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On some Inequalities of $\tau-$ Measurable Operators

K.S. Tulenov[a], D. Dauitbek[a]

[a] *Al-Farabi Kazakh National University, Almaty, Kazakhstan, and Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*

Abstract. In this paper, we extended some inequalities which were proved By F. Kittaneh in [9] to the τ-measurable operators.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space. Throughout this paper, we denote by $\mathcal{M}$ a finite von Neumann algebra in the Hilbert space $\mathcal{H}$ with a normal faithful finite trace τ. The closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, the x said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{M}) \subseteq D(x)$ and $\tau(e^{\perp}) < \varepsilon$ (where for any projection e we let $e^{\perp} = 1 - e$). The set of all τ-measure operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. Let $\mathcal{P}(\mathcal{M})$ be the lattice of projections of $\mathcal{M}$. The sets

$$\mathcal{N}(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \exists\ e \in \mathcal{P}(\mathcal{M})\ such\ that\ \|xe\| < \varepsilon\ and\ \tau(e^{\perp}) < \delta\}$$

$(\varepsilon, \delta > 0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(\mathcal{M})$ called the measure topology. Equipped with the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$-algebra (see [10, 11]). For $x \in L_0(\mathcal{M})$, the *generalised singular value function* $\mu(\cdot; x) = \mu(\cdot; \|x\|)$ is defined by

$$\mu(t; x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(\ \|x\|) \leq t\}, \quad t \geq 0.$$

It follows directly that the singular value function $\mu(x)$ is a decreasing, right-continuous function on the positive half-line $[0, \infty)$. Moreover, $\mu(uxv) \leq \|u\|\|v\|\mu(x)$ for all $u, v \in \mathcal{M}$ and $x \in L_0(\mathcal{M})$ and

$$\mu(f(x)) = f(\mu(x))$$

whenever $0 \leq x \in L_0(\mathcal{M})$ and f is an increasing continuous function on $[0, \infty)$ which satisfies $f(0) = 0$. We remark that if $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then it is not difficult to see that $L_0(\mathcal{M}) = \mathcal{M}$. In particular, if $\dim(\mathcal{H}) = n < \infty$, then $L_0(\mathcal{M})$ may be identified with $M_n(\mathbb{C})$. In this case,

$$\mu(t; x) = s_j(x), \quad t \in [j-1, j), \quad j = 1, 2, \ldots.$$

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This work was supported by the target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan
Email addresses: tulenov@math.kz (K.S. Tulenov), dauitbek@math.kz (D. Dauitbek)

The space $L_0(\mathcal{M})$ is a partially ordered vector space with the ordering defined by setting $x \geq 0$ if and only if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. If $0 \leq x_\alpha \uparrow x$ holds in $L_0(\mathcal{M})$, then $\sup \mu(t; x_\alpha) \uparrow_\alpha \mu(t; x)$ for each $t \geq 0$. The trace τ extends to the positive cone of $L_0(\mathcal{M})$ as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. Further,

$$\tau(x^* x) = \tau(xx^*)$$

for all $x \in L_0(\mathcal{M})$ and

$$\tau(f(x)) = \int_0^\infty f(\mu(t; x)dt$$

whenever $0 \leq x \in L_0(\mathcal{M})$ and f is any non-negative Borel function which is bounded on a neighbourhood of 0 and satisfies $f(0) = 0$. If $(\mathcal{M}, \sigma)$ is a finite von Neumann algebra, if $x \in L_0(\mathcal{M})$ and $y \in L_0(\mathcal{M})$ then x is said to be *submajorised* by y (in the sense of Hardy, Littlewood and Polya) if and only if

$$\int_0^t \mu(s; x)ds \leq \int_0^t \mu(s; y)ds$$

for all $t \geq 0$. We write $x << y$, or equivalently, $\mu(x) << \mu(y)$ (see [1]).

Given $0 < p \leq \infty$ we denote by $L_p(\mathcal{M})$ the usual non-commutative L_p-spaces associated with $(\mathcal{M}, \tau)$. Recall that $L_\infty(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm $\|\cdot\|_\infty := \|\cdot\|$ (see [11, 14, 15]). The norm of $L_p(\mathcal{M})$ will be denoted by $\|\cdot\|_p$.

It will be convenient to adopt the following terminology. A linear subspace $E \in L_0(\mathcal{M})$, equipped with a norm $\|\cdot\|_E$ will be called fully symmetrically normed if E is symmetrically normed and has the property that if $x \in E$, $y \in L_0(\mathcal{M})$ satisfy $x \in E$ and $y \leq x$ then $y \in E$ and $\|x\|_E \leq \|y\|_E$. (see [5, 6])

If a fully symmetrically normed space is Banach, then it will be simply called a fully symmetric space. in [7], authors obtained following result which we will use it:

Corollary 1.1. *Let E be a fully symmetric space on* $[0, \infty)$ *and suppose that* $x \in L_0(\mathcal{M})$ *and* $0 \leq a, b \in L_0(\mathcal{M})$. *If* $ax, xb \in E(\mathcal{M}, \tau)$, *then* $a^{\frac{1}{2}} x b^{\frac{1}{2}} \in E(\mathcal{M}, \tau)$ *and*

$$\|a^{\frac{1}{2}} x b^{\frac{1}{2}}\|_{E(\mathcal{M})} \leq \frac{1}{2}\|ax + xb\|_{E(\mathcal{M})}$$

Recall the construction of a Banach symmetric operator space $L_E(\mathcal{M}, \tau)$ (for convenience $L_E(\mathcal{M})$). Let E be a Banach symmetric function space. Set

$$L_E(\mathcal{M}, \tau) = \left\{x \in L_0(\mathcal{M}, \tau) : \ \mu(x) \in E\right\}.$$

We equip $L_E(\mathcal{M}, \tau)$ with a natural norm

$$\|x\|_{L_E(\mathcal{M},\tau)} = \|\mu(x)\|_E, \quad x \in E(\mathcal{M}, \tau).$$

It was further established in [12, 16] that $E(\mathcal{M}, \tau)$ is Banach (see [2, 13]).

We define the direct sum $x \oplus y$ as the block diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with the following norm

$$\|x \oplus y\| = \max(\|x\|, \|y\|)$$

2. Main Results

Lemma 2.1. *Let E be a fully symmetric space on* $[0, \infty)$ *and* x, y *are* $\tau-$*measurable positive operators such that* $x + y \geq a1$ *for some* $a \geq 0$, *then*

$$a\|x - y\|_{E(\mathcal{M})} \leq \|x^2 - y^2\|_{E(\mathcal{M})} \tag{1}$$

Proof. To prove (1), we need to use the identity

$$x^2 - y^2 = \frac{1}{2}(x+y)(x-y) + \frac{1}{2}(x-y)(x+y).$$

Then since $f(t) = t^{\frac{1}{2}}$ is operator monotone function on $[0, \infty)$ and by Corollary 1.1, we obtain

$$\begin{aligned} a\|x-y\|_{E(\mathcal{M})} &= \|(a1)^{\frac{1}{2}}(x-y)(a1)^{\frac{1}{2}}\|_{E(\mathcal{M})} \\ &\leq \|(x+y)^{\frac{1}{2}}(x-y)(x+y)^{\frac{1}{2}}\|_{E(\mathcal{M})} \\ &\leq \frac{1}{2}\|(x+y)(x-y) + (x-y)(x+y)\|_{E(\mathcal{M})} \\ &= \|x^2 - y^2\|_{E(\mathcal{M})} \end{aligned}$$

This completes the proof. □

Lemma 2.2. *If $x, \ y$ are positive τ-measurable operators, then*

$$\|xy - yx\|_2^2 + \|(x-y)^2\|_2^2 \leq \|x^2 - y^2\|_2^2$$

.

Proof. Let $x, \ y$ be positive τ-measurable operators, then for all τ-measurable operator z, we have

$$\|xz - zy\|_2^2 \leq \|xz + zy\|_2^2 \tag{2}$$

.

Indeed, (2) follows from the identity

$$\begin{aligned} &\|xz + zy\|_2^2 = \tau((xz+zy)^*(xz+zy)) \\ &= \tau((z^*x + yz^*)(xz+zy)) = \tau(z^*x^2z + z^*xy + y^*zxz + yz^*zy) \\ &= \tau(z^*y^2z - z^*xzy - yz^*xz + yz^*zy) + 2\tau(z^*xzy + yz^*xz) \\ &= \tau((xz-zy)^*(xz-zy)) + 2\tau(z^*xzy) + 2\tau(yz^*xz) \\ &= \|xz - zy\|_2^2 + 2\tau(yz^*xz) + 2\tau(yz^*xz) = \|xz - zy\|_2^2 + 4\tau(y^{\frac{1}{2}}y^{\frac{1}{2}}z^*xz) \\ &= \|xz - zy\|_2^2 + 4\tau(y^{\frac{1}{2}}z^*x^{\frac{1}{2}}x^{\frac{1}{2}}zy^{\frac{1}{2}}) = \|xz - zy\|_2^2 + 4\|x^{\frac{1}{2}}zy^{\frac{1}{2}}\|_2^2. \end{aligned}$$

Let $z = x - y$; then we conclude that

$$\|x(x-y) + (x-y)y\|_2^2 \geq \|x(x-y) - (x-y)y\|_2^2$$

Thus

$$\|x^2 - y^2\|_2^2 \geq \|x^2 - 2xy + y^2\|_2^2. \tag{3}$$

Now observe that

$$Re(x^2 - 2xy + y^2) = (x-y)^2,$$

$$Im(x^2 - 2xy + y^2) = i(xy - yx).$$

Since

$$\|h\|_2^2 = \tau(h^*h) = \tau((Reh + iImh)^*(Reh + iImh))$$
$$= \tau((Reh - iImh)(Reh + iImh)) = \tau(Reh \cdot Reh) - i\tau(Imh \cdot Reh) + i\tau(Reh \cdot Imh)$$
$$+ \tau(Imh \cdot Imh) = \tau(|Reh|^2) + \tau(|Imh|^2) = \|Reh\|_2^2 + \|Imh\|_2^2$$

we get

$$\|x^2 - 2xy + y^2\|_2^2 = \|(x-y)^2\|_2^2 + \|xz - zx\|_2^2 \tag{4}$$

Applying (3) and (4), we obtain the desired result. □

Remark 2.3. Both Lemma 2.1 and 2.2 hold for the case $\mathcal{M}$ is semifinite.

Theorem 2.4. *Let x is τ-measurable operator with a polar decomposition $x = u|x|$ then*

$$\|u|x| - |x|u\|_\infty^2 \leq \|x^*x - xx^*\|_\infty^2 \leq \|u|x| + |x|u\|_\infty^2 \cdot \|u|x| - |x|u\|_\infty^2 \tag{5}$$

$$\| |x|u|x|u^* - u|x|u^*|x| \|_2^2 + \left\| |u|x| - |x|u|^2 \right\|_2^2 \leq \|x^*x - xx^*\|_2^2$$
$$\leq \|u|x| + |x|u\|_\infty^2 \cdot \|u|x| - |x|u\|_2^2 \tag{6}$$

Proof. We have $|x|^2 - (u|x|u^*)^2 = x^*x - xx^*$. So, applying Lemmas 3.1 and 3.2 in [4] to the positive τ-measurable operators $|x|$ and $u|x|u^*$, we obtain

$$\|(|x| - u|x|u^*)^2\|_{E(\mathcal{M})} = \|x^*x - xx^*\|_{E(\mathcal{M})} \leq \||x| + u|x|u^*\|_\infty^2 \cdot \||x| - u|x|u^*\|_2^2 \tag{7}$$

Using the unitary invariance of these norms and the fact that $\||x|^2\|_{E(\mathcal{M})} = \||x^*|^2\|_{E(\mathcal{M})}$ for every x is τ-measurable operator, we have

$$\|(|x| - u|x|u^*)^2\|_{E(\mathcal{M})} = \| \, ||x| - u|x|u^*|^2\|_{E(\mathcal{M})} = \| \, |u(u^*|x| - |x|u^*)|^2\|_{E(\mathcal{M})}$$
$$= \| \, |u^*|x| - |x|u^*|^2\|_{E(\mathcal{M})} = \| \, |u|x| - |x|u|^2\|_{E(\mathcal{M})}$$

$$\| \, |x| - u|x|u^*\|_{E(\mathcal{M})} = \|(|x|u - u|x|)u^*\|_{E(\mathcal{M})} = \|u|x| - |x|u\|_{E(\mathcal{M})}$$

and

$$\| \, |x| + u|x|u^*\|_{E(\mathcal{M})} = \|(|x|u + u|x|)u^*\|_{E(\mathcal{M})} = \|u|x| + |x|u\|_{E(\mathcal{M})}.$$

These relations, together with (7), yield inequality (5), and the second inequality in (6). The first inequality in (6), which is a refinement of that Corollary 3.1. in [4] for the Hilbert-Schmidt norm, can be obtained from Lemma 2.2 by a similar argument. Indeed,

$$\| \, |x|u|x|u^* - u|x|u^*|x| \, \|_2^2 + \left\| \, |u|x| - |x|u|^2 \right\|_2^2 = \| \, |x|u|x|u^* - u|x|u^*|x| \, \|_2^2 + \left\| (|x| - u|x|u^*)^2 \right\|_2^2$$
$$\leq \left\| |x|^2 - |x^*|^2 \right\|_2^2 = \|x^*x - xx^*\|_2^2 .$$

□

Lemma 2.5. *If x, y are positive τ-measurable operators, then*

$$\|(x+y)\oplus 0\|_{E(\mathcal{M})} \leq \|x\oplus y\|_{E(\mathcal{M})} + \|x^{\frac{1}{2}}y^{\frac{1}{2}} \oplus x^{\frac{1}{2}}y^{\frac{1}{2}}\|_{E(\mathcal{M})}. \tag{8}$$

In particular, for the operator norm

$$\|x+y\| \leq \max(\|x\|, \|y\|) + \|x^{\frac{1}{2}}y^{\frac{1}{2}}\|. \tag{9}$$

Proof. We have

$$(x+y)\oplus 0 = \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ y^{\frac{1}{2}} & 0 \end{pmatrix} = T^*T$$

$$TT^* = \begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ y^{\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix} = (x+y)\oplus 0$$

$$\begin{aligned}
\|(x+y)\oplus 0\|_{E(\mathcal{M})} &= \|TT^*\|_{E(\mathcal{M})} = \|T^*T\|_{E(\mathcal{M})} \\
&= \|\begin{pmatrix} x^{\frac{1}{2}} & 0 \\ y^{\frac{1}{2}} & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\|_{E(\mathcal{M})} \\
&= \|\begin{pmatrix} x & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & y \end{pmatrix}\|_{E(\mathcal{M})} \\
&= \|\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & 0 \end{pmatrix}\|_{E(\mathcal{M})} \\
&\leq \|x\oplus y\|_{E(\mathcal{M})} + \|\begin{pmatrix} 0 & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & 0 \end{pmatrix}\|_{E(\mathcal{M})} \\
&= \|x\oplus y\|_{E(\mathcal{M})} + \|\begin{pmatrix} x^{\frac{1}{2}}y^{\frac{1}{2}} & 0 \\ 0 & y^{\frac{1}{2}}x^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\|_{E(\mathcal{M})} \\
&= \|x\oplus y\|_{E(\mathcal{M})} + \|x^{\frac{1}{2}}y^{\frac{1}{2}} \oplus y^{\frac{1}{2}}x^{\frac{1}{2}}\|_{E(\mathcal{M})}
\end{aligned}$$

This completes the proof. □

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