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A Study on Certain Köthe Spaces

Medine Yeşilkayagil[a], Feyzi Başar[b]

[a]*Department of Mathematics, Uşak University, 1 Eylül Campus, 64200 - Uşak, Turkey*
[b]*Inönü University, Malatya 44280, Turkey*

Abstract. Let $A = (a_{nk})$ be a Köthe matrix. In this paper, we introduce the space $\lambda^{bs}(A)$ and we emphasize on some topological properties of the spaces $c_0(A)$, $\lambda^{bs}(A)$ and $\lambda^p(A)$ together with some inclusion relations, where $1 \leq p \leq \infty$.

1. Introduction

Let ω be the vector space of all real or complex valued sequences. Any vector subspace of ω is called a *sequence space*. A sequence space λ with linear topology is called a $K-$*space* if each of the maps $P_n : \lambda \to \mathbb{C}$ defined by $P_n(x) = x_n$ is continuous for all $x = (x_n) \in \lambda$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the complex field and the set of natural numbers, respectively. A *Fréchet space* is a complete linear metric space. A K-space λ is called an FK-*space* if λ is a complete linear metric space. A normed FK-space is called a BK-space.

Given a BK-space λ we denote the n^{th} section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} = \sum_{k=0}^{n} x_k e^k$ and we say that x is; AK (abschnittskonvergent) when $\lim_{n\to\infty} \left\| x - x^{[n]} \right\|_\lambda = 0$, AB (abschnittsbeschränkt) when $\sup_{n\in\mathbb{N}} \left\| x^{[n]} \right\|_\lambda < \infty$ and AD (abschnittsdicht) when ϕ is dense in λ, where e^n is a sequence whose only non-zero term is 1 in n^{th} place for each $n \in \mathbb{N}$ and ϕ is the set of all finitely non-zero sequences. If one of these properties holds for every $x \in \lambda$, then we said that the space λ has that property. It is trivial that AK implies AB and AD.

The $\alpha-$, $\beta-$, $\gamma-$ and $f-$duals λ^α, λ^β, λ^γ and λ^f of a sequence space λ are defined as follows;

$$\begin{aligned}
\lambda^\alpha &= \{x = (x_k) \in \omega : xy = (x_k y_k) \in \ell_1 \text{ for all } y = (y_k) \in \lambda\}, \\
\lambda^\beta &= \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \lambda\}, \\
\lambda^\gamma &= \{x = (x_k) \in \omega : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \lambda\}, \\
\lambda^f &= \left\{(f(e^k)) : f \in \lambda'\right\},
\end{aligned}$$

where λ' is the continuous dual of the space λ.

A matrix $A = (a_{nk})$ of non-negative numbers is called a *Köthe matrix* if it satisfies the following conditions:

(i) For each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $a_{nk} > 0$.

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Email addresses: medine.yesilkayagil@usak.edu.tr (Medine Yeşilkayagil), feyzibasar@gmail.com (Feyzi Başar)

(ii) $a_{nk} \leq a_{n,k+1}$ for all $n, k \in \mathbb{N}$.

The spaces $\lambda^p(A)$ with $1 \leq p < \infty$, $\lambda^\infty(A)$ and $c_0(A)$ are defined, as follows;

$$\lambda^p(A) := \left\{ x = (x_n) \in \omega : \|x\|_k = \left(\sum_{n=0}^{\infty} |x_n a_{nk}|^p \right)^{1/p} < \infty \text{ for each } k \in \mathbb{N} \right\},$$

$$\lambda^\infty(A) := \left\{ x = (x_n) \in \omega : \|x\|_k = \sup_{n \in \mathbb{N}} |x_n a_{nk}| < \infty \text{ for each } k \in \mathbb{N} \right\},$$

$$c_0(A) := \left\{ x = (x_n) \in \lambda^\infty(A) : \lim_{n \to \infty} x_n a_{nk} = 0 \text{ for each } k \in \mathbb{N} \right\}.$$

For every Köthe matrix A, the spaces $\lambda^p(A)$ with $1 \leq p \leq \infty$ and $c_0(A)$ are Fréchet spaces, [1, 8]. A Fréchet sequence space λ is called a *Köthe space* if $\lambda = \lambda^1(A)$ for some Köthe matrix A. The spaces $\lambda^p(A)$, $1 < p \leq \infty$ are called as generalized Köthe spaces by Bierstedt et al. [1]. In some sources, for example [3, 7], the spaces $\lambda^p(A)$ denoted by $K^{\ell_p}(A)$ and called by ℓ_p-Köthe space for $1 \leq p < \infty$.

Let ℓ be a Banach space of scalar sequences with a norm $\|\cdot\|_\ell$ such that

(i) $a = (a_n) \in \ell_\infty$, $x = (x_n) \in \ell \Rightarrow ax = (a_n x_n) \in \ell$, $\|ax\|_\ell \leq \|a\|_\infty \|x\|_\ell$
(ii) $\|e^n\|_\ell = 1$ for all $n \in \mathbb{N}$.

The space $(\ell, \|\cdot\|_\ell)$ is called *admissible*, [7]. With the usual dual norm, the space ℓ^α is also admissible.

For a given Banach sequence space ℓ and a Köthe matrix A, the $\ell-$Köthe space $K^\ell(A)$ is the space of all scalar sequences $x = (x_n)$ such that

$$\|x\|_k = \|(x_n a_{nk})\|_\ell < \infty \text{ for each } k = 1, 2, \ldots. \tag{1}$$

Equipped with semi-norms given by (1) $K^\ell(A)$ is a Fréchet space, [3].

It is well-known that the space bs of bounded series is defined by

$$bs := \left\{ x = (x_k) \in \omega : \|x\|_{bs} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} x_k \right| < \infty \right\}$$

and is an admissible space with the norm $\|\cdot\|_{bs}$.

Following [3, 7], we define the new space $\lambda^{bs}(A)$ by

$$\lambda^{bs}(A) := \left\{ x = (x_n) \in \omega : \left\| x \right\|_k^{bs} = \sup_{m \in \mathbb{N}} \left| \sum_{n=0}^{m} x_n a_{nk} \right| < \infty \text{ for each } k \in \mathbb{N} \right\}.$$

One can easily see that the space $\lambda^{bs}(A)$ is a Fréchet space with the norm $\|\cdot\|_k^{bs}$.

A sequence space λ is called

(i) *solid* if $\widetilde{\lambda} = \{u = (u_n) \in \omega : \exists x \in \lambda, \forall n \in \mathbb{N} \text{ such that } |u_n| \leq |x_n|\} \subset \lambda$.
(ii) *monotone* if $ux = (u_k x_k) \in \lambda$ for every $x = (x_k) \in \lambda$ and $u = (u_k) \in \chi$,

where χ denotes the set of all sequences of zeros and ones, [2].
Obviously, each solid space is monotone.

Let λ be an $FK-$space. Then, λ is a conservative space if $c \subset \lambda$, [10].

A $BK-$space λ is said to have monotone norm if $\left\|x^{[m]}\right\| \geq \left\|x^{[r]}\right\|$ for $m > r$ and $\|x\| = \sup \left\|x^{[m]}\right\|$, [10].

Let λ be a locally convex space. Then,

(i) λ is called *bornological* if every circled, convex subset $A \subset \lambda$ that absorbs every bounded set in λ is a neighborhood of 0, [6].
(ii) A subset is called *barrel* if it is absolutely convex, absorbing and closed in λ. Moreover, λ is called a *barrelled space* if each barrel is a neighbourhood of zero, [2].

Lemma 1.1. ([2, Theorem 7.1.10 (a), p. 343]) *If λ is a solid sequence space, then $\lambda^\alpha = \lambda^\beta = \lambda^\gamma$.*

Lemma 1.2. ([10, Theorem 7.2.7, p. 106]) *Let $\lambda \supset \phi$ be an FK−space. Then, the following statements hold:*

(i) $\lambda^\beta \subset \lambda^\gamma \subset \lambda^f$.

(ii) *If λ has AK−property, then $\lambda^\beta = \lambda^f$*

(iii) *If λ has AD−property, $\lambda^\beta = \lambda^\gamma$.*

Lemma 1.3. ([6, Corollary 7.1, p. 60]) *Every Banach space and every Fréchet space is a barrelled space.*

Lemma 1.4. *[6, p. 61] Every Fréchet space and hence every Banach space is bornological.*

Lemma 1.5. *Let $y_n = y(e^n)$ for each $n \in \mathbb{N}$. Then, the following statements hold:*

(i) ([5, Lemma 27.11, p. 332]) *$\lambda' = \lambda^\alpha$ for every Köthe matrix A and $\lambda = \lambda^p(A)$, $1 \leq p < \infty$, respectively, $\lambda = c_0(A)$; where the duality is given by $y(x) = \sum_n x_n y_n$.*

(ii) ([5, Proposition 27.13, p. 332]) *For every Köthe matrix A and $\lambda = \lambda^p(A)$, $1 \leq p < \infty$, respectively, $\lambda = c_0(A)$ $(\|\cdot\|_b)_{b\in\lambda^\infty(A)}$ is a fundamental system of seminorms for λ'; where for $y \in \lambda' = \lambda^\alpha$ we define*

$$\|y\|_b = \left(\sum_{n=0}^{\infty} |y_n b_n|^q\right)^{1/q} \text{ for } \lambda = \lambda^p(A) \text{ with } 1 < p < \infty,\ q = \frac{p}{p-1},$$
$$\|y\|_b = \sup_{n\in\mathbb{N}} |y_n b_n| \text{ for } \lambda = \lambda^1(A),$$
$$\|y\|_b = \sum_{n=0}^{\infty} |y_n b_n| \text{ for } \lambda = c_0(A).$$

Further we have,

$$\lambda' = \lambda^\alpha = \left\{y \in \omega : \|y\|_b < \infty \text{ for all } b \in \lambda^\infty(A)\right\}. \tag{2}$$

In this paper, we use standard terminology and notation due to [5] and [4].

2. Main Results

Theorem 2.1. *Let $1 \leq p \leq \infty$ and let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, the spaces $\lambda^p(A)$, $c_0(A)$ and $\lambda^{bs}(A)$ are BK−spaces.*

Proof. Assume that there exists a $K \in \mathbb{R}^+$ such that $a_{nk} \geq K$ for each $n, k \in \mathbb{N}$.

Let $x = (x_n) \in \lambda^p(A)$ with $1 \leq p < \infty$. Then,

$$|P_n(x)| = |x_n| \leq \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} \leq \frac{1}{K}\left(\sum_{n=0}^{\infty} |x_n a_{nk}|^p\right)^{1/p} \leq \frac{1}{K}\|x\|_k, \tag{3}$$

where $P_n : \lambda^p(A) \to \mathbb{C}$ for each $n \in \mathbb{N}$. Hence, by (3) each of the linear maps P_n is bounded and so is continuous. So, the spaces $\lambda^p(A)$ with $1 \leq p < \infty$ are K−spaces.

Let $p = \infty$. Then, one can easily see for all $x = (x_n) \in \lambda^\infty(A)$ that

$$|P_n(x)| = |x_n| \leq \frac{1}{K}|x_n a_{nk}| \leq \frac{1}{K}\sup_{n\in\mathbb{N}} |x_n a_{nk}| = \frac{1}{K}\|x\|_k, \tag{4}$$

where $P_n : \lambda^\infty(A) \to \mathbb{C}$ for each $n \in \mathbb{N}$. Hence, by (4), each of the linear maps P_n is bounded and so is continuous. Therefore, the space $\lambda^\infty(A)$ is a K−space. With the similar way, we see that $c_0(A)$ is a K−space.

It is easy to see that

$$\sup_{n\in\mathbb{N}}|x_n a_{nk}| = \sup_{n\in\mathbb{N}}\left|\sum_{j=0}^{n} x_j a_{jk} - \sum_{j=0}^{n-1} x_j a_{jk}\right| \leq 2\left\|x\right\|_k^{bs}$$

for all $x \in \lambda^{bs}(A)$. So, we have

$$|P_n(x)| = |x_n| \leq \sup_{n\in\mathbb{N}}|x_n| \leq \frac{1}{K}\sup_{n\in\mathbb{N}}|x_n a_{nk}| \leq \frac{2}{K}\left\|x\right\|_k^{bs}, \tag{5}$$

where $P_n : \lambda^{bs}(A) \to \mathbb{C}$ for each $n \in \mathbb{N}$. Hence, by (5) each of the linear maps P_n is bounded and so is continuous. Therefore, the space $\lambda^{bs}(A)$ is a $K-$space.

In addition since these spaces are Fréchet spaces, they are $FK-$spaces and since their topology are normable, they are $BK-$spaces. □

Let $\{a_{nk}\}_{n\in\mathbb{N}}$ be a bounded sequence for each $k \in \mathbb{N}$. Then, we have the following result:

Remark 2.2. The spaces $\lambda^p(A)$ with $1 \leq p \leq \infty$, $\lambda^{bs}(A)$ and $c_0(A)$ are not $K-$spaces with every Köthe matrix A.

Let $z = \theta$ and define the sequence $x = (x_n)$ and the matrix $A = (a_{nk})$ by $x_n = 2^n$ and $a_{nk} = 1/8^{n+1}$ for all $n, k \in \mathbb{N}$, respectively. Then, $x \in \lambda^p(A)$. Suppose that there exists a $\delta > 0$ for every $\varepsilon > 0$ such that for $x \in \lambda^p(A)$, $1 \leq p \leq \infty$ the inequalities $\|x - z\|_k^p = \sum_{n=0}^{\infty}|x_n a_{nk}|^p \leq 1/6 < \delta$ and $\|x - z\|_k = \sup_{n\in\mathbb{N}}|x_n a_{nk}| \leq 1/8 < \delta$ hold. Also, we see that

$$\left|P_n(x) - P_n(z)\right| = \left|x_n\right|, \tag{6}$$

where $P_n : \lambda^p(A) \to \mathbb{C}$, $1 \leq p \leq \infty$. By (6), we have $|P_n(x) - P_n(z)| = 2^n \geq K \in \mathbb{R}^+$ for every $n \in \mathbb{N}$. Hence, each of the linear maps P_n is not continuous at 0. Therefore, the spaces $\lambda^p(A)$ are not $K-$spaces with the matrix A. Similarly, $c_0(A)$ is not a $K-$space.

With above choosing, we have $x \in \lambda^{bs}(A)$ and $\|x - z\|_k^p = \sup_{m\in\mathbb{N}}\left|\sum_{n=0}^{m} x_n a_{nk}\right| \leq 1/6 < \delta$. But, we conclude by (6) that each of the linear maps $P_n : \lambda^{bs}(A) \to \mathbb{C}$ is not continuous at 0. Therefore, the space $\lambda^{bs}(A)$ is not a $K-$space.

Theorem 2.3. *Let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, the following statements hold:*

(i) Let $1 \leq p < \infty$. Then, the spaces $\lambda^p(A)$ are $AK-$spaces.

(ii) The space $c_0(A)$ is an $AK-$space.

(iii) The $AK-$section of the space $\lambda^\infty(A)$ is the space $c_0(A)$.

Proof. Let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, the spaces $\lambda^p(A)$ and $c_0(A)$ are $FK-$spaces, where $1 \leq p \leq \infty$.

(i) Let $1 \leq p < \infty$ and let $x = (x_n) \in \lambda^p(A)$. Then, we derive that

$$\lim_{m\to\infty}\left\|x - x^{[m]}\right\|_k^p = \lim_{m\to\infty}\left(\sum_{n\geq m+1}|x_n a_{nk}|^p\right) = 0.$$

Hence, the spaces $\lambda^p(A)$ are $AK-$spaces.

(ii) Let $x = (x_n) \in c_0(A)$. That is, $x_n a_{nk} \to 0$, as $n \to \infty$, for each $k \in \mathbb{N}$. Therefore, we obtain that

$$\lim_{m\to\infty}\left\|x - x^{[m]}\right\|_k = \lim_{n\to\infty}\left(\sup_{n\geq m+1}|x_n a_{nk}|\right) = 0.$$

Hence, the space $c_0(A)$ is an $AK-$space.

(iii) For $x = (x_n) \in \lambda^\infty(A)$, we see that

$$\lim_{m\to\infty}\left\|x - x^{[m]}\right\|_k = \lim_{n\to\infty}\left(\sup_{n\geq m+1}|x_n a_{nk}|\right). \tag{7}$$

If $x \in c_0(A)$, we have $\lim_{m\to\infty} \left\|x - x^{[m]}\right\|_k = 0$ for each $k \in \mathbb{N}$ in the relation (7).
This completes the proof. □

A direct consequence of the definition of the $AB-$property, we have the following result:

Corollary 2.4. *Let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, the space $\lambda^{bs}(A)$ is an $AB-$space.*

Theorem 2.5. *The following inclusions hold:*

(i) $\lambda^1(A) \subset \lambda^{bs}(A) \subset \lambda^{\infty}(A)$.
(ii) $\lambda^p(A) \subset \lambda^r(A)$ *for* $1 \leq p < r < \infty$.

Proof. (i) Let us take any $x \in \lambda^1(A)$. Then, for each $k \in \mathbb{N}$ we have $\sum_n |x_n a_{nk}| < \infty$ and so from the triangle inequality we have $\left|\sum_{n=0}^m x_n a_{nk}\right| \leq \sum_{n=0}^m |x_n a_{nk}|$. By taking supremum over $m \in \mathbb{N}$ in this inequality, we obtain $x \in \lambda^{bs}(A)$, that is, the inclusion $\lambda^1(A) \subset \lambda^{bs}(A)$ holds.

Now, let $x = (x_n) \in \lambda^{bs}(A)$. Since there exists a $L \in \mathbb{R}^+$ such that $\left|\sum_{n=0}^m x_n a_{nk}\right| \leq L$ for each $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} |x_m a_{mk}| &= \left|\sum_{n=0}^{m} x_n a_{nk} - \sum_{n=0}^{m-1} x_n a_{nk}\right| \\ &\leq \left|\sum_{n=0}^{m} x_n a_{nk}\right| + \left|\sum_{n=0}^{m-1} x_n a_{nk}\right| \leq 2L \end{aligned} \tag{8}$$

for each $k \in \mathbb{N}$. Taking supremum over $m \in \mathbb{N}$ in (8), we have $x \in \lambda^{\infty}(A)$, as desired.
(ii) This follows applying Jensen's inequality. □

Also, Meise and Vogt [5] have the following result:

Lemma 2.6. ([5, Proposition 27.16, p. 334]) *The following statements are equivalent for every Köthe matrix A:*

(i) There are $p, r \in [1, \infty]$ with $p \neq r$, so that $\lambda^p(A) = \lambda^r(A)$.
(ii) $\lambda^p(A) = \lambda^r(A)$ as Fréchet spaces, for all $p, r \in [1, \infty]$.
(iii) For each $k \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} a_{nk} a_{nm}^{-1} < \infty$.

Although Lemma 2.6 is nowhere used in this paper, we record it for the reader.

Theorem 2.7. *Let $1 \leq p < \infty$. Then, the following statements hold:*

(i) Let $\{a_{nk}\}_{n\in\mathbb{N}} \in \ell_p$ for each $k \in \mathbb{N}$. Then, $\ell_\infty \subset \lambda^p(A)$.
(ii) Let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, $\lambda^p(A) \subset c_0$.

Proof. Let $1 \leq p < \infty$.
(i) Let $\{a_{nk}\}_{n\in\mathbb{N}} \in \ell_p$ for each $k \in \mathbb{N}$ and let $x = (x_n) \in \ell_\infty$. Then, we have

$$\sum_{n=0}^{\infty} |x_n a_{nk}|^p \leq \left\|x\right\|_\infty^p \sum_{n=0}^{\infty} |a_{nk}|^p < \infty,$$

i.e, $x \in \lambda^p(A)$.
(ii) Let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$ and let $x = (x_n) \in \lambda^p(A)$. Then, the series $\sum_{n=0}^{\infty} |x_n a_{nk}|^p$ converges for each $k \in \mathbb{N}$. Hence, the general term of this series tends to zero, as $n \to \infty$. Therefore, for each $k \in \mathbb{N}$ there exists an $\varepsilon > 0$ and an $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n|K \leq |x_n a_{nk}| < \varepsilon$ when $n > n_0$. So, $x \in c_0$. □

Remark 2.8. For $p = \infty$, depending on the choice of the Köthe matrix A we have the following statements:

(i) Define the Köthe matrix $A = (a_{nk})$ by $a_{nk} = 1/2^n$ for each $n, k \in \mathbb{N}$ and let $x = (x_n) \in \ell_\infty$. Hence, there exists a $L \in \mathbb{R}^+$ such that $\sup_{n\in\mathbb{N}} |x_n| \leq L$ and so $|x_n a_{nk}| = |x_n/2^n| \leq L$ for each $n, k \in \mathbb{N}$, that is, $x \in \lambda^\infty(A)$. Therefore, the inclusion $\ell_\infty \subset \lambda^\infty(A)$ holds for the matrix A. Also, if we define the unbounded sequence $x = (x_n)$ by $x_n = 2^n$ for all $n \in \mathbb{N}$ then we obtain that $\sup_{n\in\mathbb{N}} |x_n a_{nk}| = 1$. Hence, the inclusion $\ell_\infty \subset \lambda^\infty(A)$ is strict.

(ii) Define the Köthe matrix $A = (a_{nk})$ by $a_{nk} = r \in \mathbb{R}^+ \setminus \{1\}$ for each $n, k \in \mathbb{N}$ and let $x = (x_n) \in \lambda^\infty(A)$. Then, we have $r \sup_{n\in\mathbb{N}} |x_n| = \sup_{n\in\mathbb{N}} |x_n a_{nk}| < \infty$ and so the inclusion $\lambda^\infty(A) \subset \ell_\infty$ holds.

Since $\lambda^1(A) = \lambda^\infty(A)$ if and only if $\lambda^1(A)$ is nuclear (see Terzioğlu and Zahariuta [9]), Theorem 2.5 gives the following:

Corollary 2.9. *The equalities $\lambda^1(A) = \lambda^{bs}(A) = \lambda^\infty(A)$ hold if and only if $\lambda^1(A)$ is nuclear.*

Theorem 2.10. *Let λ denotes any of the spaces $c_0(A)$ or $\lambda^p(A)$ with $1 \leq p \leq \infty$. Then, the space λ is solid.*

Proof. Let $u = (u_n) \in \widetilde{\lambda}$. Then, there exists a sequence $x = (x_n) \in \lambda$ such that $|u_n| \leq |x_n|$ for all $n \in \mathbb{N}$. Since $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}$ by the definition of a Köthe matrix, we have

$$0 < |u_n| a_{nk} \leq |x_n| a_{nk} \tag{9}$$

for all $n, k \in \mathbb{N}$. If $\lambda = c_0(A)$, by letting $n \to \infty$ in the relation (9), we obtain $u \in c_0(A)$. Taking supremum or sum over $n \in \mathbb{N}$ in the relation (9) for each $k \in \mathbb{N}$, we have $u \in \lambda^p(A)$ with $1 \leq p \leq \infty$.

This completes the proof. □

Corollary 2.11. *Let λ be as in Theorem 2.10. Then, the space λ is monotone.*

Corollary 2.12. *Let λ be as in Theorem 2.10. Then, since the space λ is Fréchet, it is barrelled and bornological.*

Remark 2.13. Consider the sequence $x = (x_n)$ and the Köthe matrix $A = (a_{nk})$ defined by $x_n = (-1)^n$ and $a_{nk} = 1$ for each $n, k \in \mathbb{N}$. Then, since

$$\sup_{m\in\mathbb{N}} \left| \sum_{n=0}^{m} x_n a_{nk} \right| = \sup_{m\in\mathbb{N}} \frac{1 + (-1)^m}{2} = 1$$

for each $k \in \mathbb{N}$, $x \in \lambda^{bs}(A)$. Then, the following statements hold:

(i) Let $u = (u_n) \in \chi$. Define the sequence $u = (u_n)$ by

$$u_n := \begin{cases} 1 & , \text{ n is even,} \\ 0 & , \text{ n is odd} \end{cases}$$

for every $n \in \mathbb{N}$. Therefore, we see for each $k \in \mathbb{N}$ that

$$\sup_{m\in\mathbb{N}} \left| \sum_{n=0}^{m} u_n x_n a_{nk} \right| = \sup_{m\in\mathbb{N}} \left| \sum_{n=0}^{m/2} u_{2n} x_{2n} \right| = \sup_{m\in\mathbb{N}} \left(\frac{m}{2} + 1 \right) = \infty,$$

where m is even. Also, we derive same result when m is odd. Hence, $ux \notin \lambda^{bs}(A)$. That is to say that the space $\lambda^{bs}(A)$ is not monotone.

(ii) Let $u = (u_n) = (0, 1, 1, 1, \ldots) \in \widetilde{\lambda^{bs}(A)}$. Then, $|u_n| \leq |x_n|$ for all $n \in \mathbb{N}$. But $u \notin \lambda^{bs}(A)$, since

$$\sup_{m\in\mathbb{N}} \left| \sum_{n=0}^{m} u_n a_{nk} \right| = \sup_{m\in\mathbb{N}} \left| \sum_{n=1}^{m} 1 \right| = \sup_{m\in\mathbb{N}} m = \infty.$$

Hence, the inclusion $\widetilde{\lambda^{bs}(A)} \subset \lambda^{bs}(A)$ does not hold. So, the space $\lambda^{bs}(A)$ is not solid.

Corollary 2.14. *Let λ^α be as in (2) and let $a_{nk} \geq K \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$, and $1 \leq p < \infty$. Then, the following statements hold:*

(i) Combining Lemma 1.1 and Theorem 2.10 gives that $\lambda^{\alpha} = \lambda^{\beta} = \lambda^{\gamma}$ *whenever* $\lambda \in \{c_0(A), \lambda^p(A)\}$.

(ii) Combining Lemma 1.2 and Part (i) of Theorem 2.3 gives that $\lambda^{f} = \lambda^{\alpha}$ *whenever* $\lambda = \lambda^p(A)$.

Corollary 2.15. *The following statements hold:*

(i) If $\{a_{nk}\}_{n\in\mathbb{N}} \in \ell_\infty$ *for each* $k \in \mathbb{N}$*, then* $\ell_1 \subset \lambda^1(A)$.

(ii) If there exits a $K \in \mathbb{R}^+$ *such that* $a_{nk} \geq K$ *for each* $n, k \in \mathbb{N}$*, then* $\lambda^1(A) \subset \ell_1$.

Theorem 2.16. *Let* $a_{nk} \geq K \in \mathbb{R}^+$ *for each* $n, k \in \mathbb{N}$*. Then, the following statements hold:*

(i) For $1 \leq p < \infty$ *the spaces* $\lambda^p(A)$ *have monotone norm.*

(ii) The spaces $\lambda^\infty(A)$ *and* $c_0(A)$ *have not monotone norm.*

Proof. Assume that there exits a $K \in \mathbb{R}^+$ such that $a_{nk} \geq K$ for each $n, k \in \mathbb{N}$. Then, the spaces $\lambda^p(A)$ and $c_0(A)$ are $BK-$spaces, where $1 \leq p \leq \infty$. Let $m > r$, where $m, r \in \mathbb{N}$.

(i) Let $x \in \lambda^p(A)$ for $1 \leq p < \infty$. Then, we have

$$\begin{aligned} \left\|x^{[m]}\right\|_k^p &= \sum_{n=0}^{m} |x_n a_{nk}|^p = \sum_{n=0}^{r} |x_n a_{nk}|^p + \sum_{n=r+1}^{m} |x_n a_{nk}|^p \\ &= \left\|x^{[r]}\right\|_k^p + \sum_{n=r+1}^{m} |x_n a_{nk}|^p. \end{aligned} \tag{10}$$

From (10), we obtain that $\left\|x^{[m]}\right\|_k \geq \left\|x^{[r]}\right\|_k$. Also,

$$\left\|x\right\|_k^p = \sum_{n=0}^{\infty} |x_n a_{nk}|^p = \sup_{m\in\mathbb{N}} \sum_{n=0}^{m} |x_n a_{nk}|^p = \sup_{m\in\mathbb{N}} \left\|x^{[m]}\right\|_k^p,$$

as desired.

(ii) Let $x \in \lambda^\infty(A)$. Since

$$\begin{aligned} \{|x_1 a_{1k}|, |x_2 a_{2k}|, \ldots, |x_r a_{rk}|, 0, 0, \ldots\} &\subset \{|x_1 a_{1k}|, |x_2 a_{2k}|, \ldots, |x_m a_{mk}|, 0, 0, \ldots\} \\ &\subset \{|x_1 a_{1k}|, \ldots, |x_m a_{mk}|, |x_{m+1} a_{m+1,k}|, \ldots\}, \end{aligned} \tag{11}$$

we have $\left\|x^{[m]}\right\|_k \geq \left\|x^{[r]}\right\|_k$. But, we obtain by the second part of the relation (11) that $\|x\|_k \geq \left\|x^{[m]}\right\|_k$. Hence, the space $\lambda^\infty(A)$ does not have monotone norm. Since the spaces $\lambda^\infty(A)$ and $c_0(A)$ are endowed with same norm, $c_0(A)$ does not have monotone norm.

This step completes the proof. □

Remark 2.17. Consider the sequence $x = (x_n)$ and the Köthe matrix $A = (a_{nk})$ defined by $x_n = 2$ and $a_{nk} = n + k + 2$ for each $n, k \in \mathbb{N}$. It is immediate that $a_{nk} \geq 2 \in \mathbb{R}^+$ for each $n, k \in \mathbb{N}$. Then, the spaces $\lambda^p(A)$, $c_0(A)$ and $\lambda^{bs}(A)$ are $FK-$spaces by Theorem 2.1, where $1 \leq p \leq \infty$. Obviously, $x \in c$ but

$$\begin{aligned} &\sum_{n=0}^{\infty} |x_n a_{nk}|^p = \sum_{n=0}^{\infty} [2(n + k + 2)]^p = \infty, \\ &\sup_{n\in\mathbb{N}} |x_n a_{nk}| = \sup_{n\in\mathbb{N}} 2(n + k + 2) = \infty, \end{aligned}$$

i.e., $x \notin \lambda^p(A)$, where $1 \leq p \leq \infty$. Hence, x does not belong to the spaces $c_0(A)$ and $\lambda^{bs}(A)$ by the definition of the space $c_0(A)$ and by Theorem 2.5. Therefore, the spaces $\lambda^p(A)$ with $1 \leq p \leq \infty$, $c_0(A)$ and $\lambda^{bs}(A)$ are not conservative for the matrix A. That is to say that the spaces $\lambda^p(A)$, $c_0(A)$ and $\lambda^{bs}(A)$ are not conservative for every Köthe matrix A.

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