Filomat 32:3 (2018), 885–892
https://doi.org/10.2298/FIL1803885K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Isoperimetric Inequalities for the Cauchy-Dirichlet Heat Operator

Tynysbek Sh. Kal'menov[a], Aidyn Kassymov[b], Durvudkhan Suragan[c]

[a]*Institute of Mathematics and Mathematical Modeling 125 Pushkin str., 050010 Almaty, Kazakhstan*
[b]*Institute of Mathematics and Mathematical Modeling 125 Pushkin str., 050010 Almaty, Kazakhstan, and Al-Farabi Kazakh National University, 71 Al-Farabiave, 050040 Almaty, Kazakhstan*
[c]*Department of Mathematics School of Science and Technology, Nazarbayev University 53 Kabanbay Batyr Ave, Astana 010000, Kazakhstan, and Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*

Abstract. In this paper we prove that the first s-number of the Cauchy-Dirichlet heat operator is minimized in a circular cylinder among all Euclidean cylindric domains of a given measure. It is an analogue of the Rayleigh-Faber-Krahn inequality for the heat operator. We also prove a Hong-Krahn-Szegö and a Payne-Pólya-Weinberger type inequalities for the Cauchy-Dirichlet heat operator.

1. Introduction

The classical Rayleigh-Faber-Krahn inequality asserts that the first eigenvalue of the Laplacian with the Dirichlet boundary condition in $\mathbb{R}^d$, $d \geq 2$, is minimized in a ball among all domains of the same measure. However, the minimum of the second Dirichlet Laplacian eigenvalue is achieved by the union of two identical balls. This fact is called a Hong-Krahn-Szegö inequality. In this paper analogues of both inequalities are proved for the heat operator. That is, we prove that the first s-number of the Cauchy-Dirichlet heat operator is minimized in the circular cylinder among all Euclidean cylindric domains of a given measure and the second s-number of the Cauchy-Dirichlet heat operator is minimized in the union of two identical circular cylinders among all Euclidean cylindric domains of a given measure.

Payne, Pólya and Weinberger (see [6] and [7]) studied the ratio $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$ for the Dirichlet Laplacian and conjectured that the ratio $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$ is maximized in the disk among all domains of the same area. In 1991 Ashbaugh and Benguria [1] proved this conjecture for any bounded domain $\Omega \subset \mathbb{R}^d$. In the present paper we also investigate that the same ratio for $s-$numbers of the Cauchy-Dirichlet heat operator and prove an analogue of this Payne-Pólya-Weinberger inequality for the heat operator. These isoperimetric inequalities have been mainly studied for the Laplacian related operators, for example, for the p-Laplacians and bi-Laplacians. However, there are also many papers on this subject for other type of compact operators. For

2010 *Mathematics Subject Classification*. Primary 35P05; Secondary 58J50
Keywords. Cauchy-Dirichlet heat operator, s-number, Rayleigh-Faber-Krahn inequality, Hong-Krahn-Szegö inequality, Payne-Pólya-Weinberger inequality
Received: 26 December 2016; Revised: 19 April 2017; Accepted: 19 April 2017
Communicated by Allaberen Ashyralyev
This paper was published under project AP05133239 and target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan. The third author was partially supported by NU SPG grant "Spectral geometric inequalities in PDE".
Email addresses: kalmenov@math.kz (Tynysbek Sh. Kal'menov), kassymov@math.kz (Aidyn Kassymov), durvudkhan.suragan@nu.edu.kz (Durvudkhan Suragan)

instance, in the recent work [8] the authors proved Rayleigh-Faber-Krahn type inequality and Hong-Krahn-Szegö type inequality for the Riesz potential (see also [9], [10] and [11]). All these works were for self-adjoint operators. Our main goal is to extend those known isoperimetric inequalities for non-self-adjoint operators (see, e.g. [4]). The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce *a priori* bounds for spectral invariants of operators on arbitrary domains.

Summarizing our main results of the present paper, we prove the following facts:

- Rayleigh-Faber-Krahn type inequality: the first s-number of the Cauchy-Dirichlet heat operator is minimized on the circular cylinder among all Euclidean cylindric domains of a given measure;
- Hong-Krahn-Szegö type inequality: the minimizer domain of the second s-number of the Cauchy-Dirichlet heat operator among cylindric bounded open sets with a given measure is achieved by the union of two identical circular cylinders ;
- Payne-Pólya-Weinberger type inequality: the ratio $\frac{s_2}{s_1}$ is maximized in the circular cylinder among all cylindric domains of a given measure;

In Section 2 we discuss some necessary tools. In Section 3 we present main results of this paper and their proofs.

2. Preliminaries

Let $D = \Omega \times (0,T)$ be a cylindrical domain, where $\Omega \subset \mathbb{R}^d$ is a simply-connected set with smooth boundary $\partial\Omega$. We consider the heat operator with the Cauchy-Dirichlet problem (see, for example, [12]) $\diamond : L^2(D) \to L^2(D)$ in the form

$$\diamond u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \ \forall t \in (0,T). \end{cases} \tag{1}$$

The operator $\diamond$ is a non-selfadjoint operator in $L^2(D)$. An adjoint operator $\diamond^*$ to operator $\diamond$ is

$$\diamond^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,T) = 0, \ \ x \in \Omega, \\ v(x,t) = 0, \ \ x \in \partial\Omega, \ \ \forall t \in (0,T). \end{cases} \tag{2}$$

Recall that if A is a compact operator, then the eigenvalues of the operator $(A^*A)^{1/2}$, where A^* is the adjoint operator to A, are called s-numbers of the operator A (see e.g. [2]). A direct calculation gives that the operator $\diamond^*\diamond$ has the formula

$$\diamond^*\diamond u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ \ x \in \partial\Omega, \ \ \forall t \in (0,T). \end{cases} \tag{3}$$

3. Main Results and their Proofs

We consider a (circular) cylinder $C = B \times (0,T)$ where $B \subset \mathbb{R}^d$ is an open ball. Let Ω be a simply-connected set with smooth boundary $\partial\Omega$ with $|B| = |\Omega|$, where $|\Omega|$ is the Lebesgue measure of the domain Ω.

Let us introduce operators $T, L : L^2(\Omega) \to L^2(\Omega)$

$$Tz(x) = \begin{cases} -\Delta z(x), \\ z(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{4}$$

and we denote an eigenvalue of T by μ.

Similarly,

$$Lz(x) = \begin{cases} \Delta^2 z(x), \\ z(x) = 0, \quad x \in \partial\Omega, \\ \Delta z(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{5}$$

and we denote an eigenvalue of L by λ.

Lemma 3.1. *The first eigenvalue of the operator L is minimized in the ball B among all domains Ω of the same measure with $|B| = |\Omega|$.*

Proof. The Rayleigh-Faber-Krahn inequality is valid for the Dirichlet Laplacian, that is, the ball is a minimizer of the first eigenvalue of the operator T among all domains Ω with $|B| = |\Omega|$. A straightforward calculation from (4) gives that

$$T^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \quad x \in \partial\Omega, \\ \Delta z(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{6}$$

Thus, $T^2 = L$ and $\mu^2 = \lambda$. Now using the Rayleigh-Faber-Krahn inequality we establish $\lambda_1(B) = \mu_1^2(B) \leq \mu_1^2(\Omega) = \lambda_1(\Omega)$, i.e. $\lambda_1(B) \leq \lambda_1(\Omega)$. □

Theorem 3.2. *The first s-number of the operator $\diamond$ is minimized in the circular cylinder C among all cylindric domains of a given measure, that is,*

$$s_1(C) \leq s_1(D),$$

for all D with $|D| = |C|$.

Proof. Recall that $D = \Omega \times (0, T)$ is a bounded measurable set in $\mathbb{R}^{d+1}$. Its symmetric rearrangement $C = B \times (0, T)$ is the circular cylinder with the measure equals to the measure of D, i.e. $|D| = |C|$. Let u be a nonnegative measurable function in D, such that all its positive level sets have finite measure. With the definition of the symmetric-decreasing rearrangement of u we can use the layer-cake decomposition [5], which expresses a nonnegative function u in terms of its level sets as

$$u(x, t) = \int_0^\infty \chi_{\{u(x,t)>z\}} dz, \quad \forall t \in (0, T), \tag{7}$$

where χ is the characteristic function of the domain. The function

$$u^*(x, t) = \int_0^\infty \chi_{\{u(x,t)>z\}^*} dz, \quad \forall t \in (0, T), \tag{8}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

Consider the following spectral problem

$$\diamond^* \diamond u = su,$$

$$\diamond^*\diamond u(x,t)=\begin{cases}-\frac{\partial^2 u(x,t)}{\partial t^2}+\Delta_x^2 u(x,t)=su(x,t),\\ u(x,0)=0,\;\; x\in\Omega,\\ \frac{\partial u(x,t)}{\partial t}|_{t=T}-\Delta_x u(x,t)|_{t=T}=0,\;\; x\in\Omega,\\ u(x,t)=0,\;\; x\in\partial\Omega,\;\;\forall t\in(0,T),\\ \Delta_x u(x,t)=0,\;\; x\in\partial\Omega,\;\;\forall t\in(0,T).\end{cases} \tag{9}$$

Our domain D is the cylindrical domain, we can write $u(x,t)=X(x)\varphi(t)$ and $u_1(x,t)=X_1(x)\varphi_1(t)$ is the first eigenfunction of the operator $\diamond^*\diamond$. We can rewrite above fact,

$$-\varphi_1''(t)X_1(x)+\varphi_1(t)\Delta^2 X_1(x)=s_1\varphi_1(t)X_1(x). \tag{10}$$

By the variational principle for the operator $\diamond^*\diamond$, we get

$$s_1(D)=\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_\Omega X_1^2(x)dx+\int_0^T\varphi_1^2(t)dt\int_\Omega X_1(x)\Delta^2X_1(x)dx}{\int_0^T\varphi_1^2(t)dt\int_\Omega X_1^2(x)dx}$$

$$=\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_\Omega(X_1(x))^2dx+\mu_1^2(\Omega)\int_0^T\varphi_1^2(t)dt\int_\Omega(X_1(x))^2dx}{\int_0^T\varphi_1^2(t)dt\int_\Omega(X_1(x))^2dx},$$

where $\mu_1(\Omega)$ is the first eigenvalue of the operator Laplace-Dirichlet.

For each non-negative function $X_1\in L^2(\Omega)$, we obtain (see [5])

$$\int_\Omega|X_1(x)|^2dx=\int_B|X_1^*(x)|^2dx. \tag{11}$$

where X_1^* is the symmetric decreasing rearrangement of the function X_1.

Applying Lemma 3.1 and (11), we get

$$s_1(D)=\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_\Omega(X_1(x))^2dx+\mu_1^2(\Omega)\int_0^T\varphi_1^2(t)dt\int_\Omega(X_1(x))^2dx}{\int_0^T\varphi_1^2(t)dt\int_\Omega(X_1(x))^2dx}$$

$$\geq\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_B(X_1^*(x))^2dx+\mu_1^2(B)\int_0^T\varphi_1^2(t)dt\int_B(X_1^*(x))^2dx}{\int_0^T\varphi_1^2(t)dt\int_B(X_1^*(x))^2dx}$$

$$=\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_B(X_1^*(x))^2dx+\int_0^T\varphi_1^2(t)dt\int_B X_1^*(x)(\mu_1^2(B)X_1^*(x))dx}{\int_0^T\varphi_1^2(t)dt\int_B(X_1^*(x))^2dx}$$

$$=\frac{-\int_0^T\varphi_1''(t)\varphi_1(t)dt\int_B(X_1^*(x))^2dx+\int_0^T\varphi_1^2(t)dt\int_B X_1^*(x)\Delta^2X_1^*(x)dx}{\int_0^T\varphi_1^2(t)dt\int_B(X_1^*(x))^2dx}$$

$$=\frac{-\int_0^T\int_B u_1^*(x,t)\frac{\partial^2u_1^*(x,t)}{\partial t^2}dxdt+\int_0^T\int_B u_1^*(x,t)\Delta_x^2u_1^*(x,t)dxdt}{\int_0^T\int_B(u_1^*(x,t))^2dxdt}$$

$$\geq\inf_{z(x,t)\neq0}\frac{-\int_0^T\int_B z(x,t)\frac{\partial^2z(x,t)}{\partial t^2}dxdt+\int_0^T\int_B z(x,t)\Delta_x^2z(x,t)dxdt}{\int_0^T\int_B z^2(x,t)dxdt}=s_1(C).$$

The proof is complete. □

Corollary 3.3. *The norm of the operator $\diamond^{-1}$ is maximized in the circular cylinder C among all cylindric domains of a given measure, i.e.* $\|\diamond^{-1}\|_D \leq \|\diamond^{-1}\|_C$.

Theorem 3.4. *The second s-number of the operator $\diamond$ is minimized in the union of two identical circular cylinders among all cylindric domains of the same measure.*

Let $D^+ = \{(x,t) : u(x,t) > 0\}$, and $D^- = \{(x,t) : u(x,t) < 0\}$. In proofs we will use the notations

$$u_2^+(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^+, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_2^-(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^-, \\ 0, & \text{otherwise}. \end{cases}$$

To proof Theorem 3.4 we need the following lemma:

Lemma 3.5. *For the operator $\diamond^*\diamond$ we obtain the equalities*

$$s_1(D^+) = s_1(D^-) = s_2(D).$$

Proof. For the operator T we have the following equality [3]

$$\mu_1(\Omega^+) = \mu_1(\Omega^-) = \mu_2(\Omega). \tag{12}$$

Let us solve the spectral problem (9) by using Fourier's method in the domain $D^\pm$, so

$$\begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s(D^\pm)u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega^\pm, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ \ x \in \Omega^\pm, \\ u(x,t) = 0, \ \ x \in \partial\Omega^\pm, \ \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ \ x \in \partial\Omega^\pm, \ \ \forall t \in (0,T). \end{cases} \tag{13}$$

Thus, we arrive at the spectral problems for $\varphi(t)$ and $X(x)$

$$\begin{cases} \Delta^2 X(x) = \mu^2(\Omega^\pm)X(x), \ \ x \in \Omega^\pm, \\ X(x) = 0, \ \ x \in \partial\Omega^\pm, \\ \Delta X(x) = 0, \ \ x \in \partial\Omega^\pm, \end{cases} \tag{14}$$

and

$$\begin{cases} \varphi''(t) + (s(D^\pm) - \mu^2(\Omega^\pm))\varphi(t) = 0, \ \ t \in (0,T), \\ \varphi(0) = 0, \\ \varphi'(T) + \mu(\Omega^\pm)\varphi(T) = 0. \end{cases} \tag{15}$$

It also gives that

$$\tan \sqrt{s(D^\pm) - \mu^2(\Omega^\pm)}T = -\frac{\sqrt{s(D^\pm) - \mu^2(\Omega^\pm)}}{\mu(\Omega^\pm)}. \tag{16}$$

Now for the domains D and $D^{\pm}$ we have

$$\begin{cases} \tan\sqrt{s_1(D^+)-\mu_1^2(\Omega^+)}T = -\frac{\sqrt{s_1(D^+)-\mu_1^2(\Omega^+)}}{\mu_1(\Omega^+)}, \\ \tan\sqrt{s_1(D^-)-\mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^-)-\mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_2(D)-\mu_2^2(\Omega)}T = -\frac{\sqrt{s_2(D)-\mu_2^2(\Omega)}}{\mu_2(\Omega)}. \end{cases}$$

By using (12) we establish that

$$\begin{cases} \tan\sqrt{s_1(D^+)-\mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^+)-\mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_1(D^-)-\mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_1(D^-)-\mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}, \\ \tan\sqrt{s_2(D)-\mu_1^2(\Omega^-)}T = -\frac{\sqrt{s_2(D)-\mu_1^2(\Omega^-)}}{\mu_1(\Omega^-)}. \end{cases}$$

Finally, we get

$$s_1(D^+) = s_1(D^-) = s_2(D). \tag{17}$$

□

Proof. [Proof of Theorem 3.4] Let us state the spectral problem for the second $s-$number of the Cauchy-Dirichlet heat operator (that is, the second eigenvalue of (3)) in the circular cylinder C,

$$s_2(C)v_2(x,t) = -\frac{\partial^2 v_2(x,t)}{\partial t^2} + \Delta_x^2 v_2(x,t). \tag{18}$$

where $v_2(x,t)$ is the second eigenfunction of the operator $\diamond^*\diamond$ in the circular cylinder C.

Let $B = B^+ \cup B^-$. Then by multiplying $v_2^+(x,t)$ to (18) and integrating over $B^+ \times (0,T)$ we establish,

$$\begin{aligned} s_2(C)\int_0^T\int_{B^+} v_2(x,t)v_2^+(x,t)dxdt &= s_2(C)\int_0^T\int_{B^+}(v_2^+(x,t))^2dxdt \\ = -\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2(x,t)}{\partial t^2}dxdt &+ \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2(x,t)dxdt \\ &= -\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2^+(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2^+(x,t)dxdt. \end{aligned} \tag{19}$$

After we get,

$$\begin{aligned} s_2(C) &= \frac{-\int_0^T\int_{B^+} v_2^+(x,t)\frac{\partial^2 v_2^+(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} v_2^+(x,t)\Delta_x^2 v_2^+(x,t)dxdt}{\int_0^T\int_{B^+}(v_2^+(x,t))^2dxdt} \\ &\le \sup_{z(x,t)\neq 0}\frac{-\int_0^T\int_{B^+} z(x,t)\frac{\partial^2 z(x,t)}{\partial t^2}dxdt + \int_0^T\int_{B^+} z(x,t)\Delta_x^2 z(x,t)dxdt}{\int_0^T\int_{B^+} z^2(x,t)dxdt} = s_1(C^+). \end{aligned} \tag{20}$$

Similarly, if (18) multiplying by $v_2^-(x,t)$ and intergrating over $B^- \times (0,T)$, we have

$$\begin{cases} s_2(C) \le s_1(C^+) \\ s_2(C) \le s_1(C^-). \end{cases} \tag{21}$$

From the Rayleigh-Faber-Krahn inequality Theorem 3.2, we obtain

$$\begin{cases} s_1(C^+) \leq s_1(D^+) \\ s_1(C^-) \leq s_1(D^-). \end{cases} \tag{22}$$

By using Lemma 3.5 we arrive at

$$s_2(C) \leq \min(s_1(C^+), s_1(C^-)) \leq s_1(D^+) = s_1(D^-) = s_2(D).$$

□

Theorem 3.6. *The ratio* $\frac{s_2(D)}{s_1(D)}$ *is maximized in the circular cylinder C among all cylindric domains of the same measure, i.e.*

$$\frac{s_2(D)}{s_1(D)} \leq \frac{s_2(C)}{s_1(C)},$$

for all D with $|D| = |C|$.

Proof. Let us restate the second and the first s-numbers in the forms

$$s_2(D) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega \Delta^2 X_2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}$$
$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \mu_2^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}, \tag{23}$$

and

$$s_1(D) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_\Omega \Delta^2 X_1(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$
$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}. \tag{24}$$

From [1] we have

$$\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \leq \frac{\mu_2(B)}{\mu_1(B)}. \tag{25}$$

Applying this and (11) we obtain

$$\frac{s_2(D)}{s_1(D)} = \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \mu_2^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}{\int_0^T \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}} \leq \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \mu_2^2(B)\int_0^T \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \mu_1^2(B)\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}}$$

$$= \frac{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_B X_2^*(x)\Delta^2 X_2^*(x)dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}}{\frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_B (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_B X_1^*(x)\Delta^2 X_1^*(x)dx}{\int_0^T \varphi_1^2(t)dt \int_B (X_1^*(x))^2dx}} = \frac{\frac{-\int_0^T \int_B u_2^*(x,t)\frac{\partial^2 u_2^*(x,t)}{\partial t^2}dxdt + \int_0^T \int_B u_2^*(x,t)\Delta_x^2 u_2^*(x,t)dxdt}{\int_0^T \int_B (u_2^*(x,t))^2dxdt}}{\frac{-\int_0^T \int_B u_1^*(x,t)\frac{\partial^2 u_1^*(x,t)}{\partial t^2}dxdt + \int_0^T \int_B u_1^*(x,t)\Delta_x^2 u_1^*(x,t)dxdt}{\int_0^T \int_B (u_1^*(x,t))^2dxdt}} = \frac{s_2(C)}{s_1(C)}. \tag{26}$$

□

Acknowledgements

The authors also thank all the active participant of the Third International Conference on Analysis and Applied Mathematics - ICAAM 2016 (September 7–10, 2016, Almaty, Kazakhstan) for a useful discussion of the results.

References

[1] M.S. Ashbaugh, R.D. Benguria, Proof of the Payne-Polya-Weinberger conjecture, Bull. Math. Sci. 25 (1991) 19–29.
[2] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, AMS, Providence, RI, 1988.
[3] A. Henrot, Extremum problems for eigenvalues of elliptic operators, Birkhauser Verlag, Basel, 2006.
[4] A. Kassymov, D. Suragan, Some spectral geometry inequalities for generalized heat potential operators, Complex Anal. Oper. Theory, to appear (doi:10.1007/s11785-016-0605-9).
[5] E. H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, AMS, Providence, RI, Second edition, 2001.
[6] L.E. Payne, G. Polya, H. Weinberger, Sur le quotient de deux frequences propres consecutives, Comptes Rendus Acad. Sci. Paris 241 (1955) 917–919.
[7] L.E. Payne, G. Polya, H. Weinberger, On the ratio of consecutive eigenvalues, J. Math. Phys. 35 (1956) 289–298.
[8] G. Rozenblum, M. Ruzhansky, D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials, J. Funct. Anal. 271 (2016) 224–239.
[9] M. Ruzhansky, D. Suragan, Isoperimetric inequalities for the logarithmic potential operator, J. Math. Anal. Appl. 434 (2016) 1676–1689.
[10] M. Ruzhansky, D. Suragan, Schatten's norm for convolution type integral operator, Russ. Math. Surv. 71 (2016) 157–158.
[11] M. Ruzhansky, D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries, Bull. Math. Sci. 6 (2016) 325–334.
[12] V.S. Vladimirov, Equations of Mathematical Physics, Moscow, 1996 (In Russian).