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Existence and Uniqueness Results for an Inverse Problem for a Semilinear Equation with Final Overdetermination

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Abstract. In the present paper, unique solvability of a source identification inverse problem for a semilinear equation with a final overdetermination in a Banach space is investigated. Moreover, the first order of accuracy Rothe difference scheme is presented for numerically solving this problem. The existence and uniqueness result for this difference scheme is given. The efficiency of the proposed method is evaluated by means of computational experiments.

1. Introduction and Problem Formulation

The practical problems in the process of diffusion and conduction of materials are induced to the source identification inverse problems (SIPs). The SIPs are the most frequently encountered inverse problems because of their importance in the applied sciences [2, 4–6, 8, 9, 12–15]. In many papers, the unknown source term is considered as time- or space-dependent only. Many researchers have paid attention to the investigation of SIPs governed by linear or nonlinear equations. Actually, the SIPs for linear equations are of greater interest comparing for the nonlinear ones. In [4, 6], well-posedness of SIPs for linear partial differential equations have been studied. Kamynin [8] investigated the existence and uniqueness of the solution of a space-dependent SIP for linear parabolic equation. Orazov and Sadybekov [9] proved the existence and uniqueness of classical solutions to a class of problems which models the process of determining the temperature and density of heat sources for a given initial and finite temperature.

On the other hand, existence and uniqueness results for SIPs governed by semilinear and nonlinear partial differential equations have been investigated using different techniques. For example, in [12, 13], method of semigroups is applied to prove existence and uniqueness of the solution of a time-dependent SIP for semilinear parabolic equations. Also in these papers, finite difference method is applied for the

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numerical solution of these problems. Borukhov and Zayats [5] studied a time-dependent SIP for a nonlinear hyperbolic or parabolic equation.

It is known that by exact solution of a problem one means a solution that satisfies the governing equation and initial/boundary/overdetermined conditions. Several methods for obtaining the solutions of SIPs can be found in [7, 10]. Unfortunately, due to the nonlinearity and variable coefficients in a number of SIPs it is very difficult to obtain their exact solutions. At this point, the numerical methods play a vital role and there are considerably many investigations on the numerical aspects of these problems [4, 6, 14, 15].

In this paper, the following inverse problem of identifying the pair (u, p), governed by a semilinear equation with a final overdetermination

$$\begin{cases} \frac{du}{dt} + Au(t) = f(t, u(t)) + p, \ 0 < t < T, \\ u(0) = \varphi, \ u(T) = \psi \end{cases} \tag{1}$$

is considered in an arbitrary Banach space E. Here, A is a linear operator acting in E whose domain is $D(A)$, with the assumption that $-A$ is the generator of the analytic semigroup $\exp\{-tA\}$ with an exponentially decreasing norm

$$\left\|e^{-tA}\right\|_{E\to E} \le Me^{-\delta t}, \ t\left\|Ae^{-tA}\right\|_{E\to E} \le M, \ t \ge 0. \tag{2}$$

We may note that the main results of this paper were presented in [13] without their proofs.

The rest of the paper is organized as follows: In Section 2, unique solvability of inverse problem (1) is established. Moreover, the first order of accuracy Rothe difference scheme for the numerical solution of problem (1) is presented and the unique solvability of this difference scheme is given. In Section 3, an application of the main results of the paper is considered. In Section 4, the theoretical results are evaluated by some computational results.

2. Main Results

2.1. Differential Problem

For the theoretical considerations in this subsection, we shall introduce the functional space $C([0,T], E)$ that is the space of all abstract continuous functions $\phi(t)$ defined on $[0, T]$ with values in a Banach space E, equipped with the norm

$$\|\phi\|_{C([0,T],E)} = \max_{0\le t\le T} \|\phi(t)\|_E .$$

Definition 2.1. The pair $(u(t), p)$ is said to be the solution of problem (1) in $C([0,T], E) \times E_1$ if the following conditions are satisfied:

(i) $\frac{du}{dt}, Au(t) \in C([0,T], E), p \in E_1 \subset E$;

(ii) $(u(t), p)$ satisfies the equation and boundary conditions in (1).

Now, we establish the following theorem on the unique solvability of problem (1) in $C([0,T], E) \times D(A)$.

Theorem 2.2. *Suppose that $\varphi, \psi \in D(A)$ and the abstract function $f \in C([0,T], E)$ satisfies the Lipschitz condition*

$$\|f(t,u) - f(t,v)\|_E \le K\|u - v\|_E, \ K > 0 \tag{3}$$

for all $t \in [0,T]$, $u, v \in E$ with $\alpha = \frac{2K}{\delta} \max\{1, M\}(1 + M) < 1$. Then, a unique solution of problem (1) exists in $C([0,T], E) \times D(A)$.

Proof. If $u(t)$ is a solution of problem (1), then applying the Cauchy formula, we get

$$u(t) = e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s, u(s))\, ds + A^{-1}\left(I - e^{-tA}\right) p. \tag{4}$$

Since the semigroup $\exp\{-tA\}$ obeys exponential decay estimate (2), the operator $I - e^{-TA}$ has a bounded inverse. Hence, using the condition $u(T) = \psi$, we get

$$A^{-1}p = \left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right). \tag{5}$$

Applying formulas (4) and (5), we can write

$$u(t) = e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s, u(s))\, ds$$
$$+ \left(I - e^{-tA}\right)\left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right) \tag{6}$$

for all $0 \le t \le T$. Now, let us introduce the continuous mapping $B : C([0, T], E) \times E \to C([0, T], E) \times E$ such that

$$\begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix} = B \begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix}$$

defined by

$$B \begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix} = \left\{ \begin{array}{l} e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s, u(s)\, ds + \beta \left(I - e^{-tA}\right)\left(I - e^{-TA}\right)^{-1} A^{-1}p \\ + (1-\beta)\left(I - e^{-tA}\right)\left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right) \\ \left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right) \end{array} \right\}.$$

Here $\beta \in [0, 1]$. Putting $\beta = 0$, and denoting

$$B_1 u(t) = e^{-tA}\varphi + \int_0^t e^{-(t-s)A} f(s, u(s)\, ds$$
$$+ \left(I - e^{-tA}\right)\left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right),$$

$$B_2 u(t) = \left(I - e^{-TA}\right)^{-1} \left(\psi - e^{-TA}\varphi - \int_0^T e^{-(T-s)A} f(s, u(s))\, ds \right),$$

we get

$$B\begin{pmatrix} u(t) \\ A^{-1}p \end{pmatrix} = \begin{Bmatrix} B_1u(t) \\ B_2u(t) \end{Bmatrix}.$$

Applying estimate (2) and formula

$$\left(I - e^{-TA}\right)^{-1} = I - e^{-TA} + e^{-2TA} - \cdots,$$

we get

$$\left\|\left(I - e^{-TA}\right)^{-1}\right\|_{E\to E} \leq 1 + Me^{-\delta T} + Me^{-2\delta T} + \cdots \leq \max\{1, M\} \frac{1}{1 - e^{-\delta T}}. \tag{7}$$

Thus, applying the triangle inequality, estimates (2) and (7), we get

$$\begin{aligned}
&\|B_1u(t) - B_1v(t)\|_E \leq \left\|\left(I - e^{-TA}\right)^{-1}\right\|_{E\to E} \\
&\times \left\{ \int_0^t \left\|e^{-(t-s)A} - e^{-(T-s)A}\right\|_{E\to E} \left\|f(s, u(s)) - f(s, v(s))\right\|_E ds \right. \\
&\left. + \int_t^T \left\|\left(I - e^{-tA}\right)e^{-(T-s)A}\right\|_{E\to E} \left\|f(s, u(s)) - f(s, v(s))\right\|_E ds \right\} \\
&\leq \max\{1, M\} \frac{K}{1 - e^{-\delta T}} (1 + M) \left\{ \int_0^t e^{-\delta(t-s)} \|u(s) - v(s)\|_E ds \right. \\
&\left. + \int_t^T e^{-\delta(T-s)} \|(u(s) - v(s))\|_E ds \right\} \leq \frac{2K}{\delta} \max\{1, M\} (1 + M) \max_{0\leq s\leq T} \|(u(s) - v(s))\|_E \\
&= \frac{2K}{\delta} \max\{1, M\} (1 + M) \|u - v\|_{C([0,T],E)},
\end{aligned}$$

$$\begin{aligned}
\|B_2u(t) - B_2v(t)\|_E &\leq \left\|\left(I - e^{-TA}\right)^{-1}\right\|_{E\to E} \int_0^T \left\|e^{-(T-s)A}\right\|_{E\to E} \left\|f(s, u(s)) - f(s, v(s))\right\|_E ds \\
&\leq \max\{1, M\} \frac{K}{1 - e^{-\delta T}} \int_0^T e^{-\delta(T-s)} \|u(s) - v(s)\|_E ds \\
&\leq \frac{K}{\delta} \max\{1, M\} \max_{0\leq s\leq T} \|(u(s) - v(s))\|_E = \frac{K}{\delta} \max\{1, M\} \|u - v\|_{C([0,T],E)}
\end{aligned}$$

for any $t \in [0, T]$. Hence, we get

$$\max\{\|B_1u(t) - B_1v(t)\|_E, \|B_2u(t) - B_2v(t)\|_E\} \leq \frac{2K}{\delta} \max\{1, M\} (1 + M) \|u - v\|_{C([0,T],E)}.$$

Therefore, if $\alpha = \frac{2K}{\delta} \max\{1, M\} (1 + M) < 1$, then by Banach fixed point theorem there exists a unique solution of problem (1). Moreover, $\left(u(t), A^{-1}p\right) = \lim_{m\to\infty} \left({}_mu(t), A_m^{-1}p\right)$, where $\left({}_mu(t), A_m^{-1}p\right)$, $m = 1, 2, \ldots$ are defined by the formulas

$$ {}_{m}u\left(t\right) = e^{-tA}\varphi + \int_{0}^{t} e^{-(t-s)A} f\left(s,{}_{m-1}u(s)\right) ds $$

$$ + \left(I - e^{-tA}\right)\left(I - e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_{0}^{T} e^{-(T-s)A} f\left(s,{}_{m-1}u(s)\right) ds\right), $$

$$ A_{m}^{-1}p = \left(I - e^{-TA}\right)^{-1}\left(\psi - e^{-TA}\varphi - \int_{0}^{T} e^{-(T-s)A} f\left(s,{}_{m-1}u(s)\right) ds\right), $$

where ${}_{0}u\left(t\right) \in C\left(\left[0,T\right], D\left(A\right)\right)$ is given. □

2.2. Difference Problem

For the theoretical considerations in this subsection, we shall introduce the functional space $C\left(\left[0,T\right]_{\tau}, E\right)$ that is the space of grid functions $\phi^{\tau} = \left\{\phi_k\right\}_{0}^{N}$ defined on $[0,T]_{\tau}$ with values in a Banach space E and is endowed with the norm

$$ \left\|\phi^{\tau}\right\|_{C([0,T]_{\tau},E)} = \max_{1\le k\le N}\left\|\phi_k\right\|_{E}. $$

For numerically solving problem (1), the first order of accuracy Rothe difference scheme

$$ \begin{cases} \frac{u_k-u_{k-1}}{\tau} + Au_k = f(t_k,u_k) + p, \\ 1 \le k \le N,\ t_k = k\tau,\ N\tau = T, u_0 = \varphi,\ u_N = \psi \end{cases} \tag{8} $$

is constructed. Now, we establish the following theorem on the unique solvability of difference scheme (8) in $C\left(\left[0,T\right]_{\tau}, E\right)$.

Theorem 2.3. *Suppose that $\varphi, \psi \in D\left(A\right)$ and f satisfies Lipschitz condition (3) with the assumption $\alpha = \frac{2K}{\delta}\max\left\{1,M\right\}\left(1+M\right) < 1$, then a unique solution of difference scheme (8) exists in $C\left(\left[0,T\right]_{\tau}, E\right)$.*

Proof. From difference scheme (8) it follows that

$$ u_k = Ru_{k-1} + R\left[f(t_k,u_k) + p\right]\tau $$

for all $1 \le k \le N$, where $R = (I+\tau A)^{-1}$. Using this recurrence relation one can see that

$$ u_k = R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i,u_i)\tau + A^{-1}\left(I - R^k\right)p $$

for all $1 \le k \le N$. Applying the condition $u_N = \psi$, we obtain

$$ A^{-1}p = \left(I - R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i,u_i)\tau\right\}. $$

Then, we can write that

$$ u_k = R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_i,u_i)\tau + \left(I - R^k\right)\left(I - R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_i,u_i)\tau\right\}. $$

Now, let us introduce the mapping F^τ that maps $C([0,T]_\tau, E)\times E$ onto $C([0,T]_\tau, E)\times E$ defined by

$$F^\tau\begin{pmatrix} u_k \\ A^{-1}p\end{pmatrix} = \left\{\begin{array}{c} R^k\varphi + \sum_{i=1}^k R^{k-i+1}f(t_i,u_i)\tau \\ +\left(I-R^k\right)\left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^N R^{N-i+1}f(t_i,u_i)\tau\right\} \\ \\ \left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^N R^{N-i+1}f(t_i,u_i)\tau\right\}\end{array}\right\}.$$

Denoting

$$\begin{aligned} B_1^\tau u_k &= R^k\varphi + \sum_{i=1}^k R^{k-i+1}f(t_i,u_i)\tau \\ &+\left(I-R^k\right)\left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^N R^{N-i+1}f(t_i,u_i)\tau\right\}, \\ B_2^\tau u_k &= \left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^N R^{N-i+1}f(t_i,u_i)\tau\right\}, \end{aligned}$$

we get

$$F^\tau\begin{pmatrix} u_k \\ A^{-1}p\end{pmatrix} = \left\{\begin{array}{c} B_1^\tau u_k \\ B_2^\tau u_k\end{array}\right\}.$$

Applying estimate (2), we get

$$\left\|R^k\right\|_{E\to E} \le \frac{M}{(1+\delta\tau)^k},\ k\ge 1;\ \left\|\left(I-R^N\right)^{-1}\right\|_{E\to E} \le \frac{\max\{1,M\}}{1-\frac{1}{(1+\delta\tau)^N}}. \tag{9}$$

Applying the triangle inequality, Lipschitz condition (3) and estimate (9), we reach

$$\begin{aligned} &\left\|B_1^\tau u_k - B_1^\tau v_k\right\|_E \le \left\|\left(I-R^N\right)^{-1}\right\|_{E\to E} \\ &\times\left\{\sum_{i=1}^k \left\|R^{k-i+1} - R^{N-i+1}\right\|_{E\to E}\left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E\tau \right.\\ &\left.+\sum_{i=k+1}^N \left\|\left(I-R^k\right)R^{N-i+1}\right\|_{E\to E}\left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E\tau\right\} \\ &\le \max\{1,M\}\frac{K}{1-\frac{1}{(1+\delta\tau)^N}}(1+M)\left\{\sum_{i=1}^k \frac{1}{(1+\delta\tau)^{k-i+1}}\left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E\tau\right. \\ &\left.+\sum_{i=k+1}^N \frac{1}{(1+\delta\tau)^{N-i+1}}\left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E\tau\right\} \le \frac{2K}{\delta}\max\{1,M\}(1+M)\max_{1\le i\le N}\|u_i-v_i\|_E \\ &= \frac{2K}{\delta}\max\{1,M\}(1+M)\|u^\tau - v^\tau\|_{C([0,T]_\tau,E)}, \end{aligned}$$

$$\left\|B_2^\tau u_k - B_2^\tau v_k\right\|_E \le \left\|\left(I-R^N\right)^{-1}\right\|_{E\to E} \sum_{i=1}^{N} \left\|R^{N-i+1}\right\|_{E\to E} \left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E \tau$$

$$\le \max\{1,M\} \frac{K}{1-\frac{1}{(1+\delta\tau)^N}} \sum_{i=1}^{N} \frac{1}{(1+\delta\tau)^{N-i+1}} \left\|f(t_i,u_i)-f(t_i,v_i)\right\|_E \tau$$

$$\le \frac{K}{\delta}\max\{1,M\} \max_{1\le i\le N} \|u_i - v_i\|_E = \frac{K}{\delta}\max\{1,M\} \|u^\tau - v^\tau\|_{C([0,T]_\tau,E)}$$

for any $k = 1,2,...,N$. Hence, we get

$$\max\left\{\left\|B_1^\tau u_k - B_1^\tau v_k\right\|_E, \left\|B_2^\tau u_k - B_2^\tau v_k\right\|_E\right\} \le \frac{2K}{\delta}\max\{1,M\}(1+M)\|u^\tau - v^\tau\|_{C([0,T]_\tau,E)}.$$

Thus, if $\alpha = \frac{2K}{\delta}\max\{1,M\}(1+M) < 1$, then by Banach fixed point theorem there exists a unique solution of problem (8) in $C([0,T]_\tau, E)$. Moreover, $\left(u^\tau, A^{-1}p\right) = \lim_{m\to\infty}\left({}_m u^\tau, A_m^{-1}p\right)$, where $\left({}_m u^\tau, A_m^{-1}p\right)$, $m = 1,2,...$ are defined by the formulas

$$\begin{aligned} B_1^\tau u_k &= R^k\varphi + \sum_{i=1}^{k} R^{k-i+1} f(t_{i,m}\, u_i)\tau \\ &+ \left(I-R^k\right)\left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f(t_{i,m}\, u_i)\tau\right\}, \\ A_m^{-1}p &= \left(I-R^N\right)^{-1}\left\{\psi - R^N\varphi - \sum_{i=1}^{N} R^{N-i+1} f\left(t_{i,m}\, u_i\right)\tau\right\}, \end{aligned}$$

where ${}_0u_i \in C([0,T]_\tau, D(A))$ is given. □

3. Application

We shall investigate an application of Theorems 2.2-2.3. We consider the inverse problem for a semilinear parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma u(t,x) = p(x) + f(t,x,u(t,x)), \\ x \in (0,1),\ t \in (0,T), \\ u(0,x) = \varphi(x),\ u(T,x) = \psi(x),\ x \in [0,1], \\ u(t,0) = u(t,1) = 0,\ t \in [0,T], \end{cases} \tag{10}$$

where $(u(t,x), p(x))$ are the unknown vector functions, $\varphi(x)$, $\psi(x)$ and $a(x)$ are given sufficiently smooth functions with $a(x) \ge a > 0$ and $\sigma \ge 0$. It is well-known that (see, for example [1, 3])

$$-A = a(x)\frac{d^2}{dx^2} - \sigma I \tag{11}$$

is the generator of the analytic semigroup $\exp\{-tA\}$ with an exponentially decreasing norm

$$\left\|e^{-tA}\right\|_{C[0,1]\to C[0,1]} \le Me^{-\delta t},\ t\left\|Ae^{-tA}\right\|_{C[0,1]\to C[0,1]} \le M,\ t \ge 0.$$

Here $\delta > \sigma$.

Theorem 3.1. *Suppose that $\varphi(x), \psi(x) \in C^2[0,1]$ and the continuous function $f(t,x,u)$ on $[0,T]\times[0,1]$ satisfies the Lipschitz condition*

$$\left\|f(t,\cdot,u(\cdot)) - f(t,\cdot,v(\cdot))\right\|_{C[0,1]} \le K\|u(\cdot) - v(\cdot)\|_{C[0,1]},\ K > 0 \tag{12}$$

for all $t \in [0,T]$, $u, v \in C[0,1]$ with $\alpha = \frac{2K}{\delta}\max\{1,M\}(1+M) < 1$. Then, a unique solution of problem (10) exists in $C([0,T], C[0,1]) \times C^2[0,1]$.

For numerically solving (10) the following Rothe difference scheme is constructed:

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\tau} - a(x_n)\dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \sigma u_n^k = \phi_n^k, \\ \phi_n^k = f(t_k, x_n, u_n^k) + p(x_n),\ t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\ x_n = nh,\ 1 \le n \le M-1,\ Mh = 1, \\ u_n^0 = \varphi(x_n),\ u_n^N = \psi(x_n), x_n = nh,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N. \end{cases} \tag{13}$$

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied, then problem (14) has a unique solution.*

Theorem 3.3. *[2, Theorem 2] In implementation of difference scheme (14) the formula*

$$p^h(x) = A\psi^h(x) - Av_N^h,$$

where A is the difference analogue of the operator defined in (11), holds. Here values of v_N^h are obtained from the solution of auxiliary difference scheme

$$\begin{cases} \dfrac{v_n^k - v_n^{k-1}}{\tau} - a(x_n)\dfrac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} + \sigma v_n^k = \Psi_n^k, \\ \Psi_n^k = f(t_k, x_n, u_n^k),\ t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\ x_n = nh,\ 1 \le n \le M-1,\ Mh = 1, \\ u_n^0 - u_n^N = \varphi(x_n) - \psi(x_n), x_n = nh,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N. \end{cases} \tag{14}$$

4. Numerical Discussion

To support the theoretical considerations by a numerical example, let us consider the space-dependent SIP governed by a semilinear parabolic equation subject to a final overdetermination

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - e^{-x}\frac{\partial^2 u(t,x)}{\partial x^2} + 2u(t,x) = p(x) + f(t,x,u(t,x)), x \in (0,1),\ t \in (0,T], \\ f(t,x,u(t,x)) = \left(x^2 - x - 2e^{-x}\right)e^{-t} - \cos(\pi x) + \frac{1}{2}\sin\left(e^t u(t,x)\right) - \frac{1}{2}\sin\left(x^2 - x\right), \\ u(0,x) = x^2 - x,\ u(T,x) = e^{-T}\left(x^2 - x\right),\ x \in [0,1], \\ u(t,0) = u(t,1) = 0,\ t \in [0,T], \end{cases} \tag{15}$$

where the exact solution pair of the problem is $(u(t,x), p(x)) = \left(e^{-t}\left(x^2 - x\right), \cos(\pi x)\right)$.

Numerical solution of this test problem (15) can be obtained by the following iterative difference scheme of first order of accuracy:

$$\begin{cases} \dfrac{{}_mu_n^k -_m u_n^{k-1}}{\tau} - e^{-x_n}\dfrac{{}_mu_{n+1}^k - 2\,{}_mu_n^k +_m u_{n-1}^k}{h^2} + 2_m u_n^k =_m \phi_n^k, \\ {}_m\phi_n^k = f(t_k, x_{n},{}_{m-1} u_n^k) +_m p\left(x_n\right), \ t_k = k\tau, \ 1 \le k \le N, \ N\tau = T, \\ x_n = nh, 1 \le n \le M-1, \ Mh = 1, \\ {}_mu_n^0 = x^2 - x, \ {}_mu_n^N = e^{-T}\left(x_n^2 - x_n\right), x_n = nh, 0 \le n \le M, \\ {}_mu_0^k =_m u_M^k = 0, 0 \le k \le N, m = 1, 2, \cdots . \end{cases} \tag{16}$$

Here m denotes the iteration number and an initial guess ${}_0u_n^k, 0 \le k \le N, 0 \le n \le M$ is to be made.

For solving difference scheme (16), we follow the numerical steps of article [2]. For $0 \le k \le N, 0 \le n \le M$, the algorithm is as follows:

1. $m = 1$,
2. ${}_{m-1}u_n^k$ is known (initially given for $m = 1$ or calculated in step 5),
3. ${}_mv_n^k$ is calculated,
4. ${}_mp\left(x_n\right)$ is calculated,
5. ${}_mu_n^k$ is calculated,
6. if the max absolute error between ${}_{m-1}u_n^k$ and ${}_mu_n^k$ is greater than the given tolerance value, take m=m+1 and go to step 2. Otherwise, terminate the iteration process and take the ${}_mu_n^k$ as the result of the given problem.

The values ${}_mv_s^N$, $s = n \pm 1, n$ are obtained from the solution of the first order of accuracy auxiliary difference scheme

$$\begin{cases} \dfrac{{}_mv_n^k -_m v_n^{k-1}}{\tau} - e^{-x_n}\dfrac{{}_mv_{n+1}^k - 2_m v_n^k +_m v_{n-1}^k}{h^2} + 2\,{}_mv_n^k =_m \Psi_n^k, \\ {}_m\Psi_n^k = f(t_k, x_{n},{}_{m-1} u_n^k), \ t_k = k\tau, \ 1 \le k \le N, \ N\tau = T, \\ x_n = nh, 1 \le n \le M-1, \ Mh = 1, \\ {}_mv_n^N -_m v_n^0 = \left(e^{-T} - 1\right)\left(x_n^2 - x_n\right), x_n = nh, 0 \le n \le M, \\ {}_mv_0^k =_m v_M^k = 0, 0 \le k \le N. \end{cases} \tag{17}$$

The matrix representation of auxiliary difference scheme is

$$A_n\ {}_mv_{n+1} + B_n\ {}_mv_n + C_n\ {}_mv_{n-1} = F_n, \ 1 \le n \le M-1, \ v_0 = v_M = \overrightarrow{0}. \tag{18}$$

Here, F_n is an $(N+1)\times 1$ column vector, A_n, B_n, C_n are $(N+1)\times(N+1)$ matrices given below:

$$A_n = C_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

$$B_n = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 1 \\ d & b_n & 0 & 0 & \cdots & 0 \\ 0 & d & b_n & 0 & \cdots & 0 \\ 0 & 0 & d & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d & b_n \end{bmatrix},$$

$$F_n = \begin{bmatrix} \left(e^{-T}-1\right)\left(x_n^2-x_n\right) \\ {}_m\Psi_n^1 \\ {}_m\Psi_n^2 \\ \vdots \\ {}_m\Psi_n^N \end{bmatrix},$$

where

$$a_n = \frac{e^{-x_n}}{h^2},\ b_n = \frac{1}{\tau} + \frac{2e^{-x_n}}{h^2} + 2,\ d = -\frac{1}{\tau},\ 1 \le n \le M.$$

For finding the solution of matrix equation (18) we apply the modified Gauss elimination method [11], in which the solution is searched in the form

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1},\ n = M-1, \cdots 1,\ v_0 = v_M = \overrightarrow{0},$$

where α_n 's are $(N+1)\times(N+1)$ matrices and β_n 's are $(N+1)\times 1$ column vectors. In order to find α_n 's and β_n 's we employ the formulas

$$\begin{aligned} \alpha_{n+1} &= -\left(B_n + C_n\alpha_n\right)^{-1} A_n,\ n = 1, \cdots, M-1 \\ \beta_{n+1} &= -\left(B_n + C_n\alpha_n\right)^{-1}\left(F_n - C_n\beta_n\right),\ n = 1, \cdots, M-1, \end{aligned}$$

where α_1 is the $(N+1)\times(N+1)$ zero matrix and β_1 is the $(N+1)\times 1$ zero vector.

Second, for finding the values of ${}_mp(x_n)$ at the grid points, by Theorem 3.3, we employ the formula

$$\begin{aligned} &{}_mp(x_n) = 2e^{-T}\left(x_n^2 - x_n - e^{-x_n}\right) \\ &+e^{-x_n}\frac{{}_mv_{n+1}^N - 2{}_mv_n^N + {}_mv_{n-1}^N}{h^2} - 2\,{}_mv_n^N, x_n = nh, 1 \le n \le M-1. \end{aligned} \tag{19}$$

Once we find the values of ${}_mp(x_n)$ at the grid points, we put them into difference scheme (16) and difference scheme becomes direct. Then, we write it in the matrix form as

$$A_n\,{}_mu_{n+1} + D_n\,{}_mu_n + C_n\,{}_mu_{n-1} = G_n,\ 1 \le n \le M-1,\ u_0 = u_M = \overrightarrow{0}, \tag{20}$$

where A_n, C_n were the matrices defined above and

$$D_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ d & b_n & 0 & 0 & \cdots & 0 \\ 0 & d & b_n & 0 & \cdots & 0 \\ 0 & 0 & d & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d & b_n \end{bmatrix},$$

$$G_n = \begin{bmatrix} x_n^2 - x_n \\ {}_m\phi_n^1 \\ {}_m\phi_n^2 \\ \vdots \\ {}_m\phi_n^N \end{bmatrix}.$$

Applying the modified Gauss elimination method given above, matrix equation (20) is solved. In computations the initial guess is chosen as identical zero grid function ${}_0u_n^k = 0$ and when the maximum error between two consecutive results of iterative difference scheme (16) becomes less than 10^{-8}, the iterative process is terminated.

In Tables 1-3, the numerical results where errors are computed by the formulas

$$E_{u_m} = \max_{\substack{1\le k\le N \\ 1\le n\le M-1}} \left|{}_m u_n^k - u(t_k, x_n)\right|,$$

$$E_{p_m} = \max_{1\le n\le M-1} \left|{}_m p_n - p(x_n)\right|$$

are presented. Tables are constructed for $T = 1, T = 2$ and $T = 3$, respectively. As can be seen from tables, these numerical experiments support the theoretical the statements. The number of iterations and maximum errors are decreasing with the increase of grid points.

Table 1. The error analysis for difference scheme (16) when $T = 1$

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	7	7	6
E_p	0.1744	0.0871	0.0435
E_u	0.0052	0.0025	0.0012

Table 2. The error analysis for difference scheme (16) when $T = 2$

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	10	9	9
E_p	0.1249	0.0632	0.0317
E_u	0.0073	0.0038	0.0019

Table 3. The error analysis for difference scheme (16) when $T = 3$

	$N = M = 20$	$N = M = 40$	$N = M = 80$
Number of iterations (m)	22	21	20
E_p	0.1008	0.0529	0.0270
E_u	0.0075	0.0040	0.0021

5. Summary and Concluding Remarks

In the present study, the unique solvability of a source identification inverse abstract problem governed by a semilinear equation under the Lipschitz condition is established. Furthermore, the unique solvability of the corresponding first order of accuracy Rothe difference scheme is investigated. As an application, a source identification inverse problem for a semilinear parabolic equation with final overdetermination is considered. For numerically solving this problem the Rothe difference scheme is proposed. For showing the efficiency of this difference scheme some numerical results are recorded. Obviously, these results in error analysis validate the theoretical considerations of the paper. As expected, in the case when the value of T is increased, so does the number of iterations. However, this situation does not cause any problem on convergence of the iterations.

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