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A Note on the Nonlocal Boundary Value Problem for a Third Order Partial Differential Equation

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Abstract. The nonlocal boundary-value problem for a third order partial differential equation in a Hilbert space with a self-adjoint positive definite operator is considered. Applying operator approach, the theorem on stability for solution of this nonlocal boundary value problem is established. In applications, the stability estimates for the solution of three nonlocal boundary value problems for third order partial differential equations are obtained.

1. Introduction

It is known that various problems in fluid mechanics (dynamics, electricity) and other areas of physics lead to third order partial differential equations, we derive these equations as models of physical systems and consider methods for solving boundary value problems. This type of equations with constant coefficients can be solved by classical methods like Fourier transform method, and Laplace transform method (see [1, 11, 14, 16–18] and the references there in).

In the paper [18] the authors investigated the boundary value problem for the third order differential equation in the domain $\Omega\{0 < x < p, 0 < y < q\}$:

$$\begin{cases} \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} = f(x,y), \\ u(x,0) = \psi_1(x), \quad u(x,q) = \psi_2(x), u(0,y) = g_1(y), \quad u(p,y) = g_2(y), \quad \frac{\partial u}{\partial x}(0,y) = g_3(y), \end{cases} \tag{1}$$

where $\psi_1(x)$, $\psi_2(x)$, $g_1(y)$, $g_2(y)$, and $g_3(y)$ are sufficiently smooth functions and some compatibility conditions are fulfilled. The authors applied the method of lines to boundary value problem (1). The explicit

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expression and order of convergence for the approximate solution were obtained. It is well known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is operator method. The method of operator as a tool for investigation of the stability of partial differential equations in Hilbert and Banach spaces, has been systematically applied by several authors (see for example [2, 3, 7–10, 12, 14, 15, 20] and the references there in).

In the present paper we consider the boundary value problem for third order partial differential equation

$$\begin{cases} \frac{d^3u(t)}{dt^3} + A\frac{du(t)}{dt} = f(t), \quad 0 < t < 1, \\ u(0) = \gamma u(\lambda) + \varphi, \quad u'(0) = \alpha u'(\lambda) + \psi, \left|\gamma\right| < 1, \\ u''(0) = \beta u''(\lambda) + \xi, \quad \left|1 + \beta\alpha\right| > \left|\alpha + \beta\right|, 0 < \lambda \leq 1 \end{cases} \tag{2}$$

in a Hilbert space H with a self-adjoint positive definite operator A.

We are interested in studying the stability of solutions of problem (2). A function $u(t)$ is a solution of problem (2) if the following conditions are satisfied:

(i) $u(t)$ is thrice continuously differentiable on the interval $(0, 1)$ and twice continuously differentiable on the segment $[0, 1]$.

(ii) The element $u'(t)$ belongs to $D(A)$, for all $t \in [0, 1]$, and the function $Au'(t)$ is continuous on $[0, 1]$.

(iii) $u(t)$ satisfies the equation and boundary nonlocal conditions (2).

Let H be a Hilbert space, let A be a self-adjoint positive definite operator with $A \geq \delta I$, where $\delta > 0$. Throughout this paper, $C(t)$ and $S(t)$ are operator-functions defined by formulas [13]

$$C(t)u = \frac{e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}}{2}u, \quad S(t)u = \int_0^t C(s)u ds. \tag{3}$$

The paper are organized as follows. In section 2 main theorem on stability of problem (2) is obtained. In section 3, the stability estimates on t for the solution of three problems for a third order partial differential equation are obtained. Finally, section 4 is conclusion.

2. Main Theorem on Stability

Let us give some lemmas that will be needed bellow

Lemma 2.1. ([13]) *For* $t \geq 0$ *the following estimates hold*

$$\left\|\exp\left\{\pm itA^{\frac{1}{2}}\right\}\right\|_{H\to H} \leq 1, \quad \|C(t)\|_{H\to H} \leq 1, \quad \left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H} \leq 1. \tag{4}$$

Lemma 2.2. ([2]) *Assume that* $\left|1 + \beta\alpha\right| > \left|\alpha + \beta\right|$. *Then the operator* Δ *defined by the following formula*

$$\Delta = (1 + \alpha\beta) I - (\alpha + \beta) C(\lambda) \quad 0 \leq \lambda \leq 1.$$

has a bounded inverse $T = \Delta^{-1}$ *and the following estimate holds*

$$\|T\|_{H\to H} \leq \frac{1}{\left|1 + \beta\alpha\right| - \left|\alpha + \beta\right|}. \tag{5}$$

Lemma 2.3. *Suppose that* $\varphi \in D(A)$, $\psi \in D\left(A^{\frac{1}{2}}\right)$, $\xi \in D\left(A^{\frac{1}{2}}\right)$ *and* $f(t)$ *is continuously differentiable on* $[0, 1]$. *Then there is a unique solution of problem (2) and the following formula holds*

$$u(t) = \gamma u(\lambda) + \varphi + S(t)\left[\psi + \alpha u'(\lambda)\right] + A^{-1}(I - C(t))\left[\xi + \beta u''(\lambda)\right]$$

$$+\int_0^t A^{-1}\left(I-C\left(t-s\right)\right)f\left(s\right)ds, \tag{6}$$

$$u\left(\lambda\right)=\frac{1}{1-\gamma}\left\{\varphi+S\left(\lambda\right)\left[\alpha u'\left(\lambda\right)+\psi\right]+A^{-1}\left(I-C\left(\lambda\right)\right)\left[\xi+\beta u''\left(\lambda\right)\right]\right.$$

$$\left.+\int_0^\lambda A^{-1}\left(I-C\left(\lambda-s\right)\right)f\left(s\right)ds\right\}, \tag{7}$$

$$u'\left(\lambda\right)=T\left\{\left(I-\beta C\left(\lambda\right)\right)\left[C\left(\lambda\right)\psi+S\left(\lambda\right)\xi+\int_0^\lambda S\left(\lambda-s\right)f\left(s\right)ds\right]\right.$$

$$\left.+\beta S\left(\lambda\right)\left[-AS\left(\lambda\right)\psi+C\left(\lambda\right)\xi+\int_0^\lambda C\left(\lambda-s\right)f\left(s\right)ds\right]\right\}, \tag{8}$$

$$u''\left(\lambda\right)=T\left\{\left(I-\alpha C\left(\lambda\right)\right)\left[-AS\left(\lambda\right)\psi+C\left(\lambda\right)\xi+\int_0^\lambda C\left(\lambda-s\right)f\left(s\right)ds\right]\right.$$

$$\left.-\left(\alpha AS\left(\lambda\right)\right)\left[C\left(\lambda\right)\psi+S\left(\lambda\right)\xi+\int_0^\lambda S\left(\lambda-s\right)f\left(s\right)ds\right]\right\}. \tag{9}$$

Proof. It can be obviously rewritten (2) as the equivalent nonlocal boundary value problem for the system of linear differential equations

$$\begin{cases} \frac{du(t)}{dt}=v(t), 0<t<1, u\left(0\right)=\gamma u\left(\lambda\right)+\varphi, \\ \frac{d^2v(t)}{dt^2}+Av(t)=f(t),\ \ v\left(0\right)=\alpha v\left(\lambda\right)+\psi, v'\left(0\right)=\beta v'\left(\lambda\right)+\xi. \end{cases} \tag{10}$$

Integrating these equations, we can write

$$\begin{cases} u(t)=u(0)+\int_0^t v(s)ds, \\ v(t)=C\left(t\right)v\left(0\right)+S\left(t\right)v'\left(0\right)+\int_0^t S\left(t-s\right)f\left(s\right)ds. \end{cases} \tag{11}$$

Applying (3), we can write

$$\int_0^t S(s)ds\, u=-A^{-1}\left(C\left(t\right)-I\right)\ u,\quad u\in D\left(A\right).$$

From that and conditions $v\left(0\right)=u'\left(0\right), v'\left(0\right)=u''\left(0\right)$ it follows

$$u(t)=u\left(0\right)+S\left(t\right)u'\left(0\right)-A^{-1}\left(C\left(t\right)-I\right)u''\left(0\right)+\int_0^t A^{-1}\left(I-C\left(t-s\right)\right)f\left(s\right)ds. \tag{12}$$

Applying (12) and nonlocal conditions

$$u\left(0\right)=\gamma u\left(\lambda\right)+\varphi, u'\left(0\right)=\alpha u'\left(\lambda\right)+\psi, u''\left(0\right)=\beta u''\left(\lambda\right)+\xi,$$

we get

$$\begin{array}{l} u(\lambda)=\gamma u\left(\lambda\right)+\varphi+S\left(\lambda\right)\left[\alpha u'\left(\lambda\right)+\psi\right]-A^{-1}\left(C\left(\lambda\right)-I\right)\left[\beta u''\left(\lambda\right)+\xi\right] \\ +\int_0^\lambda A^{-1}\left(I-C\left(\lambda-s\right)\right)f\left(s\right)dsu'(\lambda), \\ =C\left(\lambda\right)\left[\alpha u'\left(\lambda\right)+\psi\right]+S\left(\lambda\right)\left[\beta u''\left(\lambda\right)+\xi\right]+\int_0^\lambda S\left(\lambda-s\right)f\left(s\right)ds, \\ u''\left(\lambda\right)=-AS\left(\lambda\right)\left[\alpha u'\left(\lambda\right)+\psi\right]+C\left(\lambda\right)\left[\beta u''\left(\lambda\right)+\xi\right]+\int_0^\lambda C\left(\lambda-s\right)f\left(s\right)ds, \end{array}$$

we have that

$$u(\lambda) = \frac{1}{1-\gamma}\{\varphi + S(\lambda)(\alpha u'(\lambda) + \psi)$$

$$+A^{-1}(C(\lambda) - I)\left[(\beta u''(\lambda) + \xi) + \int_0^{\lambda} A^{-1}(I - C(\lambda - s)) f(s)\, ds\right]\Bigg\}. \tag{13}$$

Therefore, we will obtain $u'(\lambda)$ and $u''(\lambda)$. For obtaining $u'(\lambda)$ and $u''(\lambda)$, we have the following system of equations

$$[I - \alpha C(\lambda)] u'(\lambda) - \beta S(\lambda) u''(\lambda) = C(\lambda)\psi + S(\lambda)\xi + \int_0^{\lambda} S(\lambda - s) f(s)\, ds,$$
$$\alpha A S(\lambda) u'(\lambda) + (I - \beta C(\lambda)) u''(\lambda) = -AS(\lambda)\psi + C(\lambda)\xi + \int_0^{\lambda} C(\lambda - s) f(s)\, ds.$$

It is clear that

$$(I - \alpha C(\lambda))(I - \beta C(\lambda)) + \alpha\beta A S^2(\lambda) = (1 + \alpha\beta) I - (\alpha + \beta) C(\lambda)$$

and by lemma 2.2 the operator Δ has the bounded inverse $T = \Delta^{-1}$. Therefore, we can get (8), (9). From that it follows (13). Applying (8), (9) and (13) and the conditions we get formula (6) for the solution of (2), where $u'(\lambda)$ and $u''(\lambda)$ are defined by (8), (9). Lemma 2.3 is proved. □

Now we will formulate the main theorem

Theorem 2.4. *Suppose that $\psi \in D(A)$, $\xi \in D\left(A^{1/2}\right)$ and $f(t)$ is continuously differentiable on $[0,1]$. Then there is a unique solution of problem (2) and the following inequalities hold*

$$\max_{0\le t\le 1} \|u(t)\|_H \le M(\gamma)\left\{\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \left\|A^{-1}\xi\right\|_H + \max_{0\le t\le 1}\left\|A^{-1} f(t)\right\|_H\right\}, \tag{14}$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\le t\le 1}\left\|A\frac{du}{dt}\right\|_H \le M\left\{\|A\psi\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\left\|f'(t)\right\|_H\right\}, \tag{15}$$

where $M, M(\gamma)$ do not depend on $f(t)$, φ, ψ, ξ.

Proof. First, we estimate $\|u(t)\|_H$ for $t \in [0,1]$. Applying (12), triangle inequality and estimates (7), (9), we get

$$\|u(t)\|_H \le |\gamma|\,\|u(\lambda)\|_H + \|\varphi\|_H + \left\|A^{\frac{1}{2}} S(t)\right\|_{H\to H}\left[\left\|A^{-\frac{1}{2}}\psi\right\|_H + |\alpha|\left\|A^{-\frac{1}{2}} u'(\lambda)\right\|_H\right]$$

$$+ \|I - c(t)\|_{H\to H}\left[\left\|A^{-1}\xi\right\|_H + |\beta|\left\|A^{-1}u''(\lambda)\right\|_H\right] + \int_0^t \|I - C(t-s)\|_{H\to H}\left\|A^{-1}f(s)\right\|_H ds$$

$$\le |\gamma|\,\|u(\lambda)\|_H + \|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + |\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H$$
$$+2\left\|A^{-1}\xi\right\|_H + 2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H + 2\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H$$

for any $t \in [0,1]$. Then, the proof of estimate (14) is based on the inequalities

$$\|u(\lambda)\|_H \le \frac{1}{|1-\gamma|}\left\{\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + |\alpha|\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H + 2\left\|A^{-1}\xi\right\|_H + 2|\beta|\left\|A^{-1}u''(\lambda)\right\|_H + 2\max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\},$$

$$\left\|A^{-\frac{1}{2}}u'(\lambda)\right\|_H \le M\left\{\left\|A^{-\frac{1}{2}}\psi\right\|_H + \left\|A^{-1}\xi\right\|_H + \max_{0\le t\le 1}\left\|A^{-1}f(t)\right\|_H\right\},$$

$$\left\|A^{-1}u''(\lambda)\right\|_H \leq M\left\{\left\|A^{-\frac{1}{2}}\psi\right\|_H + \left\|A^{-1}\xi\right\|_H + \max_{0\leq t\leq 1}\left\|A^{-1}f(t)\right\|_H\right\}.$$

Second, we estimate $\left\|\frac{d^3u(t)}{dt^3}\right\|_H$ for $t \in [0,1]$. Applying (6) and taking the third order derivative, we get

$$\frac{d^3u(t)}{dt^3} = -AC(t)\left[\psi + \alpha u'(\lambda)\right] - AS(t)\left[\xi + \beta u''(\lambda)\right] + C(t)f(0) + \int_0^t C(t-s)f'(s)\,ds.$$

Using the triangle inequality and estimates (4), we get

$$\left\|\frac{d^3u(t)}{dt^3}\right\|_H \leq \left[\|C(t)\|_{H\to H}\left[\left\|A\psi\right\|_H + |\alpha|\,\|Au'(\lambda)\|_H\right]\right.$$

$$+\left\|A^{\frac{1}{2}}S(t)\right\|_{H\to H}\left[\left\|A^{\frac{1}{2}}\xi\right\|_H + |\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H\right] + \|C(t)\|_{H\to H}\left\|f(0)\right\|_H$$

$$+\int_0^t \|C(t-s)\|_{H\to H}\left\|f'(s)\right\|_H ds \leq \left\|A\psi\right\|_H + |\alpha|\,\|Au'(\lambda)\|_H$$

$$+\left\|A^{\frac{1}{2}}\xi\right\|_H + |\beta|\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H + \left\|f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|f'(t)\right\|_H$$

for any $t \in [0,1]$. In similarly manner, we can obtain the following estimates

$$\|Au'(\lambda)\|_H \leq M\left\{\left\|A\psi\right\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \left\|f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\},$$

$$\left\|A^{\frac{1}{2}}u''(\lambda)\right\|_H \leq M\left\{\left\|A\psi\right\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \left\|f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}.$$

Applying these estimates, we get

$$\max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H \leq M\left\{\left\|A\psi\right\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \left\|f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}.$$

From that and equation (2) and triangle inequality it follows that

$$\max_{0\leq t\leq 1}\left\|A\frac{du(t)}{dt}\right\|_H \leq \max_{0\leq t\leq 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\leq t\leq 1}\left\|f(t)\right\|_H$$

$$\leq M_1\left\{\left\|A\psi\right\|_H + \left\|A^{\frac{1}{2}}\xi\right\|_H + \left\|f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|f'(t)\right\|_H\right\}.$$

The proof of Theorem 2.4 is finished. □

3. Applications

In this section we will consider three applications of the main theorem 2.4. First, for the application of theorem 2.4 we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_{tx})_x + \delta u_t(t,x) = f(t,x), \quad 0 < t < 1, 0 < x < l, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), 0 \leq x \leq l, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \quad 0 \leq x \leq l, 0 < \lambda \leq 1 \\ u_t(t,0) = u_t(t,l), \quad u_{tx}(t,0) = u_{tx}(t,l), \quad 0 \leq t \leq 1. \end{cases} \tag{16}$$

Problem (16) has the unique smooth solution $u(t,x)$ for smooth $a(x) \geq a > 0$, $x \in (0,l)$, $\delta > 0$, $a(l) = a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ ($x \in [0,l]$) and $f(t,x)$ ($t \in (0,1)$, $x \in (0,l)$) functions. This allows us to reduce problem (2) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by (16). Let us give a number of corollaries of the abstract Theorem 2.4

Theorem 3.1. *For the solution of the problem (16), the stability inequalities*

$$\max_{0\le t\le 1}\|u(t,.)\|_{L_2[0,1]} \le M_1\left[\max_{0\le t\le 1}\left\|f(t,.)\right\|_{L_2[0,1]} + \left\|\varphi\right\|_{L_2[0,1]} + \left\|\psi\right\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]}\right], \tag{17}$$

$$\max_{0\le t\le 1}\left\|\frac{\partial u}{\partial t}(t,.)\right\|_{W_2^2[0,1]} + \max_{0\le t\le 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2[0,1]}$$

$$\le M_1\left[\max_{0\le t\le 1}\left\|f_t(t,.)\right\|_{L_2[0,1]} + \left\|f(0,.)\right\|_{L_2[0,1]} + \left\|\psi\right\|_{W_2^2[0,1]} + \|\xi\|_{W_2^1[0,1]}\right] \tag{18}$$

hold, where M_1 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (16) can be written in abstract form

$$\begin{cases} \frac{d^3u(t)}{dt^3} + A\frac{du(t)}{dt} = f(t), \ \ 0 \le t \le 1, \\ u(0) = \xi u(\lambda) + \varphi, \quad u_t(0) = \alpha u_t(\lambda) + \psi, \\ u_{tt}(0) = \beta u_{tt}(\lambda) + \xi \end{cases} \tag{19}$$

in Hilbert space $L_2[0,l]$ for all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\left(a(x)u_x\right)_x + \delta u(x) \tag{20}$$

with domain

$$D(A^x) = \left\{u(x) : u, u_x, (a(x)u_x)_x \in L_2[0,l], u(0) = u(l), u'(0) = u'(l)\right\}.$$

Here $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $[0,l]$ with the values in $H = L_2[0,l]$, respectively. Therefore, estimates (17)-(18) follow from estimates (14)-(15). Thus, Theorem 3.1 is proved. □

Second, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar{\Omega} = \Omega \cup S$. In $[0,1]\times\Omega$, we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{n}\left(a_r(x)u_{tx_r}\right)_{x_r} = f(t,x), \ \ x = (x_1,...,x_n) \in \Omega, \ 0 < t < 1, \\ u(0,x) = \gamma u(\lambda,x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda,x) + \psi(x), x \in \bar{\Omega}, \\ u_{tt}(0,x) = \beta u_{tt}(\lambda,x) + \xi(x), \quad x \in \bar{\Omega}, 0 < \lambda \le 1, \\ u_t(t,x) = 0, \quad x \in S, \ 0 \le t \le 1, \end{cases} \tag{21}$$

where $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $(x \in \bar{\Omega})$ and $f(t,x)$ $(x \in [0,1])$, $x \in \Omega$ are given smooth functions and $a_r(x) > 0$. We introduce the Hilbert space $L_2(\bar{\Omega})$, the space of integrable functions defined on $\bar{\Omega}$ equipped with norm

$$\left\|f\right\|_{L_2(\bar{\Omega})} = \left\{\int\cdots\int_{x\in\bar{\Omega}}\left|f(x)\right|^2 dx_1...dx_n\right\}^{1/2}.$$

Theorem 3.2. *For the solution of the problem (21) the stability inequalities*

$$\max_{0\le t\le 1}\|u(t,.)\|_{L_2(\bar{\Omega})} \le M_2\left[\max_{0\le t\le 1}\left\|f(t,.)\right\|_{L_2(\bar{\Omega})} + \left\|\varphi\right\|_{L_2(\bar{\Omega})} + \left\|\psi\right\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})}\right], \tag{22}$$

$$\max_{0\le t\le 1}\|u(t,.)\|_{W_2^2[0,1]} + \max_{0\le t\le 1}\left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar{\Omega})}$$

$$\le M_2\left[\max_{0\le t\le 1}\left\|f_t(t,.)\right\|_{L_2(\bar{\Omega})} + \left\|f(0,.)\right\|_{L_2(\bar{\Omega})} + \left\|\psi\right\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^1(\bar{\Omega})}\right] \tag{23}$$

hold, where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (21) can be written in the abstract form (19) in the Hilbert space $L_2(\bar{\Omega})$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = \sum_{r=1}^{n} \left(a_r(x) u_{x_r}\right)_{x_r} \tag{24}$$

with domain

$$D(A^x) = \{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r}) \in L_2(\bar{\Omega}), 1 \le r \le n, u(x) = 0, x \in S\}.$$

Here $f(t) = f(t, x)$ and $u(t) = u(t, x)$ are known and unknown abstract functions defined on $\bar{\Omega}$ with the value in $H = L_2(\bar{\Omega})$, respectively. So estimates (22)-(23) follow from estimates (14)-(15) and from the coercivity inequality for the solution of the elliptic differential problem in $L_2(\bar{\Omega})$. □

Third we consider the boundary value problem for a third order partial differential equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum_{r=1}^{m} \left(a_r(x) u_{tx_r}\right)_{x_r} + \delta u_t(t,x) = f(t,x), \qquad x = (x_1, ..., x_n) \in \Omega,\ 0 < t < 1, \\ u(0,x) = \gamma u(\lambda, x) + \varphi(x), \quad u_t(0,x) = \alpha u_t(\lambda, x) + \psi(x), x \in \bar{\Omega}, \\ u_{tt}(1,x) = \beta u_{tt}(\lambda, x) + \xi(x),\ x \in \bar{\Omega},\ 0 < \lambda < 1, \\ \frac{\partial^2 u}{\partial t \partial \vec{m}}(0,x) = 0, \quad x \in S,\ 0 \le t \le 1, \end{cases} \tag{25}$$

where $a_r(x)$, $x \in \Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x \in \bar{\Omega}$ and $f(t, x)$ ($x \in [0, 1]$), $x \in \Omega$ are given smooth functions and $a_r(x) > 0$, $\vec{m}$ is the normal vector to S .

Theorem 3.3. *For the solution of the problem (25), the stability inequalities*

$$\max_{0 \le t \le 1} \|u(t,.)\|_{L_2(\bar{\Omega})} \le M_3 \left[\max_{0 \le t \le 1} \left\|f(t,.)\right\|_{L_2(\bar{\Omega})} + \left\|\varphi\right\|_{L_2(\bar{\Omega})} + \left\|\psi\right\|_{L_2(\bar{\Omega})} + \|\xi\|_{L_2(\bar{\Omega})}\right] \tag{26}$$

$$\max_{0 \le t \le 1} \|u(t,.)\|_{W_2^2(\bar{\Omega})} + \max_{0 \le t \le 1} \left\|\frac{\partial^3 u}{\partial t^3}(t,.)\right\|_{L_2(\bar{\Omega})}$$

$$\le M_3 \left[\max_{0 \le t \le 1} \left\|f_t(t,.)\right\|_{L_2(\bar{\Omega})} + \left\|f(0,.)\right\|_{L_2(\bar{\Omega})} + \left\|\psi\right\|_{W_2^2(\bar{\Omega})} + \|\xi\|_{W_2^1(\bar{\Omega})}\right] \tag{27}$$

hold, where M_3 does not depend on $f(t, x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (25) can be written in the abstract form (19) in the Hilbert space $L_2(\bar{\Omega})$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\sum_{r=1}^{m} \left(a_r(x) u_{x_r}\right)_{x_r} + \delta u(x) \tag{28}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), \left(a_r(x)u_{x_r}\right)_{x_r} \in L_2(\bar{\Omega}), 1 \le r \le m, \frac{\partial u}{\partial \vec{m}} = 0, x \in S\right\}.$$

Here $f(t) = f(t, x)$ and $u(t) = u(t, x)$ are known and unknown abstract functions defined on $\bar{\Omega}$ with the value in $H = L_2(\bar{\Omega})$, respectively. So, estimates (26)-(27) follow from estimates (14)-(15) and from the coercivity inequality for the solution of the elliptic differential problem in $L_2(\bar{\Omega})$. □

4. Conclusion

In the present paper, we have discussed a nonlocal boundary value problem of a third order partial differential equation. Theorem on stability estimates for the solution of this problem is established. In application, stability estimates for the solution of three problems for a third order partial differential equation are obtained.

In papers [4, 5], three step difference schemes generated by Taylor's decomposition on three points for the numerical solution of local and nonlocal boundary value problems of linear ordinary differential equation of third order were investigated. Note that Taylor's decomposition on four points is applicable for the construction of difference schemes of problem (2). Operator method of [8] permits to establish the stability of this difference problem for the approximation problem of (2).

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