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14-Point Difference Operator for the Approximation of the First Derivatives of a Solution of Laplace's Equation in a Rectangular Parallelepiped

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Abstract. A 14-point difference operator is used to construct finite difference problems for the approximation of the solution, and the first order derivatives of the Dirichlet problem for Laplace's equations in a rectangular parallelepiped. The boundary functions φ_j on the faces Γ_j, $j = 1, 2, ..., 6$ of the parallelepiped are supposed to have pth order derivatives satisfying the Hölder condition, i.e., $\varphi_j \in C^{p,\lambda}(\Gamma_j)$, $0 < \lambda < 1$, where $p = \{4, 5\}$. On the edges, the boundary functions as a whole are continuous, and their second and fourth order derivatives satisfy the compatibility conditions which result from the Laplace equation. For the error $u_h - u$ of the approximate solution u_h at each grid point (x_1, x_2, x_3), $|u_h - u| \leq c\rho^{p-4}(x_1, x_2, x_3)h^4$ is obtained, where u is the exact solution, $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point to the boundary of the parallelepiped, h is the grid step, and c is a constant independent of ρ and h. It is proved that when $\varphi_j \in C^{p,\lambda}$, $0 < \lambda < 1$, the proposed difference scheme for the approximation of the first derivative converges uniformly with order $O(h^{p-1})$, $p \in \{4, 5\}$.

1. Introduction

It is well known that the use of difference operators with a low number of pattern and with the highest order of accuracy for the approximate solution of differential equations reduces the effective realization of the obtained system of finite-difference equations. Moreover, to enlarge the class of applied problems the convergence of the difference solutions are preferred to be investigated under the weakened assumptions on the smoothness of the boundary conditions. All of these become more valuable in 3D problems, especially the derivatives of the unknown solution are sought.

The application of derivatives arise in many applied problems such as problems in electrophysics in which the first derivatives of the potential function define the electrostatic field [7], and in the fracture problems where the firtst derivatives of the stress function define the components of the tangential stress [8].

The investigation of approximate derivatives started in [9], where it was proved that the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the 2D Laplace equation in any strictly interior subdomain, with the same order h with which the difference solution converges on the given domain. The uniform convergence of the difference derivatives over the

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whole grid domain to the corresponding derivatives of the solution for the 2D Laplace equation with the order $O(h^2)$ was proved in [14]. In [5], for the first and pure second derivatives of the solution of the 2D Laplace equation special finite difference problems were investigated. It was proved that the solution of these problems converge to the exact derivatives with the order $O(h^4)$.

In [17] for the 3D Laplace equation the convergence of order $O(h^2)$ of the difference derivatives to the corresponding first order derivatives of the exact solution is proved. It was assumed that ton the faces the boundary functions have third derivatives satisfying the Hölder condition. Furthermore, they are continuous on the edges, and their second derivatives satisfy the compatibility condition that is implied by the Laplace equation. Whereas in [16] when the boundary values on the faces of a parallelepiped are supposed to have the fourth derivatives satisfying the Hölder condition, the constructed difference schemes converge with order $O(h^2)$ to the first and pure second derivatives of the exact solution. In [5] it is assumed that the boundary functions on the faces have sixth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Different difference schemes with the use of the 26-point difference operator are constructed on a cubic grid with mesh size h, to approximate the first and pure second derivatives of the solution of the Dirichlet problem with order $O(h^4)$.

In this paper $O(h^{p-1})$, $p = 4, 5$ order of approximation for the first derivatives of the solution of 3D Laplace's equation is obtained under weaker assumptions on the smoothness of the boundary functions on the faces of the parallelepiped than those used in [6]. Moreover, to construct the finite difference problems the difference operator with a lower number of pattern is used.

Finally, the obtained theoretical results are supported by the illustration of numerical results.

2. Some Properties of a Solution of the Dirichlet Problem on a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < a_i,\ i = 1, 2, 3\}$ be an open rectangular parallelepiped; Γ_j, $j = 1, 2, ..., 6$ be its faces including the edges; Γ_j for $j = 1, 2, 3$ ($j = 4, 5, 6$) belongs to the plane $x_j = 0$ ($x_{j-3} = a_{j-3}$), and let $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$ be the boundary of R; $\gamma_{\mu\nu} = \Gamma_\mu \cap \Gamma_\nu$ be the edges of the parallelepiped R. $C^{k,\lambda}(E)$ is the class of functions that have continuous kth derivatives satisfying the Hölder condition with an exponent $\lambda \in (0, 1)$.

Consider the boundary value problem

$$\Delta u = 0 \text{ on } R,\ u = \varphi_j \text{ on } \Gamma_j,\ j = 1, 2, ..., 6 \tag{1}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{p,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1, 2, ..., 6,\quad p \in \{4, 5\} \tag{2}$$

$$\varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3}$$

$$\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{4}$$

$$\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\nu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}, \tag{5}$$

where $1 \le \mu < \nu \le 6$, $\nu - \mu \ne 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, and t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

The following Lemma follows from Theorem 2.1 in [12].

Lemma 2.1. *Under conditions* (2) – (5) , *the solution u of the Dirichlet problem* (1) *belong to the Hölder class* $C^{p,\lambda}(\overline{R})$, $0 < \lambda < 1$, $p \in \{4, 5\}$.

Lemma 2.2. *Let $\rho(x_1, x_2, x_3)$ be the distance from the current point of the open parallelepiped R to its boundary and let $\frac{\partial}{\partial l} \equiv \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}, \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$.*
Then the next inequality holds

$$\left| \frac{\partial^6 u(x_1, x_2, x_3)}{\partial l^6} \right| \leq c\rho^{p+\lambda-6}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R \text{ and } p \in \{4, 5\} \tag{6}$$

where u is the solution of the problem (1)*, c is a constant independent of the direction of derivative $\frac{\partial}{\partial l}$.*

Proof. Since $u \in C^{p,\lambda}(\overline{R})$, $p \in \{4, 5\}$ (Lemma 2.1) the proof Lemma 2.2 follows with the use of Lemma 3 in [10] (Chap.4, Sec.3) □

3. Finite Difference Problem

We introduce a cubic grid with a step $h > 0$ defined by the planes $x_i = 0, h, 2h, \ldots, i = 1, 2, 3$. It is assumed that the edge lengths of R and h are such that $\frac{a_i}{h} \geq 4$ $(i = 1, 2, 3)$ are integers.

Let D_h be the set of nodes of the grid constructed, $\overline{R}_h = \overline{R} \cap D_h$, $R_h = R \cap D_h$, $R_h^k \subset R_h$ be the set of nodes of R_h lying at a distance of kh away from the boundary Γ of R, and $\Gamma_h = \Gamma \cap D_h$.

The 14-point difference operator S on the grid is defined as (see [19])

$$Su(x_1, x_2, x_3) = \frac{1}{56}\left(8 \sum_{p=1^{(1)}}^{6} u_p + \sum_{q=7^{(3)}}^{14} u_q\right), \quad (x_1, x_2, x_3) \in R_h, \tag{7}$$

where $\sum_{(m)}$ is the sum extending over the nodes lying at a distance of $m^{1/2}h$ away from the point (x_1, x_2, x_3) and u_p and u_q are the values of u at the corresponding nodes.

On the boundary Γ of R, we define continuous on the entire boundary including the edges of R, the function φ as follows

$$\varphi = \begin{cases} \varphi_1 & \text{on } \Gamma_1 \\ \varphi_j & \text{on } \Gamma_j \backslash \left(\bigcup\limits_{i=1}^{j-1} \Gamma_i \right), \quad j = 2, 3, \ldots, 6. \end{cases} \tag{8}$$

Obviously,

$$\varphi = \varphi_j \text{ on } \Gamma_j, \quad j = 1, 2, \ldots, 6.$$

We consider the finite difference problem approximating Dirichlet problem (1):

$$u_h = Su_h \text{ on } R_h, \; u_h = \varphi \text{ on } \Gamma_h, \tag{9}$$

where S is the difference operator given by (7) and φ is the function defined by (8). By maximum principle, the system (9) has a unique solution (see [11], Chap. 4).

In what follows and for simplicity, we denote by $c, c_1, c_2, \ldots$ constants, which are independent of h and the nearest factors, the identical notation will be used for various constants.

Consider two systems of grid equations

$$v_h = Sv_h + g_h, \text{ on } R_h, \; v_h = 0 \text{ on } \Gamma_h, \tag{10}$$

$$\overline{v}_h = S\overline{v}_h + \overline{g}_h, \text{ on } R_h, \; \overline{v}_h = 0 \text{ on } \Gamma_h, \tag{11}$$

where g_h and $\overline{g}_h$ are given functions and $\left|\overline{g}_h\right| \leq g_h$ on R_h.

Lemma 3.1. *The solutions v_h and $\overline{v}_h$ of systems* (10) *and* (11) *satisfy the inequality*

$$\left|\overline{v}_h\right| \leq v_h \text{ on } R_h.$$

The proof of Lemma 3.1 is similar to that of the comparison theorem in [11] (Chap.4, Sec.3).

Define

$$N(h) = \left[\frac{\min\{a_1, a_2, a_3\}}{2h}\right], \tag{12}$$

where $[a]$ is the integer part of a.

Consider for a fixed k, $1 \le k \le N(h)$ the systems of grid equations

$$v_h^k = Sv_h^k + g_h^k \text{ on } R_h^k, \;\; v_h^k = 0 \text{ on } \Gamma_h, \tag{13}$$

where

$$g_h^k = \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh. \end{cases}$$

Lemma 3.2. *The solution v_h^k of the system* (13) *satisfies the inequality*

$$v_h^k(x_1, x_2, x_3) \le T_h^k, \;\; 1 \le k \le N(h), \tag{14}$$

where T_h^k is defined as

$$T_h^k = T_h^k(x_1, x_2, x_3) = \begin{cases} \frac{5\rho}{h}, & 0 \le \rho(x_1, x_2, x_3) \le kh, \\ 5k, & \rho(x_1, x_2, x_3) > kh. \end{cases} \tag{15}$$

Proof. By the direct calculation of the expression ST_h^k, we obtain

$$T_h^k - ST_h^k \ge \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh, \end{cases} \tag{16}$$

on R_h. On the basis of (13), inequalities (16) and taking the boundary condition $T_h^k = 0$ on Γ_h into account, by Lemma 3.1, we get (14). □

Let $x_0 = (x_{10}, x_{20}, x_{30})$, be some point in R_h. By Taylor's formula for the solution u of the problem (1) around the point x_0, we have

$$u(x_1, x_2, x_3) = p_5(x_1, x_2, x_3; x_0) + r_5(x_1, x_2, x_3; x_0), \tag{17}$$

where p_5 is fifth-degree Taylor polynomial and r_5 is remainder.

Since u is a harmonic function and S is linear, by taking into account that $Sp_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30})$ from (17) follows

$$Su(x_{10}, x_{20}, x_{30}) = u(x_{10}, x_{20}, x_{30}) + Sr_5(x_{10}, x_{20}, x_{30}; x_0). \tag{18}$$

Lemma 3.3. *The following estimation holds*

$$\max_{(x_1,x_2,x_3)\in R_h^k} |Su - u| \le c_4 \frac{h^{p+\lambda}}{k^{6-p-\lambda}}, \; k = 1, 2, ..., N(h), \; p \in \{4, 5\}, \tag{19}$$

where u is the solution of the Dirichlet problem (1) *, S is the difference operator defined by* (7) *, and $N(h)$ is given by* (12) *.*

Proof. Let $x_0 = (x_{10}, x_{20}, x_{30})$ be some point in R_h^1, and let $P_{mnq} = (x_{10} + mh, x_{20} + nh, x_{30} + qh)$, where $m, n, q = 0, \pm 1$, $m^2 + n^2 + q^2 \neq 0$, be any point in the pattern of operator S. Then by using the integral form of the remainder term of Taylor's formula for each point P_{mnq} by virtue of Lemma 2.1 and Lemma 2.2, we obtain

$$\left| r_5 (x_{10} + mh, x_{20} + nh, x_{30} + qh; x_0) \right| \leq ch^{p+\lambda},\ p \in \{4, 5\}. \tag{20}$$

From the structure (7) of the operator S follows that its norm in the uniform metric is equal to one, then using by (20), we have

$$|Sr_5 (x_{10}, x_{20}, x_{30}; x_0)| \leq ch^{p+\lambda},\ p \in \{4, 5\}. \tag{21}$$

On the basis of (18) and (21) follows the inequality (19), for $k = 1$. Let $x_0 \in R_h^k$ be an arbitrary point for $2 \leq k \leq N(h)$, and let $r_5 (x_1, x_2, x_3; x_0)$ be remainder term of the Taylor formula (17) in the Lagrange form. Then $Sr_6 (x_{10}, x_{20}, x_{30}; x_0)$ can be expressed linearly in terms of the 14 number of sixth derivatives of u at some points on the open intervals connecting the points of pattern of the operator S with the point x_0. The sum of the absolute values of the coefficients multiplying the sixth derivatives does not exceed ch^6 which is independent k_0 $(2 \leq k_0 \leq N(h))$ or the point $x_0 \in R_h^{k_0}$. Using the estimation of the sixth derivatives by Lemma 2.2, for all k, $2 \leq k \leq N(h)$, we obtain

$$|Sr_5 (x_{10}, x_{20}, x_{30}; x_0)| \leq c_1 \frac{h^6}{(kh)^{6-p-\lambda}} = c_1 \frac{h^{p+\lambda}}{k^{6-p-\lambda}}. \tag{22}$$

By virtue of (18) and (22) follows the estimation (19). □

Theorem 3.4. *Assume that the boundary functions* φ_j *satisfy conditions* (2)−(5). *Then at each point* $(x_1, x_2, x_3) \in R_h$

$$|u_h - u| \leq c_0 h^4 \rho^{p-4},\ p \in \{4, 5\}, \tag{23}$$

where u_h *is the solution of the finite difference problem* (9), u *is the exact solution of problem* (1), *and* $\rho = \rho(x_1, x_2, x_3)$ *is the distance from the current point* $(x_1, x_2, x_3) \in R_h$ *to the boundary of the rectangular parallelepiped* R.

Proof. Let ε_h^k, $1 \leq k \leq N(h)$, be a solution of the system

$$\varepsilon_h^k = S\varepsilon_h^k + \mu_h^k \text{ on } R_h,\ \varepsilon_h^k = 0 \text{ on } \Gamma_h, \tag{24}$$

where

$$\mu_h^k = \begin{cases} Su - u \text{ on } R_h^k \\ 0 \qquad \text{on } R_h \backslash R_h^k. \end{cases} \tag{25}$$

Let

$$\varepsilon_h = u_h - u \text{ on } \overline{R}_h. \tag{26}$$

By (9) and (26) the error function ε_h satisfies the system of equations

$$\varepsilon_h = S\varepsilon_h + (Su - u) \text{ on } R_h,\ \varepsilon_h = 0 \text{ on } \Gamma_h. \tag{27}$$

We represent a solution of the system (27) as follows

$$\varepsilon_h = \sum_{k=1}^{N(h)} \varepsilon_h^k, \tag{28}$$

where $N(h)$ defined by (12), ε_h^k, $1 \leq k \leq N(h)$, is a solution of the system

$$\varepsilon_h^k = S\varepsilon_h^k + \sigma_h^k \text{ on } R_h,\ \varepsilon_h^k = 0 \text{ on } \Gamma_h, \tag{29}$$

when

$$\sigma_h^k = \begin{cases} Su - u \text{ on } R_h^k \\ 0 \quad\quad \text{on } R_h \backslash R_h^k. \end{cases} \tag{30}$$

Then on the basis of (28), (29), (30), Lemma 3.2 and Lemma 3.3, for the solution of (27), we have

$$\begin{aligned} |\varepsilon_h| &\leq \sum_{k=1}^{N(h)} \left|\varepsilon_h^k\right| \leq \sum_{k=1}^{N(h)} T_h^k \max_{(x_1,x_2,x_3)\in R_h^k} |Su - u| \leq c_1 h^6 \sum_{k=1}^{N(h)} \frac{T_h^k}{(kh)^{6-p-\lambda}} \\ &\leq 5c_1 h^{p+\lambda} \sum_{k=1}^{\rho/h-1} \frac{k}{(k)^{6-p-\lambda}} + 5c_1 h^6 \sum_{k=\rho/h}^{N(h)} \frac{\rho/h}{(kh)^{6-p-\lambda}} \\ &\leq 5c_1 h^{p+\lambda} \sum_{k=1}^{\rho/h-1} k^{-5+p+\lambda} + 5c_1 h^{p-1+\lambda}\rho \sum_{k=\rho/h}^{N(h)} k^{-6+p+\lambda} \\ &\leq c_2 h^4 \rho^{-4+p+\lambda} + c_3 h^4 \rho \leq c_4 h^4 \rho^{p-4}, p \in \{4,5\}. \end{aligned} \tag{31}$$

From (26) and (31), for any point $(x_1, x_2, x_3) \in R_h$, we obtain

$$|u_h - u| = |\varepsilon_h| \leq c_0 h^4 \rho^{p-4}(x_1, x_2, x_3),\ p \in \{4,5\}.$$

□

4. Approximation of the First Derivative

4.1. Boundary Function is from $C^{5,\lambda}$

Let the boundary functions φ_j, $j = 1, 2, ..., 6$, in problem (1) on the faces Γ_j be satisfied the conditions

$$\varphi_j \in C^{5,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1, 2, ..., 6, \tag{32}$$

i.e., $p = 5$ in (2). Let u be a solution of the problem (1) with the conditions (32) and (3) – (5).

We put $v = \frac{\partial u}{\partial x_1}$, and $\Phi_j = \frac{\partial u}{\partial x_1}$ on Γ_j, $j = 1, 2, ..., 6$. It is obvious that the function v is a solution of the following boundary value problem

$$\Delta v = 0 \text{ on } R,\ v = \Phi_j \text{ on } \Gamma_j,\ j = 1, 2, ..., 6, \tag{33}$$

where u is a solution of the problem (1) for $p = 5$.

We define an approximate solution of problem (33) as a solution of the following finite difference problem

$$\nu_h = S\nu_h \text{ on } R_h,\ \nu_h = \Phi_{jh}(u_h) \text{ on } \Gamma_j^h,\ j = 1, 2, ..., 6, \tag{34}$$

where u_h is the solution of the problem (9), Φ_{1h} (Φ_{4h}) is the fourth order forward (backward) numerical differentiation operator (see [1], [2]) used in [6] with the 26-point difference operator. On the nodes Γ_p^h, the boundary values are defined as $\Phi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x_1}$, $p = 2, 3, 5, 6$.

Theorem 4.1. *The estimation is true*

$$\max_{(x_1,x_2,x_3)\in \overline{R}_h} \left|\nu_h - \frac{\partial u}{\partial x_1}\right| \leq ch^4, \tag{35}$$

where u is the solution of the problem (1), *ν_h is the solution of the finite difference problem* (34).

Proof. Let

$$\epsilon_h = \nu_h - \nu \text{ on } \overline{R}^k, \tag{36}$$

where $\nu = \frac{\partial u}{\partial x_1}$. From (34) and (36) , we have

$$\begin{aligned} \epsilon_h &= S\epsilon_h + (S\nu - \nu) \text{ on } R_h, \\ \epsilon_h &= \Phi_{kh}(u_h) - \nu \text{ on } \Gamma_k^h,\ k = 1,4,\ \epsilon_h = 0 \text{ on } \Gamma_p^h,\ p = 2,3,5,6. \end{aligned}$$

We put

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2, \tag{37}$$

where

$$\epsilon_h^1 = S\epsilon_h^1 \text{ on } R_h, \tag{38}$$

$$\epsilon_h^1 = \Phi_{kh}(u_h) - \nu \text{ on } \Gamma_k^h,\ k = 1,4,\ \epsilon_h^1 = 0 \text{ on } \Gamma_q^h,\ q = 2,3,5,6; \tag{39}$$

$$\epsilon_h^2 = S\epsilon_h^2 + (S\nu - \nu) \text{ on } R_h,\ \epsilon_h^2 = 0 \text{ on } \Gamma_j^h,\ j = 1,2,...,6. \tag{40}$$

First, we estimate the difference $\Phi_{kh}(u_h) - \nu$ on Γ_k^h, $k = 1,4$ using the representation

$$\Phi_{kh}(u_h) - \nu = (\Phi_{kh}(u_h) - \Phi_{kh}(u)) + (\Phi_{kh}(u) - v) \tag{41}$$

Since $\Phi_{kh}(u)$, $k = 1,4$ are the fourth order approximation of $\partial u/\partial x_1$ on Γ_k and by Lemma 2.1 the fifth order partial derivatives of the solution u are bounded in $\overline{R}$, the difference $\Phi_{kh}(u) - v$ has estimation (see [1], [2])

$$\max_{k=1,4} \max_{(x_1,x_2,x_3)\in\Gamma_k^k} |\Phi_{kh}(u) - \nu| \le c_1 h^4. \tag{42}$$

To estimate $\Phi_{kh}(u_h) - \Phi_{kh}(u)$, we take the fourth order forward formula ($k = 1$),

$$\begin{aligned} \Phi_{1h}(u_h) = {} & \frac{1}{12h}[-25\varphi_1(x_2, x_3) + 48u_h(h, x_2, x_3) - 36u_h(2h, x_2, x_3) \\ & +16u_h(3h, x_2, x_3) - 3u_h(4h, x_2, x_3)] \text{ on } \Gamma_1^h. \end{aligned} \tag{43}$$

Using the pointwise estimation (24) in Theorem 3.4, when $p = 5$, and taking into account the values of the distance function $\rho(x_1, x_2, x_3)$ in the formula (43), we have

$$|\Phi_{1h}(u_h) - \Phi_{1h}(u)| \le c_2 h^4. \tag{44}$$

The estimation (44) is true for the backward formula ($k = 4$), also. On the basis of (41), (42), (44), by using the maximum principle, for the solution of system (38), (39) , we have

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} \left|\epsilon_h^1\right| \le c_3 h^4. \tag{45}$$

The solution ε_h^2 of system (40) is the error function of the finite difference solution for problem (33) , when the boundary functions $\Phi_j = \partial u/\partial x_1$, $j = 1,2,...,6$, as follows from (2) – (5) satisfy the conditions

$$\Phi_j \in C^{4,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1,2,...,6,$$

$$\Phi_\mu = \Phi_\nu \text{ on } \gamma_{\mu\nu},$$

$$\frac{\partial^2\Phi_\mu}{\partial t_\mu^2} + \frac{\partial^2\Phi_\nu}{\partial t_\nu^2} + \frac{\partial^2\Phi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}$$

Then, on the basis of Theorem 4 in [19] for the error ε_h^2, we have

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} \left|\epsilon_h^2\right| \le c_4 h^4. \tag{46}$$

By virtue of (37) , (45) , and (46) follows the inequality (35) . □

4.2. Boundary Function From $C^{4,\lambda}$

Let the boundary functions $\varphi_j \in C^{4,\lambda}\left(\Gamma_j\right)$, $0<\lambda<1$, $j=1,2,...,6$, in $(1)-(5)$, i.e., $p=4$ in (2), and let $v=\frac{\partial u}{\partial x_1}$ and let $\Phi_j=\frac{\partial u}{\partial x_1}$ on Γ_j, $j=1,2,...,6$, and consider the boundary value problem:

$$\Delta v=0 \text{ on } R,\ v=\Phi_j \text{ on } \Gamma_j,\ j=1,2,...,6, \tag{47}$$

where u is a solution of the boundary value problem (1).

We define the following third order numerical differentiation operators $\Phi_{\nu h}$, $\nu=1,4$

$$\begin{aligned}\Phi_{1h}(u_h) &= \frac{1}{6h}[-11\varphi_1(x_2,x_3)+18u_h(h,x_2,x_3)-9u_h(2h,x_2,x_3)\\ &\quad +2u_h(3h,x_2,x_3)] \text{ on } \Gamma_1^h, \end{aligned}\tag{48}$$

$$\begin{aligned}\Phi_{4h}(u_h) &= \frac{1}{6h}[11\varphi_4(x_2,x_3)-18u_h(a_1-h,x_2,x_3)+9u_h(a_1-2h,x_2,x_3)\\ &\quad -2u_h(a_1-3h,x_2,x_3)] \text{ on } \Gamma_4^h, \end{aligned}\tag{49}$$

and we put

$$\Phi_{ph}(u_h)=\frac{\partial\varphi_p}{\partial x_1},\ \text{on } \Gamma_p^h,\ p=2,3,5,6 \tag{50}$$

where u_h is the solution of the finite difference problem (9).

It is obvious that Φ_j, $j=1,2,...,6$, satisfy the conditions

$$\Phi_j\in C^{3,\lambda}\left(\Gamma_j\right),\ 0<\lambda<1,\ j=1,2,...,6, \tag{51}$$

$$\Phi_\mu=\Phi_\nu \text{ on } \gamma_{\mu\nu}, \tag{52}$$

$$\frac{\partial^2\Phi_\mu}{\partial t_\mu^2}+\frac{\partial^2\Phi_\nu}{\partial t_\nu^2}+\frac{\partial^2\Phi_\mu}{\partial t_{\mu\nu}^2}=0 \text{ on } \gamma_{\mu\nu}. \tag{53}$$

Let v_h be the solution of the following finite difference problem

$$v_h=Sv_h \text{ on } R_h,\ v_h=\Phi_{jh} \text{ on } \Gamma_j^h,\ j=1,2,...,6, \tag{54}$$

where Φ_{jh}, $j=1,2,...,6$, are defined by $(48)-(50)$.

Theorem 4.2. *Let the boundary function $\varphi_j\in C^{4,\lambda}(\Gamma_j)$, $j=1,...,6$. The estimation is true*

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|v_h-\frac{\partial u}{\partial x_1}\right|\le ch^3, \tag{55}$$

where u is the solution of the problem (1), *v_h is the solution of the finite difference problem* (54).

Proof. The proof of Theorem 4.2 is similar to that of Theorem 4.1, with the following differences in estimation for the errors ϵ_h^1 and ϵ_h^2 in (37) : (i) putting $p=4$ in Lemma 2.1 and Theorem 3.4, and taking into account that the formulae (48) and (49) are the third order, the estimation

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\epsilon_h^1\right|\le c_5h^3.$$

is proved. (ii) on the basis of (51)-(53) and Theorem 2 in [19], we obtain

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\epsilon_h^1\right|\le c_6h^3.$$

□

Remark 4.3. We have investigated the method of high order approximations of the first derivative $\partial u/\partial x_1$. The same results are obtained for the derivatives $\partial u/\partial x_l$, $l=2,3$ analogously, by using the same order forward and backward formulae in appropriate faces of the parallelepiped.

5. Numerical Results

Example 5.1. Let $R = \{(x_1, x_2, x_3) : 0 < x_i < 1,\ i = 1, 2, 3\}$, and let $\Gamma_j,\ j = 1, ..., 6$ be its faces. We consider the following problem:

$$\Delta u = 0 \ \text{ on } R,\ \ u = \varphi(x_1, x_2, x_3) \ \text{ on } \Gamma_j,\ j = 1, ..., 6, \tag{56}$$

where

$$\varphi(x_1, x_2, x_3) = \left(x_3 - \frac{1}{2}\right)^2 - \left(\frac{x_1^2 + x_2^2}{2}\right) + \left(x_1^2 + x_2^2\right)^{\frac{\left(5+\frac{1}{30}\right)}{2}} . \cos\left(\left(5 + \frac{1}{30}\right) . \arctan\left(\frac{x_2}{x_1}\right)\right) \tag{57}$$

is the exact solution of this problem,which is in $C^{5,1/30}$.

We solve the system (9) and (34) to find the approximate solution u_h for u and approximate first derivative v_h for $\frac{\partial u}{\partial x_1}$ respectively.

In Tables 1 and 2 the maximum errors are given. Table 1 shows that the convergence order more than 4 which is corresponds to the product ρ in Theorem 3.4. Table 2 justified estimation (35) in Theorem 4.1, i.e., the fourth order convergence.

h	$\|\|u_h - u\|\|_{\overline{R}_h}$	E_u^m
2^{-3}	$7,5172E-09$	$32,13$
2^{-4}	$2,3396E-10$	$32,66$
2^{-5}	$7,1637E-12$	$32,74$
2^{-6}	$2,1883E-13$	$32,75$
2^{-7}	$6,6827E-15$	

Table1 Errors for the solution in maximum norm

h	$\|\|v_h - v\|\|_{\overline{R}_h}$	E_v^m
2^{-3}	$4.5436E-03$	$13,40$
2^{-4}	$3.3909E-04$	$14,76$
2^{-5}	$2.2975E-05$	$15,40$
2^{-6}	$1.4922E-06$	$15,70$
2^{-7}	$9.5053E-08$	

Table 2 Errors for the first derivative in maximum norm with the fourth-order formulae

Example 5.2. Let u be a solution of problem (56) when the boundary function φ is chosen from $C^{4,\frac{1}{30}}$ as

$$\varphi(x_1, x_2, x_3) = \left(x_3 - \frac{1}{2}\right)^2 - \left(\frac{x_1^2 + x_2^2}{2}\right) + \left(x_1^2 + x_2^2\right)^{\frac{\left(4+\frac{1}{30}\right)}{2}} . \cos\left(\left(4 + \frac{1}{30}\right) . \arctan\left(\frac{x_2}{x_1}\right)\right).$$

Table 3 and 4 give the fourth order convergence when boundary function is from $C^{4,\frac{1}{30}}$ for both, solution and first derivative which are the numerical justification of Theorem 3.4 and 4.2 respectively.

h	$\|\|u_h - u\|\|_{\overline{R}_h}$	E_u^m
2^{-3}	$3,4801E-08$	$16,20$
2^{-4}	$2,1486E-09$	$16,36$
2^{-5}	$1,3135E-10$	$16,37$
2^{-6}	$8,0228E-12$	$16,37$
2^{-7}	$4,8998E-13$	

Table 3 Errors for the solution in maximum norm

h	$\|\|v_h - v\|\|_{\overline{R}_h}$	E_v^m
2^{-3}	$8,5126E-03$	$6,49$
2^{-4}	$1,3161E-03$	$7,29$
2^{-5}	$1,8065E-04$	$7,66$
2^{-6}	$2,3598E-05$	$7,83$
2^{-7}	$3,0144E-06$	

Table 4 Errors for the first derivative in maximum norm with the third-order formulae

In Tables 1-4 we have used the following notations:

$\|U_h - U\|_{\overline{R}_h} = \max_{\overline{R}_h} |U_h - U|$ and $E_U^m = \frac{\|U - U_{2^{-m}}\|_{\overline{R}_h}}{\|U - U_{2^{-(m+1)}}\|_{\overline{R}_h}}$ where U be the exact solution of the continuous problem, and U_h be its approximate values on $\overline{R}_h$.

6. Conclusion

Three different schemes with the 14-point difference operator are constructed on a cubic grid with mesh size h, whose solutions separately approximate the solution of the Dirichlet problem for 3D Laplace's equation with the order $O(h^4\rho^{p-4})$, $p \in \{4, 5\}$, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R_h$ to the boundary of the rectangular parallelepiped R and its first derivatives with the orders $O(h^{p-1})$.

The obtained results can be used to highly approximate the derivatives of the solution of 3D Laplace's boundary value problems on a prism with an arbitrary polygonal base and on polyhedra by developing the combined or composite grid methods [13, 15]. For the 2D case see [3, 4, 18, 20].

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