon\phi\le 0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha(u_0+\epsilon\phi)(t)dt \\
&-\epsilon\int_{\{t|u_0+\epsilon\phi\le 0\}} \left(g(t)u_0^{-\gamma}(u_0+\epsilon\phi) + \lambda f(t,u_0)(u_0+\epsilon\phi)\right)\mathrm{d}t \\
&\le \epsilon\int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha\phi(t)dt - \epsilon\int_0^T\left(g(t)u_0^{-\gamma}\phi + \lambda f(t,u_0)\phi\right)\mathrm{d}t \\
&-\int_{\{t|u_0+\epsilon\phi\le 0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha(u_0+\epsilon\phi)(t)dt.
\end{aligned}
$$

Since the measure of the domain of integration $\{x : u_0 + \epsilon\phi < 0\}$ tends to zero as $\epsilon \to 0^+$. It follows as $\epsilon \to 0^+$ that,

$$\int_{\{t|u_0+\epsilon\phi\le 0\}} |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha(u_0+\epsilon\phi)(t)dt \to 0.$$

Dividing by ϵ and letting $\epsilon \to 0^+$, we get

$$\int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha\phi(t)dt - \int_0^T\left(u_0^{-\gamma}\phi + \lambda f(t,u_0)\phi\right)\mathrm{d}t \ge 0.$$

Since the equality holds if we replace φ by $-\varphi$ which implies that u_0 is a positive solution of problem (P_λ).

Step 2: (P_λ) have a positive solution in $\mathcal{N}_\lambda^-$.

Similarly to the first Step, applying Ekeland's variational principle to the minimization problem $m^- = \inf_{v\in\mathcal{N}_\lambda^-} J_\lambda(v)$ there exists a sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ such that

(i) $J_\lambda(v_n) < m^+ + \frac{1}{n}$,

(ii) $J_\lambda(v) \ge J_\lambda(v_n) - \frac{1}{n}\|v - v_n\|$, for all $v \in \mathcal{N}^-$.

Since $J_\lambda(v) = J_\lambda(|v|)$, we can assume that $v_n(x) \ge 0$. Consequently, as J_λ is coercive on $\mathcal{N}_\lambda$, $\{v_n\}$ is a bounded sequence in $E_0^{\alpha,p}$, going to a sub-sequence denoted by $\{v_n\}$, and $v_0 \ge 0$ such that $u_n \rightharpoonup u_0$, weakly in $E_0^{\alpha,p}$, $v_n \to v_0$, strongly in $L^{1-\gamma}(\Omega)$, and $L^s(\Omega)$, for $1 \le s < p^*$, and $v_n(x) \to v_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (21) and using the weak lower semi-continuity of norm $J_\lambda(v_0) \le \liminf J_\lambda(v_n) = \inf_{\mathcal{N}^-} J_\lambda$, we see that $v_0 \not\equiv 0$ in Ω. Now, we prove that $v_0(x) > 0$ a.e. in Ω. Similarly to the arguments in Claim 1, we start by observing that, since $v_n \in \Lambda^-$, one has

$$(1+\gamma)\|v_n\|^p - \lambda(\gamma + r)\int_0^T F(t,v_n)dt < 0 \tag{37}$$

and consequently,

$$(1+\gamma)\int_0^T g(t)v_n^{1-\gamma}dt - \lambda(r-1)\int_0^T F(t,v_n)dt < 0. \tag{38}$$

Therefore, from (24) and (25) it follows that

$$\begin{aligned}&\lim_{n\to\infty}\left[(1+\gamma)\int_0^T g(t)v_n^{1-\gamma}dt - \lambda(r-1)\int_0^T F(t,v_n)dt\right]\\ &= (1+\gamma)\int_0^T g(t)v_0^{1-\gamma}dt - \lambda(r-1)\int_0^T F(t,v_0)dt \le 0.\end{aligned}$$

Now, repeating the same arguments as in Claim 1, it follows that

$$(1+\gamma)\int_0^T |v_0|^{1-\gamma}dt - \lambda(r-1)\int_0^T F(t,|v_0|)dt < 0. \tag{39}$$

Now, let $\varphi \in E_0^{\alpha,p}$, with $\varphi \ge 0$. From Lemma 3.6 with $u = v_n$, there exits a sequence of continuous functions $h_n = h_n(t)$ such that $h_n(t)(v_n + t\varphi) \in \Lambda^-$ and $h_n(0) = 1$. Therefore, using the same arguments as in Claim 1 we prove that

$$h_n'(0) \quad \text{is uniformly bounded for } n \text{ large enough.} \tag{40}$$

Then, as in Step 1 applying (ii) and (40), we conclude that $v_0(x) > 0$ a.e. in Ω and

$$\int_0^T |\,{}_0D_t^\alpha u_0(t)|^{p-2}\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha \phi(t)dt - \int_0^T \left(u_0^{-\gamma}\phi + \lambda f(t,u_0)\phi\right)\mathrm{d}t \ge 0.$$

for all $\varphi \in E_0^{\alpha,p}$. Finally, as in the arguments of Claim 2, we obtain that $v_0 \in \Lambda^-$ is a positive solution of problem (P_λ). The proof of the Theorem 1.1 is now completed. □

5. Some Examples

In this section we give some examples to illustrate the usefulness of our main results.

Example 5.1. *Let h and g be continuous functions on $[0,T]$ such that $h^+ \neq 0$ and $h^- \neq 0$. Consider the following fractional differential equation with Riemann-Liouville boundary conditions:*

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_1^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}\,{}_0D_t^\alpha u(t)\right)\\ \quad = \frac{g(t)}{u^\gamma(t)} + \lambda h(t)|u(t)|^{r-2}u(t) \quad (t \in (0,T)),\\ u(0) = u(T) = 0,\end{cases}$$

where $\frac{1}{2} < \alpha < 1$, $0 < \gamma < 1$, $1 < p < r$. It is easy to see that $f(t,x) = h(t)|x|^{r-2}x$ is positively homogeneous of degree $r-1$. Moreover, By a simple computation, we obtain $F(t,x) = h(t)|x|^r$ which is positively homogeneous of degree r. On the other hand, since $h^+ \neq 0$ and $h^- \neq 0$, all properties in hypothesis **(H_1)** *and* **(H_2)** *hold true. So all conditions of Theorem 1.1 are satisfied, and our conclusion follows from Theorem 1.1.*

Example 5.2. *Consider the following fractional differential equation with Riemann-Liouville boundary conditions:*

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_1^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}\,{}_0D_t^\alpha u(t)\right)\\ \quad = \frac{g(t)}{u^\gamma(t)} + \lambda h(t)\left(\int_0^T |u(t)|^r dt\right)^{\frac{r-q}{r}} |u(t)|^{q-2}u(t),\ t \in (0,T),\\ u(0) = u(T) = 0,\end{cases}$$

where $\frac{1}{2} < \alpha < 1$, $0 < \gamma < 1$, $1 < q < p < r$, $g \in C([0,1])$ *and the function* h *is such that* $h^{\pm} \neq 0$.
It is easy to see that the function f *defined by:*

$$f(t,u) = h(t)\left(\int_0^T |u(t)|^r dt\right)^{\frac{r-q}{r}} |u(t)|^{q-2}u(t),$$

is positively homogeneous of degree $r-1$. *Moreover, a simple calculation shows that*

$$F(t,u) = h(t)\left(\int_0^T |u(t)|^r dt\right)^{\frac{r-q}{r}} |u(t)|^{q},$$

which is positively homogeneous of degree r*, that is hypothesis* **(H**$_1$**)** *is satisfied. On the other hand, since* $h^{\pm} \neq 0$*, then, hypothesis* **(H**$_2$**)** *is also satisfied. So, Theorem 1.1 implies the existence of* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$*, problem* (P_λ) *has at least two non-trivial solutions.*

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