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Application of Thermal Potentials to the Solution of the Problem of Heat Conduction in a Region Degenerates at the Initial Moment

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Abstract. In this paper, the boundary value problem for the heat equation in the region which degenerates at the initial time is considered. Such problems arise in mathematical models of the processes occurring by opening of electric contacts, in particular, at the description of the heat transfer in a liquid metal bridge and electric arcing. The boundary value problem is reduced to a Volterra integral equation of the second kind which has a singular feature. The class of solutions for the integral equation is defined and the constructive method of its solution is developed.

1. Introduction

Thermal potentials are some convenient tools for solving boundary value problems of heat conduction in regions with variable boundaries [8]. With their help, the boundary value problems are reduced to some integral equations of Volterra type of the second kind, which are successfully solved by Picard's method of successive approximations [4].

It has been experimentally established that when the contacts of electric current circuit breakers open, a liquid metal bridge, which significantly affects the erosion of the contact material, appears for a short time [2]. Modeling the thermophysical properties of the bridge, S. N. Kharin came to the boundary-value problem, in which at the initial moment of contact opening the solution region is absent [3]. This fact affected the integral equation of the boundary value problem. It turned out that the sequence of Picard approximations of the integral equation is divergent. This feature and some results of its investigation will be considered below.

2. Formulation of the Boundary Value Problem

It is required to find the solution $u(x,t)$ of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

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in a region $\Omega = \{0 < x < \alpha_0 t,\ 0 < t < T\}$ with moving boundary $x = \alpha_0 t$, α_0 = constant, with the boundary conditions

$$u(0,t) = \varphi(t), \tag{2}$$

$$u(\alpha_0 t, t) = \psi(t). \tag{3}$$

It is assumed that the functions $\varphi(t)$ and $\psi(t)$ satisfy the Hölder condition with exponent $\sigma > \frac{1}{2}$, that is

$$\left|\varphi(t_2) - \varphi(t_1)\right| < A\left|t_2 - t_1\right|^{\sigma}, \tag{4}$$

$$\left|\psi(t_2) - \psi(t_1)\right| < A\left|t_2 - t_1\right|^{\sigma}, \tag{5}$$

and agreed at the initial moment $t = 0$

$$\varphi(0) = \psi(0) = 0. \tag{6}$$

3. Integral Representation of Solution of the Boundary Value Problem

The solution of boundary value problem (1)–(6) is sought in the form of a sum of two integrals

$$u(x,t) = u_1(x,t) + W(x,t), \tag{7}$$

where

$$u_1(x,t) = \int_0^t \frac{x\varphi(\tau)}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{x^2}{4a^2(t-\tau)}\right) d\tau,$$

$$W(x,t) = 2a^2 \int_0^t \theta(\tau)\, \frac{\partial G(x,r,t-\tau)}{\partial r}\bigg|_{r=\alpha_0\tau} d\tau.$$

Integral $u_1(x,t)$ is the solution of equation

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2}, \qquad 0 < x < \infty, \quad 0 < t < T, \tag{8}$$

satisfying boundary condition

$$u_1(0,t) = \varphi(t), \tag{9}$$

and zero initial condition $u_1(x,0) = 0$.

Integral $W(x,t)$ is a heat potential of the double layer with density $\theta(t)$ and with kernel

$$K(x,t) = 2a^2\, \frac{\partial G(x,r,t-\tau)}{\partial r}\bigg|_{r=\alpha_0\tau},$$

where

$$G(x,r,t) = \frac{1}{2a\sqrt{\pi t}} \left\{ \exp\left(-\frac{(x-r)^2}{4a^2 t}\right) - \exp\left(-\frac{(x+r)^2}{4a^2 t}\right) \right\}, \qquad 0 < x, r, t < \infty. \tag{10}$$

4. Properties of the Function $G(x, r, t)$

The function $G(x,r,t)$ defined by expression (10) is a positive, infinitely differentiable function and it is a solution of equation

$$\frac{\partial G}{\partial t} \equiv a^2 \frac{\partial^2 G}{\partial x^2}.$$

On the border region Ω the function $G(x,r,t)$ satisfies the following conditions:

a) $\lim\limits_{t\to 0} \int_0^\infty G(x,r,t)dx = 1$,

b) $G(0,r,t) \equiv 0$, $0 < r, t < \infty$,

c) $\lim_{t\to 0} G(x,r,t) = \begin{cases} 0, & x \neq r, \\ +\infty, & x = r. \end{cases}$

The derivative $\frac{\partial G(x,r,t)}{\partial r}$ has the form

$$\frac{\partial G(x,r,t)}{\partial r} = \frac{1}{2a\sqrt{\pi t}} \left\{ \frac{x-r}{2a^2 t} \exp\left(-\frac{(x-r)^2}{4a^2 t}\right) + \frac{x+r}{2a^2 t} \exp\left(-\frac{(x+r)^2}{4a^2 t}\right) \right\}.$$

It is a solution of the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial r}\right) \equiv a^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial G}{\partial r}\right) \tag{11}$$

everywhere in the domain $0 < x, r, t < \infty$ which satisfies the condition

$$\frac{\partial G(0,r,t)}{\partial r} \equiv 0, \qquad 0 < r, t < \infty. \tag{12}$$

5. Properties of the Potential $W(x, t)$

From (11) and (12) it follows that the potential $W(x,t)$ in any bounded continuous function $\theta(t)$ satisfies the equation

$$\frac{\partial W}{\partial t} \equiv a^2 \frac{\partial^2 W}{\partial x^2}, \tag{13}$$

and the boundary condition

$$W(0,t) = 0, \qquad 0 < t < \infty. \tag{14}$$

The movable boundary $x = \alpha_0 t$ of the region Ω is the set of discontinuity points of the potential $W(x,t)$ which has the jump on the boundary

$$\lim_{x\to\alpha(t)\pm 0} W(x,t) = \mp\theta(t) + \widetilde{W}(t),$$

$$\widetilde{W}(t) = W(\alpha_0 t, t) = \int_0^t \frac{\alpha_0 \theta(\tau)}{2a\sqrt{\pi}\sqrt{t-\tau}} \left[\exp\left(-\frac{\alpha_0^2 (t-\tau)}{4a^2}\right) + \frac{t+\tau}{t-\tau} \exp\left(-\frac{\alpha_0^2 (t+\tau)^2}{4a^2 (t-\tau)}\right) \right] d\tau.$$

6. Integral Equation for the Density of the Double-Layer Potential

On the basis of (8) and (13) we conclude that integral representation (7) satisfies equation (1). Expressions (9) and (14) show that function (7) satisfies boundary condition (2) with an arbitrary density $\theta(t)$. Moving an interior point (x,t) of the region Ω in (7) to the boundary $x = \alpha_0 t$ a for fixed t and taking into account boundary condition (3), we obtain the integral equation for the density $\theta(t)$

$$\theta(t) = g(t) + \int_0^t K(t,\tau)\, \theta(\tau)\, d\tau, \tag{15}$$

where

$$K(t,\tau) = \frac{\alpha_0}{2a\sqrt{\pi}\sqrt{t-\tau}} \left[\exp\left(-\frac{\alpha_0^2 (t-\tau)}{4a^2}\right) + \frac{t+\tau}{t-\tau} \exp\left(-\frac{\alpha_0^2 (t+\tau)^2}{4a^2 (t-\tau)}\right) \right], \tag{16}$$

$$g(t) = u_1(\alpha_0 t, t) - \psi(t). \tag{17}$$

7. The Method of Successive Approximations Picard's Solution of the Integral Equation

The Picard method of successive approximations for the solution of integral equation (15) is applied. Recall that the functional Picard's sequence $\{\theta_n(t)\}$, $n = 0, 1, 2, \ldots$ is based on the following recursive formula

$$\begin{cases} \theta_0(t) = g(t), \\ \theta_n(t) = g(t) + \int_0^t K(t,\tau)\theta_{n-1}(\tau)d\tau, \qquad n = 1, 2, \ldots. \end{cases} \tag{18}$$

Consecutively performing iterative procedure (18) we obtain

$$\theta_1(t) = g(t) + \int_0^t K(t,\tau)g(\tau)d\tau,$$

$$\theta_2(t) = g(t) + \int_0^t K(t,\tau)\left\{g(\tau) + \int_0^\tau K(\tau,\tau_1)g(\tau_1)d\tau_1\right\}d\tau$$

$$= g(t) + \int_0^t K(t,\tau)g(\tau)d\tau + \int_0^t K(t,\tau)d\tau \int_0^\tau K(\tau,\tau_1)g(\tau_1)d\tau_1.$$

If we substitute $\theta_2(t)$ into $\theta_3(t)$ and continue this process further, we obtain in the general case

$$\theta_n(t) = g(t) + \sum_{i=1}^{n} \int_0^t K_i(t,\tau)g(\tau)d\tau = g(t) + \int_0^t \left(\sum_{i=1}^{n} K_i(t,\tau)\right)g(\tau)d\tau, \quad n = 1, 2, \ldots, \tag{19}$$

where the functions $K_i(t,\tau)$ are called re-cores and are computed by the formula

$$K_1(t,\tau) = K(t,\tau), \quad K_{i+1}(t,\tau) = \int_\tau^t K_1(t,\tau_1)K_i(\tau_1,\tau)d\tau_1, \quad i = 1, 2, \ldots. \tag{20}$$

In the case when the functions $g(t)$ and $K(t,\tau)$ are continuous and bounded, we have the inequalities

$$\left|g(t)\right| \le A_g, \quad |K(t,\tau)| \le A_K, \quad 0 \le t \le T, \quad 0 \le \tau \le t,$$

then

$$|K_{i+1}(t,\tau)| \le A_K^{i+1}\frac{(t-\tau)^i}{i!}, \qquad i = 0, 1, 2, \ldots.$$

Thus, the integral operator in (15) is contractive. The solution of equation (15) can be written in the form

$$\theta(t) = g(t) + \int_0^t R(t,\tau)\, g(\tau)\, d\tau, \tag{21}$$

where the solving kernel

$$R(t,\tau) = \sum_{i=0}^{\infty} K_{i+1}(t,\tau) \tag{22}$$

has the estimation

$$|R(t,\tau)| \le \sum_{i=0}^{\infty} A_K^{i+1}\frac{(t-\tau)^i}{i!} = A_K e^{A_K(t-\tau)}.$$

It follows from this evaluation that series (22) converges uniformly and its sum $R(t,\tau)$ and the function $\theta(t)$ in (21) are continuous and bounded.

8. A Special Property of the Integral Operator of Equation (15)

Definition 8.1. We introduce the functional class $M_\beta(0,T)$, to which belong all continuous functions $f(t)$ defined on the interval $(0,T)$ and satisfying the condition $\left|f(t)\right| \le A_f t^\varepsilon$ where $\varepsilon > \beta$.

Generally speaking, the condition of boundedness of $K(t,\tau)$ and $g(t)$ for the convergence of Picard's method is not strictly necessary. For the convergence in the class of bounded functions $g(t)$ it is sufficient that the kernel $K(t,\tau)$ was continuous and satisfies the inequality

$$|K(t,\tau)| \le A_K(t-\tau)^\varepsilon, \quad \text{where} \quad \varepsilon > -1.$$

In particular, it follows from this inequality that

$$\lim_{t\to 0}\int_0^t K(t,\tau)d\tau \le \lim_{t\to 0}\frac{A_K t^{\varepsilon+1}}{\varepsilon+1} = 0. \tag{23}$$

We will show that in the case of problem (1)–(3) condition (23) fails. We write the integrand $K(t,\tau)$ as the sum

$$K(t,\tau) = \hat{K}(t,\tau) + N(t,\tau), \tag{24}$$

where

$$\hat{K}(t,\tau) = \frac{\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\alpha_0^2 t\tau}{a^2(t-\tau)}\right),$$

$$\begin{aligned} N(t,\tau) = {} & \frac{\alpha_0}{2a\sqrt{\pi}\sqrt{t-\tau}}\exp\left(-\frac{\alpha_0^2(t-\tau)}{4a^2}\right)\left[1-\exp\left(-\frac{\alpha_0^2 t\tau}{a^2(t-\tau)}\right)\right] \\ & -\frac{\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\left[1-\exp\left(-\frac{\alpha_0^2(t-\tau)}{4a^2}\right)\right]\exp\left(-\frac{\alpha_0^2 t\tau}{a^2(t-\tau)}\right). \end{aligned} \tag{25}$$

Let us consider the integral operator

$$\left(\hat{\mathbf{K}}g\right)(t) = \int_0^t \hat{K}(t,\tau)g(\tau)d\tau = \int_0^t \frac{g_0\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\alpha_0^2 t\tau}{a^2(t-\tau)}\right)d\tau,$$

where $g(t) \equiv g_0 \ne 0$, $g_0 = \text{constant}$. Using the substitutions $x = \frac{\alpha_0\sqrt{t\tau}}{a\sqrt{t-\tau}}$, $0 < x < \infty$, $dx = \frac{\alpha_0 t^{\frac{3}{2}}}{2a\sqrt{\tau}(t-\tau)^{\frac{3}{2}}}d\tau$, $\tau = \frac{a^2x^2}{a^2x^2+\alpha_0^2 t}$, we get

$$\left(\hat{\mathbf{K}}g_0\right)(t) = \frac{2g_0}{\sqrt{\pi}}\int_0^\infty \exp(-x^2)\frac{a\,x\,dx}{\sqrt{a^2x^2+\alpha_0^2 t}},$$

hence

$$\lim_{t\to 0}\left(\hat{\mathbf{K}}g_0\right)(t) = g_0 \ne 0. \tag{26}$$

The second term in (24) is a Volterra kernel. Indeed, given that the exponential function in (25) is positive and does not exceed unity, and also that for all $x \ge 0$ the inequality $1-e^{-x} \le x$ holds, we get the inequalities

$$\left|\exp\left(-\frac{\alpha_0^2(t-\tau)}{4a^2}\right)-\exp\left(-\frac{\alpha_0^2(t+\tau)^2}{4a^2(t-\tau)}\right)\right| \le 1, \qquad \left|1-\exp\left(-\frac{\alpha_0^2(t-\tau)}{4a^2}\right)\right| \le \frac{\alpha_0^2(t-\tau)}{4a^2}.$$

These inequalities enable us to derive the estimation

$$|N(t,\tau)| \le \frac{D_1}{\sqrt{t-\tau}}, \tag{27}$$

where the constant D_1 is determined by the constant parameters T, a and α_0 of boundary value problem (1)–(3).

In this case for an integrable function $g(t) \in M_\varepsilon(0,T)$, $\varepsilon > -\frac{1}{2}$ we get

$$\left|\int_0^t N(t,\tau)g(\tau)d\tau\right| \leq D_1 A_g \int_0^t \frac{\tau^\varepsilon}{\sqrt{t-\tau}}d\tau = D_1 A_g \mathrm{B}\left(\varepsilon+1, \frac{1}{2}\right) t^{\varepsilon+\frac{1}{2}} \xrightarrow{t\to 0} 0, \tag{28}$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function of Euler.

From (24), (26) and (28) we find that at $g(t) \equiv g_0 \neq 0$

$$\lim_{t\to 0} (\mathbf{K}g)(t) = \lim_{t\to 0} \int_0^t K(t,\tau)g(\tau)d\tau = g_0 \neq 0. \tag{29}$$

It follows from (29) that the integral operator in equation (15) with kernel (16) for boundary problem (1)–(3) in the class of bounded functions $g(t)$ is not compressive. Therefore, despite the fact that equation (15) formally refers to the type of Volterra integral equations of the second kind, it requires a separate study.

Equation (29) for the first time was obtained by S. N. Kharin [3] when he considered the asymptotic properties of the solution of integral equation (15). T. E. Omarov showed the existence and uniqueness of the solution of integral equation (15) in the class of the functions $g(t)$ decreasing with $t \to 0$ faster than any power function [5]. M. I. Ramazanov investigated the spectral properties of the operator $\left(\hat{K}\theta\right)(t)$ [6]. In particular, he found that the eigenfunction of the operator is the function $g(t) = \frac{1}{\sqrt{t}}$. The study of the integral equations with such properties continues at the present time [1, 7].

9. The Analytical Expression for $K_n(t,\tau)$

We perform the study of the kernel $K(t,\tau)$ in details. By applying an iterative procedure (20) to (24), we obtain

$$K_n(t,\tau) = \hat{K}_n(t,\tau) + N_n(t,\tau), \tag{30}$$

where

$$\hat{K}_{n+1}(t,\tau) = \int_\tau^t \hat{K}_1(t,s)\hat{K}_n(s,\tau)ds, \tag{31}$$

$$N_{n+1}(t,\tau) = \int_\tau^t \left[\hat{K}_1(t,s)N_n(s,\tau) + N_1(t,s)\hat{K}_n(s,\tau) + N_1(t,s)N_n(s,\tau)\right]ds, \quad n = 1,2,\ldots, \tag{32}$$

$$\hat{K}_1(t,\tau) = \hat{K}(t,\tau), \quad N_1(t,\tau) = N(t,\tau).$$

Now let us obtain an analytical expression for $\hat{K}_n(t,\tau)$. Using the method of the complete induction and direct evaluation of the integral in (31) we will show that

$$\hat{K}_n(t,\tau) = \frac{n\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{n^2\alpha_0^2 t\tau}{a^2(t-\tau)}\right). \tag{33}$$

Formula (33) is valid for $n = 1$. We calculate $\hat{K}_{n+1}(t,\tau)$ by the formula (31). We have

$$\hat{K}_{n+1}(t,\tau) = \int_\tau^t \frac{\alpha_0 t}{a\sqrt{\pi}(t-s)^{\frac{3}{2}}} \frac{n\alpha_0 s}{a\sqrt{\pi}(s-\tau)^{\frac{3}{2}}} \exp\left(-\frac{\alpha_0^2 ts}{a^2(t-s)} - \frac{n^2\alpha_0^2 s\tau}{a^2(s-\tau)}\right)ds$$

$$= \frac{n\alpha_0^2 t}{a^2\pi} \int_\tau^t \frac{s}{(t-s)^{\frac{3}{2}}(s-\tau)^{\frac{3}{2}}} \exp\left(-\frac{\alpha_0^2 ts}{a^2(t-s)} - \frac{n^2\alpha_0^2 s\tau}{a^2(s-\tau)}\right)ds.$$

The last integral is evaluated using the following substitutions of variables of integration. First, we apply the substitution $s = \frac{1}{x}$ and introducing the notation $t = \frac{1}{u}$, $\tau = \frac{1}{v}$.

$$\hat{K}_{n+1}(t,\tau) = \frac{n\alpha_0^2 v^{\frac{3}{2}}\sqrt{u}}{a^2\pi}\int_u^v \exp\left(-\frac{\alpha_0^2}{a^2(x-u)} - \frac{n^2\alpha_0^2}{a^2(v-x)}\right)\frac{dx}{(x-u)^{\frac{3}{2}}(v-x)^{\frac{3}{2}}}.$$

Second, we apply the substitutions

$$y = \sqrt{\frac{n(x-u)}{v-x}}, \quad 0 < y < \infty, \quad x = \frac{nu + vy^2}{n+y^2}, \quad dx = \frac{2n(v-u)y}{(n+y^2)^2}dy,$$

$$x - u = \frac{(v-u)y^2}{n+y^2}, \quad v - x = \frac{n(v-u)}{n+y^2}, \quad \frac{1}{(x-u)^{\frac{3}{2}}(v-x)^{\frac{3}{2}}} = \frac{\left(n+y^2\right)^3}{n^{\frac{3}{2}}(v-u)^3 y^3},$$

$$\frac{\alpha_0^2}{a^2(x-u)} + \frac{n^2\alpha_0^2}{a^2(v-x)} = \frac{\alpha_0^2(n+y^2)}{a^2(v-u)y^2} + \frac{n^2\alpha_0^2(n+y^2)}{a^2(v-u)n}$$

$$= \frac{\alpha_0^2}{a^2(v-u)}\left(\frac{n}{y^2} + 1 + n^2 + ny^2\right) = \frac{\alpha_0^2}{a^2(v-u)}\left(\left(\frac{n}{y^2} - 2n + ny^2\right) + \left(1 + 2n + n^2\right)\right)$$

$$= \frac{\alpha_0^2(n+1)^2}{a^2(v-u)} + \frac{\alpha_0^2 n}{a^2(v-u)}\left(y - \frac{1}{y}\right)^2.$$

$$\hat{K}_{n+1}(t,\tau) = \frac{2\sqrt{n}\alpha_0^2 v^{\frac{3}{2}}\sqrt{u}}{a^2\pi(v-u)^2}\exp\left(-\frac{\alpha_0^2(n+1)^2}{a^2(v-u)}\right)\int_0^\infty \exp\left(-\frac{\alpha_0^2 n}{a^2(v-u)}\left(y - \frac{1}{y}\right)^2\right)\frac{\left(n+y^2\right)}{y}\frac{dy}{y}.$$

Third, we apply the substitutions

$$z = \frac{1}{2}\left(y - \frac{1}{y}\right), \quad -\infty < z < \infty, \quad y = \sqrt{1+z^2} + z, \quad \frac{dy}{y} = \frac{dz}{\sqrt{1+z^2}},$$

$$\frac{n+y^2}{y} = \frac{n}{y} + y = n\left(\sqrt{1+z^2} - z\right) + \sqrt{1+z^2} + z = (n+1)\sqrt{1+z^2} - (n-1)z,$$

$$\hat{K}_{n+1}(t,\tau) = \frac{2\sqrt{n}\alpha_0^2 v^{\frac{3}{2}}\sqrt{u}}{a^2\pi(v-u)^2}\exp\left(-\frac{\alpha_0^2(n+1)^2}{a^2(v-u)}\right)$$

$$\times\int_{-\infty}^{\infty}\exp\left(-\frac{4\alpha_0^2 n}{a^2(v-u)}z^2\right)\left[(n+1)\sqrt{1+z^2} - (n-1)z\right]\frac{dz}{\sqrt{1+z^2}}.$$

Since

$$\int_{-\infty}^{\infty}\exp\left(-\frac{4\alpha_0^2 n}{a^2(v-u)}z^2\right)[(n-1)z]\frac{dz}{\sqrt{1+z^2}} = 0$$

the integrand function is odd, we get

$$\hat{K}_{n+1}(t,\tau) = \frac{2(n+1)\sqrt{n}\alpha_0^2 v^{\frac{3}{2}}\sqrt{u}}{a^2\pi(v-u)^2}\exp\left(-\frac{\alpha_0^2(n+1)^2}{a^2(v-u)}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{4\alpha_0^2 n}{a^2(v-u)}z^2\right)dz.$$

Taking into account that

$$\int_{-\infty}^{\infty}\exp\left(-\frac{4\alpha_0^2 n}{a^2(v-u)}z^2\right)dz = \frac{a\sqrt{v-u}}{2\alpha_0\sqrt{n}}\sqrt{\pi},$$

and recovery t and τ we obtain the expression

$$\hat{K}_{n+1}(t,\tau) = \frac{(n+1)\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(n+1)^2\alpha_0^2 t\tau}{a^2(t-\tau)}\right),$$

which is similar to (33).

10. Evaluation of the Expression $N_{n+1}(t,\tau)$

We show that integral (32) is an operator of Volterra type. We write (32) as the sum of three terms

$$N_{n+1}(t,\tau) = N_n^A(t,\tau) + N_n^B(t,\tau) + N_n^C(t,\tau), \tag{34}$$

where

$$N_n^A(t,\tau) = \int_\tau^t \tilde{K}_1(t,s)N_n(s,\tau)ds,$$

$$N_n^B(t,\tau) = \int_\tau^t N_1(t,s)\tilde{K}_n(s,\tau)ds,$$

$$N_n^C(t,\tau) = \int_\tau^t N_1(t,s)N_n(s,\tau)ds$$

and find their estimation. Considering the positive $\tilde{K}_n(t,\tau)$ and using inequality (27), we obtain

$$\left|N_n^A\right| \le D_n \int_\tau^t \frac{\alpha_0 t}{a\sqrt{\pi}(t-s)^{\frac{3}{2}}} \exp\left(-\frac{\alpha_0^2 ts}{a^2(t-s)}\right)\frac{ds}{\sqrt{s-\tau}}.$$

Performing the change of variable of integration $s = \frac{tz^2+\tau}{1+z^2}$, $ds = \frac{2(t-\tau)z}{(1+z^2)^2}dz$, $0 < z < \infty$, after simple calculation we can write the inequality

$$\left|N_n^A(t,\tau)\right| \le \frac{D_n}{\sqrt{t-\tau}}.$$

For the second integral in (34) we have

$$\left|N_n^B(t,\tau)\right| \le \int_\tau^t \frac{D_1}{\sqrt{t-s}} \frac{n\alpha_0 s}{a\sqrt{\pi}(s-\tau)^{\frac{3}{2}}} \exp\left(-\frac{n^2\alpha_0^2\tau s}{a^2(s-\tau)}\right)ds.$$

Changing the variable of integration $s = \frac{t+\tau z^2}{1+z^2}$, $ds = -\frac{2(t-\tau)z}{(1+z^2)^2}dz$, $0 < z < \infty$, we get

$$\left|N_n^B(t,\tau)\right| \le D_1 \frac{2n\alpha_0}{a\sqrt{\pi}(t-\tau)} \exp\left(-\frac{n^2\alpha_0^2 t\tau}{a^2(t-\tau)}\right)\int_0^\infty \exp\left(-\frac{n^2\alpha_0^2\tau^2 z^2}{a^2(t-\tau)}\right)\left(\tau + \frac{t-\tau}{1+z^2}\right)dz,$$

and after simple calculations we obtain the inequality

$$\left|N_n^B(t,\tau)\right| \le \frac{D_1}{\sqrt{t-\tau}} + \frac{D_1\alpha_0\sqrt{\pi}}{a}.$$

The third term in (34) is bounded by the constant

$$N_n^C(t,\tau) \le D_1 D_n \int_\tau^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s-\tau}}ds = \pi D_1 D_n.$$

Thus, all components of sum (34) are the Volterra type and in general they can be represented by the expression

$$N_n(t,\tau) = \frac{D_n}{\sqrt{t-\tau}} + F_n(t,\tau), \tag{35}$$

where $F_n(t,\tau)$ is a bounded continuous function.

11. Convergence of Iterative Process (18)

Theorem 11.1. *If $g(t) \in M_{\frac{1}{2}}(0,T)$, then there exists a unique solution $\theta(t) \in M_0(0,T)$ of integral equation (15), which can be calculated by the method of the Picard successive approximations.*

Proof. The considered iterative process forms a functional series

$$\hat{R}(t,\tau) = \sum_{n=1}^{\infty} \hat{K}_n(t,\tau) = \sum_{n=1}^{\infty} \frac{n\alpha_0 t}{a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{n^2\alpha_0^2 t\tau}{a^2(t-\tau)}\right), \tag{36}$$

$$Q(t,\tau) = \sum_{n=1}^{\infty} N_n(t,\tau). \tag{37}$$

Let us consider the properties of series (36) and find its evaluation. Functional series (36) converges for all $0 < \tau < t < \infty$, but at the point $\tau = 0$ convergence is not uniform, because

$$\hat{R}(t,0) = \sum_{n=1}^{\infty} \frac{n\alpha_0}{a\sqrt{\pi t}} = \infty.$$

To estimate (36) consider the numerical series

$$S(\xi) = \sum_{n=1}^{\infty} n e^{-\xi n^2}, \quad \xi > 0.$$

A number $S(\xi)$ represents the numerical approximation of the integral

$$I(\xi) = \int_0^{\infty} x e^{-\xi x^2} dx$$

according to the formula of partition of rectangles with the length of the subintervals equal to one. Given that all terms of the series $S(\xi)$ are positive, we can show that there is a constant C_R such that

$$\sum_{n=1}^{\infty} n e^{-\xi n^2} < C_R \int_0^{\infty} x e^{-\xi x^2} dx = \frac{C_R}{2\xi}.$$

Applying this estimate to (36) with $\xi = \frac{\alpha_0^2 t\tau}{a^2(t-\tau)}$ we get

$$\hat{R}(t,\tau) < \frac{C\alpha_0 t}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \frac{a^2(t-\tau)}{\alpha_0^2 t\tau} = \frac{C}{2\sqrt{\pi}} \frac{1}{\tau\sqrt{t-\tau}}. \tag{38}$$

It follows from inequality (38) that if $g(t) \in M_{\frac{1}{2}}(0,T)$, then the integral converges:

$$\int_0^t \hat{R}(t,\tau) g(\tau) d\tau \in M_0(0,T).$$

□

Every finite sum of functional series (37) based on the evaluation of (35) is the kernel of the Volterra type. A detailed study of the convergence of this series was not conducted, due to excessive complexity of the expressions for $N(t,\tau)$. However, a large number of numerical test calculations performed by the formula (18), confirm the convergence of the iterative process to the function $\theta(t) \in M_0(0,T)$.

The result is that for the function

$$R(t,\tau)=\sum_{n=1}^{\infty}K_n(t,\tau)$$

and any function $g(t)\in M_{\frac{1}{2}}(0,T)$ the integral

$$\int_0^t R(t,\tau)g(\tau)d\tau=\int_0^t\left(\sum_{n=1}^{\infty}K_n(t,\tau)\right)g(\tau)d\tau$$

exists. Thus, in expression (19) it is acceptable to change the order of summation and integration, after passing to the limit with $n\to\infty$, resulting in the assertion of Theorem 11.1. The solution of the integral equation (15) can be written in the standard Volterra equation

$$\theta(t)=g(t)+\int_0^t\left(\sum_{i=1}^{\infty}K_i(t,\tau)\right)g(\tau)d\tau=g(t)+\int_0^t R(t,\tau)g(\tau)d\tau.$$

12. The Sufficiency of Conditions (4)–(6) for $g(t)\in M_{\frac{1}{2}}(T)$

Theorem 12.1. *The function $g(t)\in M_{\frac{1}{2}}(0,T)$ under constraints (4)–(6) for the boundary functions $\varphi(t)$ and $\psi(t)$.*

Proof. Let us write expression (17) for the function as a sum

$$g(t)=\left[u_1(\alpha_0 t,t)-\varphi(t)Y(t)\right]+\left[\varphi(t)(1-Y(t)\right]+\left[\varphi(t)-\psi(t)\right], \tag{39}$$

where

$$Y(t)=\int_0^t\frac{\alpha_0 t}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\alpha_0^2t^2}{4a^2(t-\tau)}\right)d\tau. \tag{40}$$

We will show that each term in (39) belongs to the class of $M_{\frac{1}{2}}(0,T)$.

Firstly, we shall evalute the first term of sum (39). We have

$$\left|u_1(\alpha_0 t,t)-\varphi(t)Y(t)\right|\le\int_0^t\frac{\alpha_0 t\left|\varphi(\tau)-\varphi(t)\right|}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\alpha_0^2t^2}{4a^2(t-\tau)}\right)d\tau$$

$$<\int_0^t\frac{A\alpha_0 t(t-\tau)^{\sigma}}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\alpha_0^2t^2}{4a^2(t-\tau)}\right)d\tau=\frac{A\alpha_0 t}{2a\sqrt{\pi}}\int_0^t(t-\tau)^{\sigma-\frac{3}{2}}\exp\left(-\frac{\alpha_0^2t^2}{4a^2(t-\tau)}\right)d\tau.$$

After replacing the integration variable $\tau=t-tz$ we get

$$\left|u_1(\alpha_0 t,t)-\varphi(t)Y(t)\right|<\frac{A\alpha_0 t^{\sigma+\frac{1}{2}}}{2a\sqrt{\pi}}\int_0^1 z^{\sigma-\frac{3}{2}}\exp\left(-\frac{\alpha_0^2 t}{4a^2 z}\right)dz\le\frac{A\alpha_0 t^{\sigma+\frac{1}{2}}}{(2\sigma-1)a\sqrt{\pi}}. \tag{41}$$

Secondly, we shall evaluate the second term of sum (39). Applying to (40) the substitution of the integration variable $z=\frac{\alpha_0 t}{2a\sqrt{t-\tau}}$, $dz=\frac{\alpha_0 t}{4a(t-\tau)^{\frac{3}{2}}}$, we get

$$1-Y(t)=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\alpha_0\sqrt{t}}{2a}}\exp\left(-z^2\right)dz\le\frac{\alpha_0\sqrt{t}}{a\sqrt{\pi}}.$$

In this case

$$\left|\varphi(t)(1-Y(t)\right|<\frac{A\alpha_0}{a\sqrt{\pi}}t^{\frac{1}{2}+\sigma}. \tag{42}$$

Thirdly, we shall evaluate the third term of sum (39). Using the condition (6), we find

$$\left|\varphi(t)-\psi(t)\right|\le\left|\varphi(t)-\varphi(0)\right|+\left|\psi(t)-\psi(0)\right|<2At^{\sigma}. \tag{43}$$

Taking into account (41), (42) and (43) we conclude that each term in (39) belongs to the class $M_{\frac{1}{2}}(0,T)$, which implies the assertion of Theorem 12.1. □

13. An Example of a Numerical Calculation

We consider an example of constructing a numerical solution $u(x,t)$ of problem (1) - (3) for boundary values $\varphi(t) = 0$ and $\psi(t) = \sin(\alpha_0 t)\exp(-a^2 t)$, $0 \le t \le T$, and compare it with the exact analytic solution of the same problem $U(x,t) = \sin(x)\exp(-a^2 t)$ where $(x,t) \in \Omega$.

The solution $u(x,t)$ is computed in two stages. At the first stage, the solution $\theta(t)$ of integral equation (15) is found by the iteration procedure (18); in the second stage, $u(x,t)$ is calculated by the formula (7). The iterative process (18) is completed when the following inequality is satisfied:

$$\int_0^T \frac{|\theta_n(t) - \theta_{n-1}(t)|}{\sqrt{t}} dt < \varepsilon, \quad n = 1, 2, \dots, \tag{44}$$

where ε is the accuracy of calculating $\theta(t)$. We emphasize that, because of $\theta_n(t) \in M_{\frac{1}{2}}(T)$, the relations

$$\frac{|\theta_n(t) - \theta_{n-1}(t)|}{\sqrt{t}} \in M_0(T) \quad \text{and} \quad \lim_{t\to 0} \frac{|\theta_n(t) - \theta_{n-1}(t)|}{\sqrt{t}} = 0$$

for the integral expression in (44) are valid.

The results of the three calculation options performed in the MATLAB environment for $T = 10$, $a = 0.3$ and for three values of $\alpha_0 = 0.5, 1.5, 2.5$ are presented in Table 1. The N_{iter} values indicate the number of iterations required to achieve an accuracy of $\varepsilon = 0.001$ in each variant. The maximum deviations of the numerical solution $u(x,t)$ from the exact $U(x,t)$ are attained at a finite boundary point $x = \alpha_0 T$ in all three cases.

Table 1. The results of the calculations, depending on the velocity of the boundary α_0

T	a	α_0	N_{iter}	$\max_{(x,t)\in\Omega} \lvert u(x,t) - U(x,t)\rvert$
10	0.3	0.5	31	0.0013
10	0.3	1.5	162	0.0051
10	0.3	2.5	402	0.0078

The calculation results for $\alpha_0 = 1.5$ are also presented graphically in Figures 1 and 2.

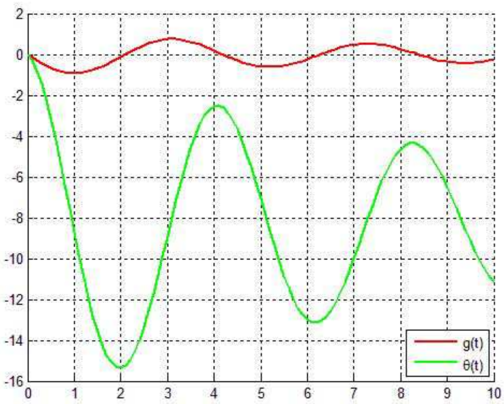

Fig.1. Density of the potential

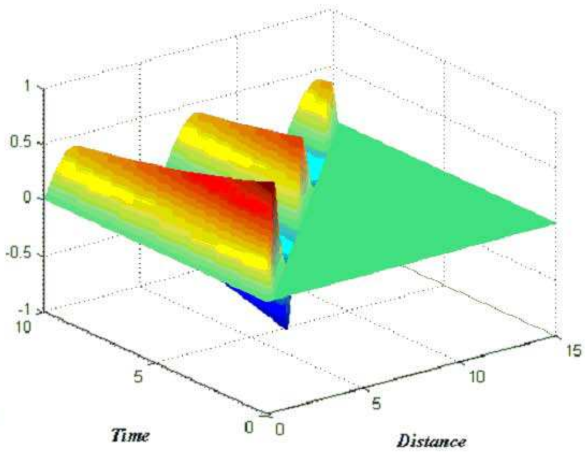

Fig.2. Temperature field

It is evident from Figure 1 that the changes in $\theta(t)$ follow with some delay from the changes in $g(t)$, but they have a large amplitude.

14. Conclusion

The main reason to consider the solution of integral equation (15) is the non-uniform convergence of the approximation by iteration in a neighborhood $t = 0$ for the partial sums $K_n(t, \tau)$.

If the boundary concordance condition (6) $\varphi(0) = \psi(0) = v_0 \neq 0$ holds, then the boundary value problem can be reduced to problem (1)–(6) by replacing $u(x, t) = v(x, t) + v_0$.

The condition $g(t) \in M_{\frac{1}{2}}(0, T)$ is too rigid, as it directly follows from conditions (4)–(6) typical for the most practical problems of the heat conduction.

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