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Spectral Properties of the Generalised Samarskii–Ionkin Type Problems

Nurgissa Yessirkegenov[a]

[a] *Institute of Mathematicsand Mathematical Modeling, Almaty, Kazakhstan,*
and
Department of Mathematics, Imperial College London, London, UK

Abstract. In this paper, we study spectral properties of the Laplace operator with generalised Samarskii-Ionkin boundary conditions in a disk. The eigenfunctions and eigenvalues of these problems are constructed in the explicit form. Moreover, we prove the completeness of these eigenfunctions.

1. Introduction

The Dirichlet, Neumann and periodic boundary value problems are one of the most important problems in the theory of harmonic functions. A new class of the boundary value problems for the Poisson equation in a unit disk $\Omega = \{z = (x, y) = x + iy \in C : |z| < 1\}$ was introduced in [4, 7] ($k = 1, 2$):

The problem P_k. Find a solution of the Poisson equation

$$-\Delta u = f(z),\ z \in \Omega \tag{1}$$

satisfying the following periodic boundary conditions

$$u(1, \varphi) - (-1)^k u(1, \varphi + \pi) = \tau(\varphi),\ 0 \le \varphi \le \pi, \tag{2}$$

$$\frac{\partial u}{\partial r}(1, \varphi) + (-1)^k \frac{\partial u}{\partial r}(1, \varphi + \pi) = \nu(\varphi),\quad 0 \le \varphi \le \pi, \tag{3}$$

where $f(z) \in C^{\alpha}(\overline{\Omega})$, $\tau(\varphi) \in C^{1+\alpha}[0, \pi]$, and $\nu(\varphi) \in [0, \pi]$, $0 < \alpha < 1$.

These problems are analogous to the classical periodic boundary value problems. In [7], the authors showed the possibility of using the method of separation of variables for the P_k problems. Furthermore, in this case, they showed the self-adjointness of these problems and constructed their all eigenvalues and eigenfunctions.

Then, in [4], the authors considered the P_k problems in the multidimensional case and proved the well-posedness of this problem. The existence and uniqueness of the solution of the problem P_1 were shown.

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Email address: n.yessirkegenov15@imperial.ac.uk (Nurgissa Yessirkegenov)

They also proved that the solution of the problem P_2 is not unique up to a constant term and exists if the necessary condition of the well-posedness is held. Using the extreme principle and Green's function, they investigated the uniqueness and existence of the solution of the P_k problems.

In [3], an analog of the Samarskii-Ionkin type boundary value problem for the Poisson equation in a disk was considered. The authors proved the well-posedness of this problem and constructed its Green's function in the explicit form. Then, in [8], the spectral properties of this problem were studied. They constructed the eigenfunctions of this problem and proved its completeness.

In [5, 6], there were investigated problems generalising the periodic P_k problems.

The Samarskii-Ionkin type non-local problems for other partial differential equations were also investigated (see e.g. [1, 2]).

In this paper, we are interested in spectral properties of the generalised Samarskii-Ionkin type boundary value problems.

2. Statement of the Problem

Let $\Omega = \{z = (x, y) = x + iy \in C : |z| < 1\}$ be a unit disk, $r = |z|, \varphi = \arctan(y/x)$, $\Omega^+ = \Omega \cap \{y > 0\}$, and $\Omega^- = \Omega \cap \{y < 0\}$. We consider the spectral problem corresponding to the Laplace operator

$$-\Delta u(z) = \lambda u(z), |z| < 1 \tag{4}$$

with the nonlocal boundary conditions

$$u(1, \varphi) - \alpha u(1, 2\pi - \varphi) = 0,\ 0 \le \varphi \le \pi,\ \alpha \in \mathbb{R}, \tag{5}$$

and

$$\frac{\partial u}{\partial r}(1, \varphi) - \frac{\partial u}{\partial r}(1, 2\pi - \varphi) = 0,\ \ 0 \le \varphi \le \pi \tag{6}$$

or

$$\frac{\partial u}{\partial r}(1, \varphi) + \frac{\partial u}{\partial r}(1, 2\pi - \varphi) = 0,\ \ 0 \le \varphi \le \pi. \tag{7}$$

The antiperiodic boundary value problem (4)-(6) for $\alpha = -1$ and the periodic boundary value problem (4)-(5), (7) for $\alpha = 1$ are investigated in [4, 7]. When $\alpha = 0$, these problems are considered in [3, 8].

3. Main Results

Let us denote by L_1 the closure in $L_2(\Omega)$ of the operator defined by the differential expression $L_1 u = -\Delta u(z)$ on the linear manifold of functions $u(z) \in C^{2+\gamma}(\Omega)$, $0 < \gamma < 1$, satisfying the boundary conditions

$$u(1, \varphi) - \alpha u(1, 2\pi - \varphi) = 0,\ \ \frac{\partial u}{\partial r}(1, \varphi) - \frac{\partial u}{\partial r}(1, 2\pi - \varphi) = 0,\ \ 0 \le \varphi \le \pi.$$

Similarly, by L_2 we denote the closure in $L_2(\Omega)$ of the operator defined by the differential expression $L_2 u = -\Delta u(z)$ on the linear manifold of functions $u(z) \in C^{2+\gamma}(\Omega)$, $0 < \gamma < 1$, satisfying the boundary conditions

$$u(1, \varphi) - \alpha u(1, 2\pi - \varphi) = 0,\ \ \frac{\partial u}{\partial r}(1, \varphi) + \frac{\partial u}{\partial r}(1, 2\pi - \varphi) = 0,\ \ 0 \le \varphi \le \pi.$$

Theorem 3.1. *Let $\alpha \ne 1$. The system of the eigenfunctions of the operator L_1 is complete in $L_2(\Omega)$ and has the following form:*

$$u_k^1(z) = J_k(r\sqrt{\lambda_D}) \cos k\varphi, 0 \le \varphi \le 2\pi, k = 0, 1, 2, ..., \tag{8}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N})\sin m\varphi + \frac{a_0}{2}J_0(r\sqrt{\lambda_N}) + \sum_{n=1,\, n\neq m}^{\infty} a_n J_n(r\sqrt{\lambda_N})\cos n\varphi, 0 \leq \varphi \leq 2\pi, m = 1, 2, ..., \tag{9}$$

with

$$a_n = \frac{(1+\alpha)J_m(\sqrt{\lambda_N})}{\pi(1-\alpha)J_n(\sqrt{\lambda_N})}\left(\frac{(-1)^{m-n}-1}{m-n} + \frac{(-1)^{m+n}-1}{m+n}\right), n \neq m,\ n = 0, 1, ...$$

Here, $J_i(x)$, $i = 0, 1, ...$ are Bessel functions, λ_D and λ_N are eigenvalues of the Dirichlet and Neumann problems for the Laplace equation in Ω, respectively.

Proof. We introduce auxiliary functions

$$c(r,\varphi) = \frac{1}{2}(u(r,\varphi) + u(r, 2\pi - \varphi)), \quad s(r,\varphi) = \frac{1}{2}(u(r,\varphi) - u(r, 2\pi - \varphi)). \tag{10}$$

It is obvious that $u(z) = c(z) + s(z)$. By a direct calculation, we find spectral problems for the functions $c(z)$ and $s(z)$: for the function $s(z)$, we obtain Neumann problem

$$-\Delta s(z) = \lambda s(z),\ z \in \Omega;\ \frac{\partial s}{\partial r}(1,\varphi) = 0, 0 \leq \varphi \leq 2\pi, \tag{11}$$

and for the function $c(z)$, we have Dirichlet problem

$$-\Delta c(z) = \lambda c(z),\ z \in \Omega;\ c(1,\varphi) = \begin{cases} -\frac{1+\alpha}{1-\alpha}s(1,\varphi), & 0 \leq \varphi \leq \pi; \\ \frac{1+\alpha}{1-\alpha}s(1,\varphi), & \pi \leq \varphi \leq 2\pi. \end{cases} \tag{12}$$

Let us consider two cases:

1) In the case $\lambda \neq \lambda_N$, one obtains $s(r,\varphi) = 0$, and the Dirichlet problem (12) has the form

$$-\Delta c(z) = \lambda c(z),\ z \in \Omega;\ c(1,\varphi) = 0,\ \ 0 \leq \varphi \leq 2\pi. \tag{13}$$

Since $c(r,\varphi) = c(r, 2\pi - \varphi)$, one of the series of the eigenfunctions of the L_1 problem has the following form:

$$u_k(z) = J_k(r\sqrt{\lambda_D})\cos k\varphi, k = 0, 1, ... \tag{14}$$

2) In the case $\lambda = \lambda_N$, by using the property $s(r,\varphi) = -s(r, 2\pi - \varphi)$, we obtain

$$s_m(z) = J_m(r\sqrt{\lambda_N})\sin m\varphi, m = 1, 2, ... \tag{15}$$

Then, we rewrite the Dirichlet problem (12) as

$$-\Delta c(z) = \lambda_N c(z),\ z \in \Omega, \tag{16}$$

$$c(1,\varphi) = \begin{cases} -\frac{1+\alpha}{1-\alpha}J_m(\sqrt{\lambda_N})\sin m\varphi, & 0 \leq \varphi \leq \pi; \\ \frac{1+\alpha}{1-\alpha}J_m(\sqrt{\lambda_N})\sin m\varphi, & \pi \leq \varphi \leq 2\pi. \end{cases} \tag{17}$$

Since $c(r,\varphi) = c(r, 2\pi - \varphi)$, we seek $c(r,\varphi)$ in the form

$$c(r,\varphi) = \frac{a_0}{2}J_0(r\sqrt{\lambda_N}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_N})\cos n\varphi. \tag{18}$$

From the boundary condition (17) we obtain

$$a_n J_n(\sqrt{\lambda_N}) = -\frac{1+\alpha}{\pi(1-\alpha)}\int_0^{\pi} J_m(\sqrt{\lambda_N})\sin m\varphi\cos n\varphi d\varphi + \frac{1+\alpha}{\pi(1-\alpha)}\int_{\pi}^{2\pi} J_m(\sqrt{\lambda_N})\sin m\varphi\cos n\varphi d\varphi$$

$$= -\frac{2(1+\alpha)}{\pi(1-\alpha)}\int_0^{\pi} J_m(\sqrt{\lambda_N})\sin m\varphi\cos n\varphi d\varphi,\ n = 0, 1, ...$$

It follows that

$$a_n = \frac{(1+\alpha)J_m(\sqrt{\lambda_N})}{\pi(1-\alpha)J_n(\sqrt{\lambda_N})}\left(\frac{(-1)^{m-n}-1}{m-n}+\frac{(-1)^{m+n}-1}{m+n}\right),\ n=0,1,...,$$

for $n \neq m$ and $a_n = 0$ for $n = m$.

Thus, we obtain the eigenfunctions of the L_1 problem as follows

$$u_k^1(z) = J_k(r\sqrt{\lambda_D})\cos k\varphi, 0 \leq \varphi \leq 2\pi, k = 0,1,2,..., \tag{19}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N})\sin m\varphi + \frac{a_0}{2}J_0(r\sqrt{\lambda_N}) + \sum_{n=1,\, n\neq m}^{\infty} a_n J_n(r\sqrt{\lambda_N})\cos n\varphi, 0 \leq \varphi \leq 2\pi, m = 1,2,.... \tag{20}$$

By asymptotic forms of the Bessel function and using Leibniz criterion, one shows the convergence of the series in (20).

Now, we prove that the eigenfunctions (19) and (20) are complete in $L_2(\Omega)$. We have

$$\int_\Omega u_k^1(z)f(z)dz = \int_0^1\int_0^\pi rJ_k(r\sqrt{\lambda_D})(f(r,\varphi)+f(r,2\pi-\varphi))\cos k\varphi dr d\varphi = 0.$$

Since $\left\{rJ_k(r\sqrt{\lambda_D})\cos k\varphi\right\}_{k=0}^{k=\infty}$ is complete in $L_2(\Omega^+)$, one has

$$f(r,\varphi) + f(r,2\pi-\varphi) = 0,\ 0 \leq \varphi \leq \pi. \tag{21}$$

Taking into account (21), we get

$$\int_\Omega u_m^2(z)f(z)dz = \int_\Omega \left(J_m(r\sqrt{\lambda_N})\sin m\varphi\right)f(z)dz$$

$$+\int_\Omega\left(\frac{a_0}{2}J_0(r\sqrt{\lambda_N}) + \sum_{n=1,\, n\neq m}^{\infty} a_n J_n(r\sqrt{\lambda_N})\cos n\varphi\right)f(z)dz$$

$$= \int_0^1\int_0^\pi rJ_m(r\sqrt{\lambda_D})\sin m\varphi(f(r,\varphi)-f(r,2\pi-\varphi))dr d\varphi = 0.$$

Since $\left\{rJ_m(r\sqrt{\lambda_D})\sin m\varphi\right\}_{m=1}^{m=\infty}$ is complete in $L_2(\Omega^+)$, we obtain

$$f(r,\varphi) - f(r,2\pi-\varphi) = 0,\ 0 \leq \varphi \leq \pi. \tag{22}$$

The formulas (21) and (22) imply $f(r,\varphi) = 0$ for $0 \leq \varphi \leq 2\pi$, which provides the completeness of the eigenfunctions (19) and (20) in $L_2(\Omega)$. □

Theorem 3.2. *Let $\alpha \neq -1$. The system of the eigenfunctions of the operator L_2 is complete in $L_2(\Omega)$ and has the following form:*

$$u_k^1(z) = J_k(r\sqrt{\lambda_D})\sin k\varphi, 0 \leq \varphi \leq 2\pi, k = 1,2,..., \tag{23}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N})\cos m\varphi + \sum_{n=1,\, n\neq m}^{\infty} b_n J_n(r\sqrt{\lambda_N})\sin n\varphi, 0 \leq \varphi \leq 2\pi, m = 0,1,..., \tag{24}$$

with

$$b_n = \frac{(1-\alpha)J_m(\sqrt{\lambda_N})}{\pi(1+\alpha)J_n(\sqrt{\lambda_N})}\left(\frac{(-1)^{n-m}-1}{n-m}+\frac{(-1)^{n+m}-1}{n+m}\right), n \neq m.$$

Here, $J_i(x)$, $i = 0,1,...$ are Bessel functions, λ_D and λ_N are eigenvalues of the Dirichlet and Neumann problems for the Laplace equation in Ω, respectively.

Proof. By a direct calculation, we find spectral problems for the functions $c(z)$ and $s(z)$: for the function $c(z)$, we obtain Neumann problem

$$-\Delta c(z) = \lambda c(z),\ z \in \Omega;\ \frac{\partial c}{\partial r}(1,\varphi) = 0, 0 \le \varphi \le 2\pi, \tag{25}$$

and for the function $s(z)$, we have Dirichlet problem

$$-\Delta s(z) = \lambda s(z),\ z \in \Omega;\ s(1,\varphi) = \begin{cases} -\frac{1-\alpha}{1+\alpha}c(1,\varphi),\ 0 \le \varphi \le \pi; \\ \frac{1-\alpha}{1+\alpha}c(1,\varphi),\ \pi \le \varphi \le 2\pi. \end{cases} \tag{26}$$

Let us consider again two cases:

1) In the case $\lambda \ne \lambda_N$, we have $c(r,\varphi) = 0$, and the Dirichlet problem (26) has the form

$$-\Delta s(z) = \lambda s(z),\ z \in \Omega;\ s(1,\varphi) = 0, 0 \le \varphi \le 2\pi. \tag{27}$$

Since $s(r,\varphi) = -s(r, 2\pi - \varphi)$, one of the series of the eigenfunctions of the L_2 problem has the following form:

$$u_k^1(z) = J_k(r\sqrt{\lambda_D})\sin k\varphi, k = 1,2,.... \tag{28}$$

2) In the case $\lambda = \lambda_N$, using the property $c(z) = c(r, 2\pi - \varphi)$, we obtain

$$c_m(z) = J_m(r\sqrt{\lambda_N})\cos m\varphi, m = 0,1,2,.... \tag{29}$$

Then, we rewrite the Dirichlet problem (26) as

$$-\Delta s(z) = \lambda_N s(z),\ z \in \Omega, \tag{30}$$

$$s(1,\varphi) = \begin{cases} -\frac{1-\alpha}{1+\alpha}J_m(\sqrt{\lambda_N})\cos m\varphi,\ 0 \le \varphi \le \pi; \\ \frac{1-\alpha}{1+\alpha}J_m(\sqrt{\lambda_N})\cos m\varphi,\ \pi \le \varphi \le 2\pi. \end{cases} \tag{31}$$

Since $s(r,\varphi) = -s(r, 2\pi - \varphi)$, we seek $s(r,\varphi)$ in the form

$$s(r,\varphi) = \sum_{n=1}^{\infty} b_n J_n(r\sqrt{\lambda_N})\sin n\varphi. \tag{32}$$

From the boundary condition (31), one calculates

$$\begin{aligned} b_n J_n(\sqrt{\lambda_N}) &= -\frac{(1-\alpha)}{\pi(1+\alpha)}\int_0^{\pi} J_m(\sqrt{\lambda_N})\cos m\varphi \sin n\varphi d\varphi + \frac{(1-\alpha)}{(1+\alpha)\pi}\int_{\pi}^{2\pi} J_m(\sqrt{\lambda_N})\cos m\varphi \sin n\varphi d\varphi \\ &= -\frac{2(1-\alpha)}{(1+\alpha)\pi}\int_0^{\pi} J_m(\sqrt{\lambda_N})\cos m\varphi \sin n\varphi d\varphi,\ n = 1,2,.... \end{aligned}$$

This yields that

$$b_n = \frac{(1-\alpha)J_m(\sqrt{\lambda_N})}{\pi(1+\alpha)J_n(\sqrt{\lambda_N})}\Big(\frac{(-1)^{n-m}-1}{n-m} + \frac{(-1)^{n+m}-1}{n+m}\Big)$$

for $n \ne m$ and $b_n = 0$ for $n = m$.

Thus, we obtain the eigenfunctions of the L_2 problem

$$u_k^1(z) = J_k(r\sqrt{\lambda_D})\sin k\varphi, 0 \le \varphi \le 2\pi, k = 1,2,..., \tag{33}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N})\cos m\varphi + \sum_{n=1,\ n\ne m}^{\infty} b_n J_n(r\sqrt{\lambda_N})\sin n\varphi, 0 \le \varphi \le 2\pi, m = 0,1,.... \tag{34}$$

As in the proof of Theorem 3.1, it is easy to show that the series in (34) converges.

Now, we show that the eigenfunctions (33) and (34) are complete in $L_2(\Omega)$. We have

$$\int_\Omega u_k^1(z) f(z) dz = \int_0^1 \int_0^\pi r J_k(r\sqrt{\lambda_D}) \sin k\varphi (f(r,\varphi) - f(r, 2\pi - \varphi)) dr d\varphi = 0.$$

Since $\left\{ r J_k(r\sqrt{\lambda_D}) \sin k\varphi \right\}_{k=1}^{k=\infty}$ is complete in $L_2(\Omega^+)$, we obtain

$$f(r,\varphi) - f(r, 2\pi - \varphi) = 0, \; 0 \le \varphi \le \pi. \tag{35}$$

Using (35) and a direct calculation, we get

$$\int_\Omega u_m^2(z) f(z) dz = \int_\Omega \left(J_m(r\sqrt{\lambda_N}\right) \cos m\varphi) f(z) dz + \int_\Omega \left(\sum_{n=1,\, n\neq m}^{\infty} b_n J_n(r\sqrt{\lambda_N}) \sin n\varphi \right) f(z) dz$$

$$= \int_0^1 \int_0^\pi r J_m(r\sqrt{\lambda_D}) \cos m\varphi (f(r,\varphi) + f(r, 2\pi - \varphi)) dr d\varphi = 0.$$

Since $\left\{ r J_m(r\sqrt{\lambda_D}) \cos m\varphi \right\}_{m=0}^{m=\infty}$ is complete in $L_2(\Omega^+)$, one has

$$f(r,\varphi) + f(r, 2\pi - \varphi) = 0, \; 0 \le \varphi \le \pi. \tag{36}$$

From (35) and (36), we obtain $f(r,\varphi) = 0, \; 0 \le \varphi \le 2\pi$, which implies that the eigenfunctions (33) and (34) are complete in $L_2(\Omega)$. □

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