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On Generalized Classes of Exponential Distribution using T-X Family Framework

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Abstract. We introduce new generalized classes of exponential distribution, called T-exponential{Y} class using the quantile functions of well-known distributions. We derive some general mathematical properties of this class including explicit expressions for the quantile function, Shannon entropy, moments and mean deviations. Some generalized exponential families are investigated. The shapes of the models in these families can be symmetric, left-skewed, right-skewed and reversed-J, and the hazard rate can be increasing, decreasing, bathtub, upside-down bathtub, J and reverse-J shaped. Two real data sets are used to illustrate the applicability of the new models.

1. Introduction

The exponential distribution is one of the first lifetime models for which statistical methods were extensively developed. Here, it is worthwhile to quote [13] "the most important one parameter family of lifetime distributions is the family of exponential distributions". This importance is partly due to the fact that several of the most commonly used families of lifetime distributions are two- or three-parameter extensions of the exponential distribution. Although, one parameter exponential distribution has several interesting properties such as lack of memory property, one of the major disadvantages of the exponential distribution is that it has a constant hazard rate function. Moreover, the density function of the exponential distribution is always a decreasing function. In many practical situations, one might observe non-monotone hazard functions, and in such cases, the exponential distribution can not be used. Due to this reason several generalizations of the exponential distribution have been suggested in the literature. For example, [9] introduced generalized exponential distribution and [12] proposed beta exponential distribution by using the beta-family proposed by [8].

The beta-generated family was extended by [3] to the T-R{W} family. The cumulative distribution function (cdf) of the T-R{W} distribution is $G(x) = \int_a^{W(F(x))} r(t)dt$, where $r(t)$ is the probability density function

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(pdf) of a random variable T with support (a,b) for $-\infty \leq a < b \leq \infty$. The link function $W : [0,1] \to R$ is monotonic and absolutely continuous with $W(0) \to a$ and $W(1) \to b$. [2] considered a special case of the T-R{W} family by taking $W(.)$ to be the quantile function of a random variable Y and defined the T-R{Y}family.

2. The T-R{Y} family

The T-R{Y} framework defined in [2] (see also [4]) is given as follows. Let T, R and Y be random variables with cdfs $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$ and $F_Y(x) = P(Y \leq x)$, respectively. The corresponding quantile functions (qfs) are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the qf is defined as $Q_Z(p) = \inf\{z : F_Z(z) \geq p\}$, $0 < p < 1$. If the densities exist, we denote them by $f_T(x)$, $f_R(x)$ and $f_Y(x)$. Further, we assume that the random variables T, $Y \in (a,b)$ for $-\infty \leq a < b \leq \infty$. The cdf of the T-R{Y} class of distributions is given by

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\,dt = P\Big[T \leq Q_Y\big(F_R(x)\big)\Big] = F_T\Big(Q_Y\big(F_R(x)\big)\Big). \tag{1}$$

The pdf and hazard rate function corresponding to (1) are given by

$$f_X(x) = f_R(x)\,\frac{f_T\Big(Q_Y\big(F_R(x)\big)\Big)}{f_Y\Big(Q_Y\big(F_R(x)\big)\Big)} \tag{2}$$

and

$$h_X(x) = h_R(x)\,\frac{h_T\Big(Q_Y\big(F_R(x)\big)\Big)}{h_Y\Big(Q_Y\big(F_R(x)\big)\Big)}. \tag{3}$$

Table 1: Quantile functions for some choices of the random variable Y.

S.No.	Y	$Q_Y(p)$
(a)	Uniform	p
(b)	Exponential	$-b\log(1-p), b > 0$
(c)	Log-logistic	$a\,(\frac{p}{1-p})^{1/b}, a, b > 0$
(d)	Logistic	$a + b\log(\frac{p}{1-p}), b > 0$
(e)	Extreme value	$a + b\log(-\log(1-p)), b > 0$
(f)	Weibull	$(-\frac{1}{\theta}\log(1-p))^{1/\gamma}, \gamma, c > 0$

Table 2: CDFs or PDFs for some choices of the random variable T.

S.No.	T	$F_T(x)$ or $f_T(x)$
1.	EE	$F_T(x) = [1 - e^{-\lambda x}]^{\alpha}$
2.	Weibull	$F_T(x) = 1 - e^{-\theta x^{\gamma}}$
3.	Logistic	$F_T(x) = 1 - [1 + e^{-(x-a)/b}]^{-1}$
4.	Log-logistic	$F_T(x) = 1 - [1 + (x/a)^b]^{-1}$
5.	Pareto	$F_T(x) = 1 - (x/\alpha)^{\beta}$
6.	Cauchy	$f_T(x) = \{\pi\,\beta[1 + \frac{x-\alpha}{\beta}]^2\}^{-1}$
7.	Pascal	$f_T(x) = 0.5\,\lambda\,e^{-\lambda\lvert x\rvert}$
8.	Gamma	$f_T(x) = \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\,x^{\alpha-1}\,e^{-x/\beta}$

Next we mention some existing generalized families of distributions that fall into the T-R{Y} framework.

1. *Gamma-G family.* This family (introduced by [18]) can be generated by taking T as gamma random variable and Q_Y is the quantile function of the exponential distribution.

2. *Weibull-X family.* This family (proposed by [3]; [5]) is obtained by taking T as Weibull random variable and Q_Y is the quantile function of the exponential distribution.
3. *Libby and Novick's generalized beta family.* This family [7] can be obtained by taking T as Libby and Novick's generalized beta distribution and Q_Y is the quantile function of the uniform distribution.

Our motivation in this paper is related to the flexibility of the new generalized family of exponential distribution to model complex positive real data sets, that is, its sub-models can present increasing, decreasing, upside-down bathtub and bathtub shaped hazard rate functions. Due to great flexibility of its hazard rate functions, it thus provides a good alternative to many existing lifetime distributions. In this article, we propose a family of generalized exponential distributions, the T-exponential{Y} family, and study some its properties and applications. A member of the T-exponential{Y} family, namely, T-exponential{loglogistic} distribution is studied in detail. The paper is unfolded as follows. In Section 2, we consider the *T-exponential{Y}* class of distributions and define some new generalized exponential families. In Section 3, we investigate some structural properties of this class. In Section 4, we define some new extended exponential distributions and study some of their properties. In Section 5, we explore the usefulness of three generalized exponential family models by means of two applications to real data sets. Finally, Section 6 offers some concluding remarks.

3. The T-exponential{Y} class

If R follows the exponential random variable with pdf $f_R(x) = \beta^{-1}\,\mathrm{e}^{-x/\beta}$ and cdf $F_R(x) = 1 - \mathrm{e}^{-x/\beta}$, the *T-exponential{Y}* (or T-E{Y} for short) class of distributions is defined from Eq. (1) as

$$F_X(x) = \int_a^{Q_Y\left(1-\mathrm{e}^{-x/\beta}\right)} f_T(x)\,dt = F_T\left(Q_Y\left[1-\mathrm{e}^{-x/\beta}\right]\right). \tag{4}$$

The pdf corresponding to Eq. (4) can be expressed as

$$f_X(x) = \frac{1}{\beta}\,\mathrm{e}^{-x/\beta} \times \frac{f_T\left(Q_Y\left(1-\mathrm{e}^{-x/\beta}\right)\right)}{f_Y\left(Q_Y\left(1-\mathrm{e}^{-x/\beta}\right)\right)}. \tag{5}$$

The hazard rate function of the T-E{Y} class becomes

$$h_X(x) = \frac{1}{\beta}\,\frac{h_T\left(Q_Y\left(1-\mathrm{e}^{-x/\beta}\right)\right)}{h_Y\left(Q_Y\left(1-\mathrm{e}^{-x/\beta}\right)\right)}. \tag{6}$$

Remark 1. If X follows the T-E{Y} class of distributions given by Eq. (4), we have the following:
(i) $X \overset{d}{=} -\beta\,\log\{1-F_Y(T)\}$,
(ii) $Q_X(p) = -\beta\,\log\left\{1-F_Y\left(Q_T(p)\right)\right\}$,
(iii) if $T \overset{d}{=} Y$, then $X \overset{d}{=}$ Exponential (β), and
(iv) if $Y \overset{d}{=}$ Exponential (β), then $X \overset{d}{=} T$.

The T-E{Y} class in Eq. (5) can generate many different extended exponential families. In the following, we define some generalized exponential families using some qfs listed in Table 1.

3.1. The T-E{log-logistic} family

Let the random variable $T \in (0,\infty)$. By using the qf (c) in Table 1, $Q_Y(p) = a\left(\frac{p}{1-p}\right)^{1/b}$, $a, b > 0$, the cdf and pdf follow from Eq. (4) and Eq. (5) as

$$F_X(x) = F_T\left(a\left[\mathrm{e}^{x/\beta}-1\right]^{1/b}\right) \tag{7}$$

and

$$f_X(x) = \frac{a\,\mathrm{e}^{x/\beta}}{b\,\beta}\,\left(\mathrm{e}^{x/\beta}-1\right)^{\frac{1}{b}-1}\,f_T\left(a\left[\mathrm{e}^{x/\beta}-1\right]^{1/b}\right). \tag{8}$$

3.2. The T-E{logistic} family

Let the random variable $T \in (-\infty, \infty)$. By using the qf (d) in Table 1, $Q_Y(p) = a + b\log\left(\frac{p}{1-p}\right), b > 0$, the cdf and pdf follow from Eq. (4) and Eq. (5) as

$$F_X(x) = F_T\Big(a + b\log\left[\mathrm{e}^{x/\beta} - 1\right]\Big) \tag{9}$$

and

$$f_X(x) = \frac{b}{\beta\left(1 - \mathrm{e}^{-x/\beta}\right)}\; f_T\Big(a + b\,\log\left[\mathrm{e}^{x/\beta} - 1\right]\Big). \tag{10}$$

3.3. The T-E{extreme value} family

Let the random variable $T \in (-\infty, \infty)$. By using the qf (e) in Table 1, $Q_Y(p) = a + b\log\left[-\log(1-p)\right], b > 0$, the cdf and pdf follow from Eq. (4) and Eq. (5) as

$$F_X(x) = F_T\Big(a + b\log\left(x/\beta\right)\Big) \tag{11}$$

and

$$f_X(x) = \frac{b}{x}\; f_T\Big(a + b\,\log\left(x/\beta\right)\Big). \tag{12}$$

4. Some general properties

In this section, some of general properties of the T-E{Y} class are investigated.

The following Lemma gives the relationships between the random variables X and T for some cases which can be used to simulate the random variable X from the random variable T.

Lemma 1. Let T be a random variable with pdf $f_T(x)$.
(i) If $T \in (0, \infty)$, then the random variable $X = \beta\log\left\{1 + \left(\frac{T}{a}\right)^b\right\}$ follows the T-E{log-logistic} family in Eq. (8).
(ii) If $T \in (-\infty, \infty)$, then the random variable $X = \beta\log\left\{1 + \mathrm{e}^{\frac{T-a}{b}}\right\}$ follows the T-E{logistic} family in Eq. (10).
(iii) If $T \in (-\infty, \infty)$, then the random variable $X = \beta\,\mathrm{e}^{\frac{T-a}{b}}$ follows the T-E{extreme value} family in Eq. (12).

Remark 2. From Lemma 1, the qfs for the (i) T-E{log-logistic}, (ii) T-E{logistic}, and (iii) T-E{extreme value} families are, respectively, given by:
(i) $Q_X(p) = \beta\log\left\{1 + \left(\frac{Q_T(p)}{a}\right)^b\right\}$,
(ii) $Q_X(p) = \beta\log\left\{1 + \mathrm{e}^{\frac{Q_T(p)-a}{b}}\right\}$,
(iii) $Q_X(p) = \beta\,\mathrm{e}^{\frac{Q_T(p)-a}{b}}$.

Theorem 1. The modes of the T-E{Y} class are the solutions of the equation:

$$x = \beta\log\left[\frac{Q_Y''\left(1 - \mathrm{e}^{-x/\beta}\right)}{Q_Y'\left(1 - \mathrm{e}^{-x/\beta}\right)} + \frac{f_T'\left(Q_Y\left(1 - \mathrm{e}^{-x/\beta}\right)\right)}{f_T\left(Q_Y\left(1 - \mathrm{e}^{-x/\beta}\right)\right)} \times Q_Y'\left(1 - \mathrm{e}^{-x/\beta}\right)\right]. \tag{13}$$

Corollary 1. The modes of the (i) T-E{log-logistic}, (ii) T-E{logistic}, and (iii) T-E{extreme value} families can be determined as the solutions of the equations:

$$
\begin{aligned}
(i)\quad x =&\ \ \beta \log\left[\frac{1}{b\,\beta}\left\{\frac{a\,\mathrm{e}^{x/\beta}\left(\mathrm{e}^{x/\beta}-1\right)^{1/b-1}\times f_T'\left(a\left(\mathrm{e}^{x/\beta}-1\right)^{1/b}\right)}{f_T\left(a\left(\mathrm{e}^{x/\beta}-1\right)^{1/b}\right)}+1\right\}\right],\\
(ii)\quad x =&\ \ \beta \log\left[\frac{1}{\beta\,(\mathrm{e}^{x/\beta}-1)}\left\{\frac{b\,f_T'\left(a+b\log\left(\mathrm{e}^{x/\beta}-1\right)\right)}{f_T\left(a+b\log\left(\mathrm{e}^{x/\beta}-1\right)\right)}-1\right\}\right],\\
(iii)\quad x =&\ \ \beta \log\left[\frac{1}{x}\left\{\frac{b\,f_T'\,(a+b\log(x/\beta))}{f_T\,(a+b\log(x/\beta))}-1\right\}\right],
\end{aligned}
$$

respectively. Note that the result in Theorem 1 does not imply that the mode is unique. It is possible that there is more than one mode for some families in the T-E{Y} class.

The entropy of a random variable X is a measure of variation of uncertainty ([15]). The Shannon's entropy ([17]) of the random variable X with pdf $g(x)$ is defined by $\eta_X = \mathbb{E}\{-\log[g(X)]\}$. It has been used in many applications in fields of engineering, physics and economics.

Theorem 2. The Shannon's entropy of the T-E{Y} class can be expressed as

$$\eta_X = \eta_T + \mathbb{E}\left[\log\, f_Y(T)\right] + \log(\beta) + \frac{1}{\beta}\,\mathbb{E}(X). \tag{14}$$

Proof. Since $X \overset{d}{=} Q_R\,(F_Y\,(T))$, we have $T \overset{d}{=} Q_Y\,(F_R\,(X))$. Hence, based on equation (2), we can write

$$f_X(x) = \frac{f_T(t)}{f_Y(t)}\, f_R(x).$$

This result implies that

$$\eta_X = \eta_T + \mathbb{E}\left[\log\, f_Y(T)\right] - \mathbb{E}\left[\log\, f_R(X)\right]. \tag{15}$$

For the T-E{Y} class, we have

$$\log[f_R(x)] = -\log(\beta) - \frac{x}{\beta}. \tag{16}$$

Eq. (14) follows from Eq. (15) and Eq. (16).

Corollary 2. The Shannon's entropies for the (i) T-E{log-logistic}, (ii) T-E{logistic}, and (iii) T-E{extreme value} families, respectively, are given by

$$
\begin{aligned}
(i)\quad \eta_X =&\ \ \eta_T + \log\left(b\,\beta\,a^{-b}\right) + (b-1)\mathbb{E}(\log T) - 2\mathbb{E}\left\{\log\left[1+(T/a)^b\right]\right\} + \beta^{-1}\mathbb{E}(X),\\
(ii)\quad \eta_X =&\ \ \eta_T + \log(\beta\lambda) - \lambda\,\mu_T - 2\mathbb{E}\left\{\log\left[1+\mathrm{e}^{-\lambda\,T}\right]\right\} + \beta^{-1}\mathbb{E}(X),\\
(iii)\quad \eta_X =&\ \ \eta_T + \log(\beta/b) - (a/b) + \mu_T - \mathrm{e}^{\frac{-a}{b}}\,M_T(1/b) + \beta^{-1}\mathbb{E}(X),
\end{aligned}
$$

where $M_T(s) = \mathbb{E}\left(\mathrm{e}^{s\,T}\right)$ is the moment generating function of T.

Proof. The results in (i)-(iii) can be easily proved using Eq. (14) and the facts that:
$f_Y(T) = (b/a)\,(T/a)^{b-1}\left[1+(T/\alpha)^{\beta}\right]^{-2}$, $f_Y(T) = \lambda\,\mathrm{e}^{-\lambda\,T}\left(1+\mathrm{e}^{-\lambda\,T}\right)^{-2}$,
$f_Y(T) = \frac{1}{b}\,\mathrm{e}^{\left(\frac{T-a}{b}\right)}\,\mathrm{e}^{-\mathrm{e}^{\left(\frac{T-a}{b}\right)}}$, for the log-logistic, logistic and extreme value families, respectively.

Some key features of a distribution such as skewness and kurtosis can be studied through its moments.

Corollary 3. The ordinary moments of the T-E{extreme value} family can be expressed as $\mathbb{E}(X^r) = \beta^r\, e^{-r\,a\,b^{-1}}\, M_T(rb^{-1})$.

Proof. Follows from Lemma 1(iii).

The central moments (μ_s) and cumulants (κ_s) of X can be determined from the ordinary moments as follows:

$$\mu_s = \sum_{k=0}^{p} \binom{s}{k} (-1)^k\, \mu_1'^s\, \mu_{s-k}' \quad \text{and} \quad \kappa_s = \mu_s' - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k\, \mu_{s-k}',$$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ of X immediately follow from the third and fourth standardized cumulants.

The nth descending factorial moment of X is

$$\mu_{(n)}' = \mathbb{E}(X^{(r)}) = E\,[X(X-1)\times\cdots\times(X-r+1)] = \sum_{k=0}^{r} s(r,k)\, \mu_k',$$

where

$$s(r,k) = (k!)^{-1} \left[\frac{d^k}{dx^k} x^{(r)} \right]_{x=0}$$

is the Stirling number of the first kind which counts the number of ways to permute a list of r items into k cycles. So, we can obtain the factorial moments from the ordinary moments given before.

Lemma 2 (Upper bound for the moments of the T-E{Y} family). For the T-R{Y} family in Eq. (2), if R is a non-negative random variable and $\mathbb{E}\left[\left(1 - F_Y(T)\right)^{-1}\right] < \infty$, we obtain

$$\mathbb{E}\,(X^n) \le \mathbb{E}\,(R^n)\ \mathbb{E}\Big[\,(1 - F_Y(T))^{-1}\,\Big].$$

Proof. The result follows from Theorem 1 in [2].

Theorem 3. If X follows the T-E{Y} family of distributions in (5), then $\mathbb{E}\,(X^n) \le n!\,\beta^n\ \mathbb{E}\Big[\left(1 - F_Y(T)\right)^{-1}\Big]$.

Proof. The result follows from Lemma 2.

Corollary 4. Assuming that the moments exist, we have:
(a) If $X \sim$ T-E{log-logistic}, then $\mathbb{E}\,(X^n) \le n!\,\beta^n\ \left[1 + a^{-b}\,\mathbb{E}\left(T^b\right)\right]$.
(b) If $X \sim$ T-E{logistic}, then $\mathbb{E}\,(X^n) \le n!\,\beta^n\ \left[1 + e^{-a/b}\, M_T\,(1/b)\right]$.
(c) If $X \sim$ T-E{extreme value}, then $\mathbb{E}\,(X^n) < n!\,\beta^n\ \left[1 + e^{-a/b}\, M_T\,(1/b)\right]$.

Proof. For (a) and (b), the results follow immediately from Theorem 3. Now, if X follows the T-E{extreme value} in Eq. (12) then $\left[\left(1 - F_Y(T)\right)^{-1}\right] = \left[1 - e^{-e^{-(T-a)/b}}\right]^{-1}$. By using the inequality $1 - e^{-x} > x/(1+x)$ for all $x > -1$ [1], we obtain $\left[\left(1 - F_Y(T)\right)^{-1}\right] < 1 + e^{(T-a)/b}$. Hence, the result follows from Theorem 3.

The deviations from the mean and from the median, say $D(\mu)$ and $D(M)$, measure the dispersion and the spread in a population from the center of the distribution.

Theorem 4. The quantities $D(\mu)$ and $D(M)$ for the (i) T-E{log-logistic}, (ii) T-E{logistic}, and (iii) T-E{extreme value} families, respectively, are obtained below.

(i) For $w = a\left(e^{x/\beta} - 1\right)^{1/b}$,

$$D(\mu) = 2\mu F_X(\mu) - 2\beta \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j\, a^{bj}}\, S_w(\mu), \tag{17}$$

$$D(M) = \mu - 2\beta \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j\, a^{bj}}\, S_w(M), \tag{18}$$

where $S_w(c) = \int_0^{Q_Y(F_T(c))} w^{bj}\, f_T(w)\, dw$ and $Q_Y\left(F_T(c)\right) = a\left(e^{c/\beta} - 1\right)^{1/b}$.

(ii) For $w = a + b\log\left(e^{x/\beta} - 1\right)$,

$$D(\mu) = 2\mu F_X(\mu) - 2\beta \sum_{j=1,k=0}^{\infty} \frac{(-1)^{j+1}(j/b)^k\, e^{-\frac{ja}{b}}}{k!}\, S_w(\mu),$$

$$D(M) = \mu - 2\beta \sum_{j=1,k=0}^{\infty} \frac{(-1)^{j+1}(j/b)^k\, e^{-\frac{ja}{b}}}{k!}\, S_w(M),$$

where $S_w(c) = \int_{-\infty}^{Q_Y(F_T(c))} w^k\, f_T(w)\, dw$ and $Q_Y\left(F_T(c)\right) = a + b\log\left(e^{c/\beta} - 1\right)$.

(iii) For $w = a + b\log(x/\beta)$,

$$D(\mu) = 2\mu F_X(\mu) - 2\beta \sum_{k=0}^{\infty} \frac{e^{-a/b}}{k!\, b^k}\, S_w(\mu),$$

$$D(M) = \mu - 2\beta \sum_{k=0}^{\infty} \frac{e^{-a/b}}{k!\, b^k}\, S_w(M),$$

where $S_w(c) = \int_{-\infty}^{Q_Y(F_T(c))} w^k\, f_T(w)\, dw$ and $Q_Y\left(F_T(c)\right) = a + b\log(c/\beta)$.

Proof. By definitions of $D(\mu)$ and $D(M)$, we can write:

$$D(\mu) = 2\mu F_X(\mu) - 2\int_0^{\mu} x\, f_X(x)\, dx \quad \text{and} \quad D(M) = \mu - 2\int_0^{M} x\, f_X(x)\, dx. \tag{19}$$

First, we prove Eq. (17) and Eq. (18) for the T-E{log-logistic} family. Consider the first incomplete moment of X defined by

$$I_c = \int_0^c x\, f_X(x)\, dx = \int_0^c x\, \frac{a\, e^{x/\beta}}{b\,\beta} \left(e^{x/\beta} - 1\right)^{\frac{1}{b}-1} f_T\!\left(a\left[e^{x/\beta} - 1\right]^{1/b}\right) dx.$$

For empirical purposes, the quantity I_c plays an important role for measuring inequality, for example, Lorenz and Bonferroni curves.

Setting $w = a\left(e^{x/\beta} - 1\right)^{1/b}$, we can write

$$I_c = \beta \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j\, a^{bj}}\, S_w(c), \tag{20}$$

where $S_w(c) = \int_0^{Q_Y(F_T(c))} w^{bj}\, f_T(w)\, dw$ and $Q_Y(F_T(c)) = a\left(\mathrm{e}^{c/\beta} - 1\right)^{1/b}$.

Equations (17) and (18) follow by inserting Eq. (20) in Eq. (19). Similarly, we can prove the results in (ii) and (iii).

For a given probability π, the Bonferroni and Lorenz curves of X are defined by $B(\pi) = I_q/(\pi\mu_1')$ and $L(\pi) = I(q)/\mu_1'$, respectively, where $q = Q_X(\pi)$ can be obtained from Lemma 2 and I_q from Theorem 4. These curves can be readily determined and have applications in economics to study income and poverty, and also in other fields like reliability, demography, insurance and medicine. In economics, if $\pi = F_X(q)$ is the proportion of units whose income is lower than or equal to q, $L(\pi)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q. The Lorenz curve is increasing and convex and given the mean income, the density function of T can be obtained from the curvature of $L(\pi)$. In a similar manner, the Bonferroni curve $B(\pi)$ gives the ratio between the mean income of this group and the mean income of the population. In summary, $L(\pi)$ yields fractions of the total income, while the values of $B(\pi)$ refer to relative income levels.

5. Some special models in the T-E{Y} class

We choose some T and Y random variables to generate four new T-E{Y} distributions, namely: the Weibull-E{log-logistic}, gamma-E{log-logistic}, normal-E{logistic} and logistic-E{extreme value} distributions. For illustrative purposes, we study some properties of the Weibull-E{log-logistic} distribution. To save space, some mathematical properties of the other distributions are not given here. One can follow similar algebra to study some properties of them.

5.1. The Weibull-E{log-logistic} distribution

If $T \sim$ Weibull(θ, γ), then $f_T(t) = \theta\gamma\, t^{\gamma-1}\,\mathrm{e}^{-\theta\, t^{\gamma}}$ and $F_T(t) = 1 - \mathrm{e}^{-\theta\, t^{\gamma}}$. Using Eq. (7), the cdf of the Weibull-E{log-logistic} distribution is given by

$$F_X(x) = 1 - \mathrm{e}^{-\theta\left[a\left(\mathrm{e}^{x/\beta}-1\right)^{1/b}\right]^{\gamma}}.$$

To reduce the redundancy of the scale and shape parameters, we set $a = \theta = 1$ and $c = \gamma/b$. Therefore, the cdf and pdf of the Weibull-E{log-logistic} distribution, respectively, are

$$F_X(x) = 1 - \mathrm{e}^{-\left(\mathrm{e}^{x/\beta}-1\right)^{c}}$$

and

$$f_X(x) = \frac{c}{\beta}\,\mathrm{e}^{x/\beta}\left[\mathrm{e}^{x/\beta} - 1\right]^{c-1}\,\mathrm{e}^{-\left(\mathrm{e}^{x/\beta}-1\right)^{c}}.$$

Next, some properties of the Weibull-E{log-logistic} distribution are obtained using the general properties discussed in Section 3.

(1) *Quantile function.* By using Lemma 2, the qf of the Weibull-E{LL} distribution is given by

$$Q_X(p) = \beta\,\log\left\{1 + (-\log(1-p))^{c}\right\}.$$

(2) *Mode.* By using Corollary 1, the modes of the Weibull-E{log-logistic} distribution are the solutions of the following equation:

$$x = \log\left[\frac{1}{(1-\mathrm{e}^{-x/\beta})}\left\{c - c\left(\mathrm{e}^{x/\beta} - 1\right)^{c} - \mathrm{e}^{-x/\beta}\right\}\right].$$

(3) *Moments.* The following Lemma shows that the moments of the Weibull-E{log-logistic} exist.

Lemma 3. The nth moment of the Weibull-E{log-logistic} distribution exists for all n and satisfies the following inequality $\mathbb{E}(X^n) \leq n!\, \beta^n \left[1 + \Gamma\left(1 + 1/c\right)\right]$.

Proof. The results follow from Corollary 4.

(4) *Shannon entropy.* By using Corollary 2 and the fact that $\eta_T = 1 - \log(\gamma) + \xi(1 - \gamma^{-1})$, the Shannon entropy of the Weibull-E{log-logistic} distribution follows as

$$\eta_x = 1 - \log(\beta/c) + \xi\,(1 - 1/c) + \beta^{-1}\,\mathbb{E}(X),$$

where ξ is an Euler constant.

5.2. *The gamma-E{log-logistic} distribution*

If $T \sim$Gamma$(\alpha, 1)$, then $f_T(t) = \frac{1}{\Gamma(\alpha)}\, t^{\alpha-1}\, \mathrm{e}^{-t}$ and $F_T(t) = \gamma(\alpha, t)/\Gamma(\alpha)$, where $\gamma(p, x) = \int_0^\infty x^{p-1}\, \mathrm{e}^{-x} dx$ is the incomplete gamma function. Using Eq. (7) and Eq. (8), the cdf of the gamma-E{log-logistic} distribution is given by

$$F_X(x) = \gamma\left(\alpha, a\left[\mathrm{e}^{x/\beta} - 1\right]^{1/b}\right)/\Gamma(\alpha).$$

Setting $a = 1$, the corresponding pdf is

$$f_X(x) = \frac{1}{b\,\beta\,\Gamma(\alpha)}\; \mathrm{e}^{x/\beta}\; \left(\mathrm{e}^{x/\beta} - 1\right)^{\frac{\alpha}{b}-1}\; \mathrm{e}^{-\left(\mathrm{e}^{x/\beta}-1\right)^{\frac{1}{b}}}.$$

5.3. *The normal-E{logistic} distribution*

If $T \sim$Normal(μ, σ), then $f_T(t) = \sigma^{-1}\, \phi\left(\frac{x-\mu}{\sigma}\right)$ and $F_T(t) = \Phi\left(\frac{x-\mu}{\sigma}\right)$. Using Eq. (9) and Eq. (10), the cdf and pdf of the Normal-E{logistic} distribution are, respectively, given by

$$F_X(x) = \Phi\left(a + b\log\left(\mathrm{e}^{x/\beta} - 1\right)\right)$$

and

$$f_X(x) = \frac{b}{\beta\,(1 - \mathrm{e}^{-x/\beta})}\; \phi\left(a + b\log\left(\mathrm{e}^{x/\beta} - 1\right)\right),$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter and $\sigma > 0$ is a scale parameter.

5.4. *The logistic-E{extreme value} distribution*

If $T \sim$logistic(λ) and $\lambda \in \mathbb{R}^+$, then $f_T(t) = \lambda\, \mathrm{e}^{-\lambda t}\,(1 + \mathrm{e}^{-\lambda t})^{-2}$ and $F_T(t) = (1 + \mathrm{e}^{-\lambda t})^{-1}$. Using Eq. (11) and Eq1. (12), the cdf of the logistic-E{extreme value} distribution is given by

$$F_X(x) = \left\{1 + \mathrm{e}^{-\lambda\left[a + b\log(x/\beta)\right]}\right\}^{-1}.$$

Setting $b = 1$, the pdf of the logistic-E{extreme value} distribution is

$$f_X(x) = \frac{\lambda}{x}\; \mathrm{e}^{-\lambda\left[a + \log(x/\beta)\right]}\; \left\{1 + \mathrm{e}^{-\lambda\left[a + \log(x/\beta)\right]}\right\}^{-2}.$$

Figures 1 and 2 display some plots of the pdf and hrf of the Weibull-E{log-logistic} (W-E{LL}), gamma-E{log-logistic} (Ga-E{LL}), normal-E{logistic} (N-E{L}) and logistic-E{extreme value} (L-E{EV}) distributions for selected parameter values. Figure 1 indicates that the T-E{Y} class generates distributions with various shapes such as symmetric, left-skewed, right-skewed and reversed-J. Further, Figure 2 reveals that this class produces flexible hazard rate shapes such as increasing, decreasing, bathtub, upside-down bathtub, J and reversed-J. In fact, the T-E{Y} class is very useful for fitting data sets with various shapes.

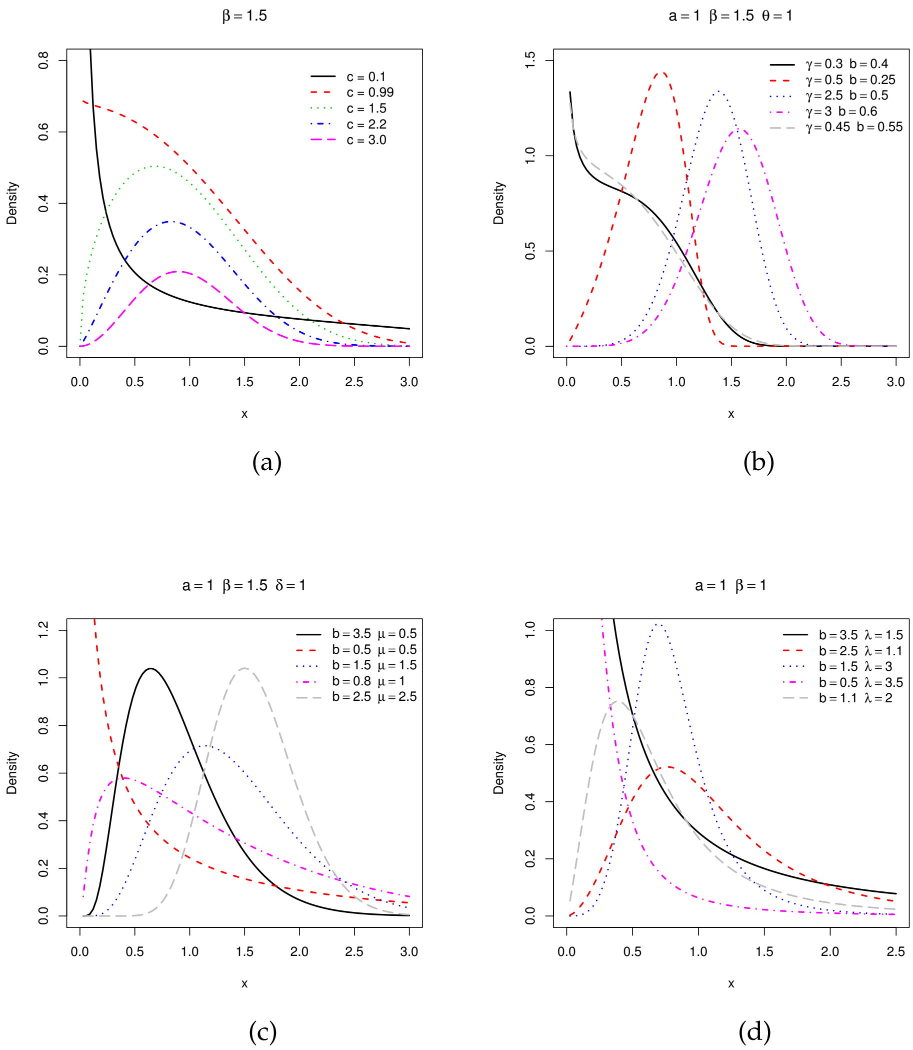


Figure 1: Density plots: (a) W-E{LL} (b) Ga-E{LL} (c) N-E{L} and (d) L-E{EV} models.

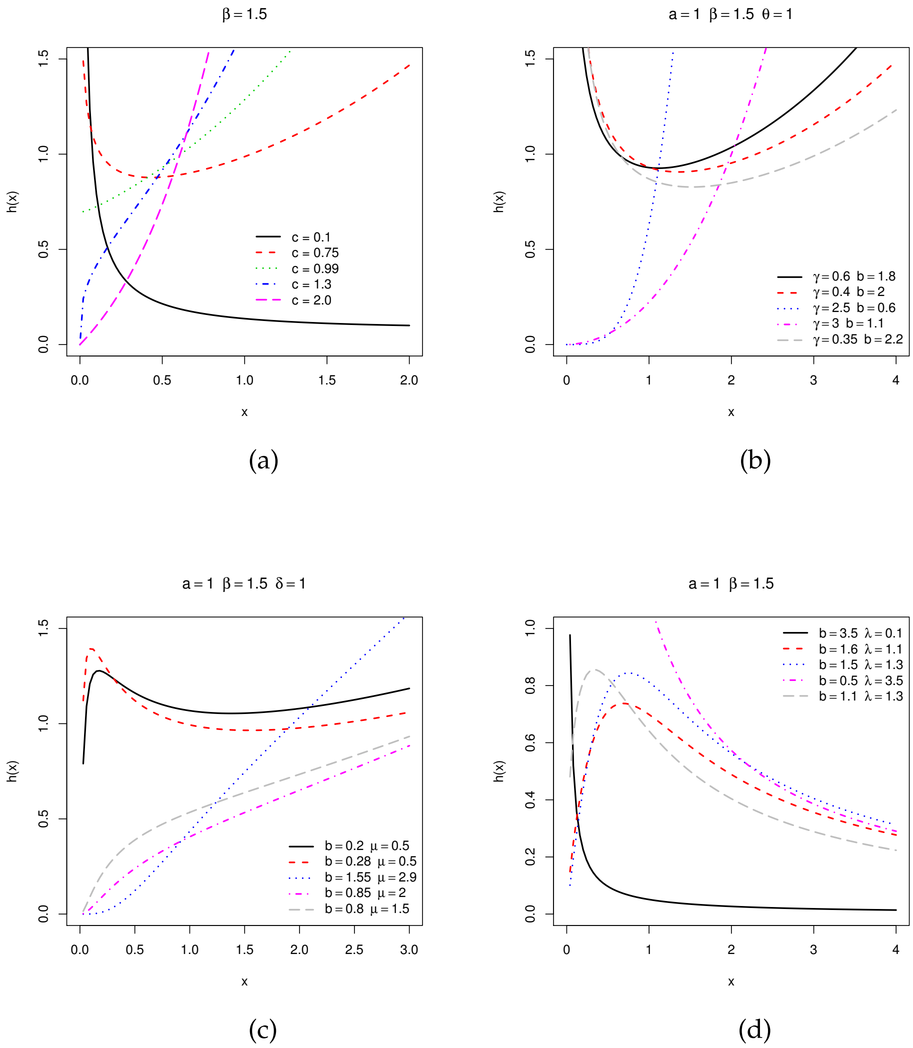


Figure 2: Hazard rate plots: (a) W-E{LL} (b) Ga-E{LL} (c) N-E{L} and (d) L-E{EV} models.

6. Applications

In this section, three sub-model of the T-E{Y} class are fitted to two real data set, namely the W-E{LL}, Ga-E{LL} and N-E{L} distributions. For comparison purposes, gamma exponentiated-exponential (GEE) ([16]), beta-exponential (BE) ([12]) and exponentiated exponential (EE) ([9]) distributions are fitted to the real data sets. The first data set represents the failure time of 20 components ([11]). The data set are: 0.072, 4.763, 8.663, 12.089, 0.477, 5.284, 9.511, 13.036, 1.592, 7.709, 10.636, 13.949, 2.475, 7.867, 10.729, 16.169, 3.597, 8.661, 11.501, 19.809. The second data set represents the breaking stress of carbon fibres ([14]). The data set are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The maximum likelihood method is used to estimate the model parameters. The goodness of fit statistics including the maximum log-likelihood ($\hat{\ell}_{max}$), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics A^* and W^* are described in details in [6]. In general, the smaller the values of these statistics, the better the fit to the data.

The required computations are carried out in the R-language. Tables 3 and 4 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The numerical values of the model selection statistics $\hat{\ell}_{max}$, AIC, CAIC, BIC, HQIC, A^*, W^* and K-S are listed in Tables 5 and 6.

Table 3: MLEs and their standard errors (in parentheses) for the first data set.

Distribution	a	b	c	β	θ	γ	α	λ
W-E{LL}	1.000	-	0.618	12.375	1.000	-	-	-
	-	-	(0.141)	(2.072)	-	-	-	-
Ga-E{LL}	1.000	4.334	-	2.242	-	-	2.682	-
	-	(2.723)	-	(1.561)	-	-	(0.513)	-
GEE	-	-	-	0.193	-	-	1.087	0.677
	-	-	-	(0.258)	-	-	(0.621)	(0.974)
BE	1.274	4.905	-	0.030	-	-	-	-
	(0.363)	(16.879)	-	(0.100)	-	-	-	-
EE	-	-	-	0.135	-	-	1.235	-
	-	-	-	(0.036)	-	-	(0.361)	-
W	-	-	-	-	1.351	0.110	-	-
	-	-	-	-	(0.261)	(0.019)	-	-

Table 4: MLEs and their standard errors (in parentheses) for the second data set.

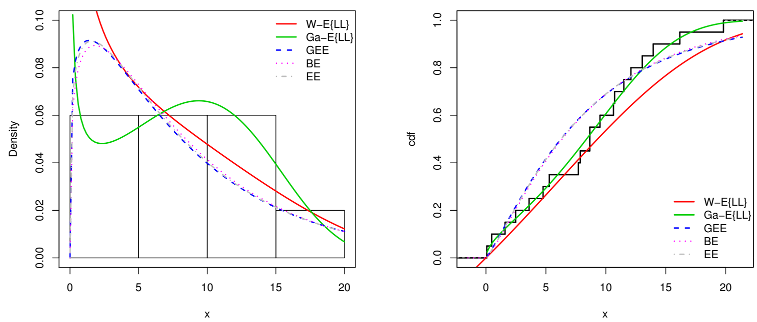
Distribution	a	b	β	μ	σ	α	λ
N-E{L}	0.000	0.891	1.014	2.182	1.000	-	-
	-	(0.355)	(0.533)	(0.506)	-	-	-
Ga-E{LL}	1.000	0.862	2.740	-	-	2.048	-
	-	(0.397)	(1.255)	-	-	(1.005)	-
GEE	-	-	0.270	-	-	8.067	6.138
	-	-	(0.254)	-	-	(2.171)	(6.700)
BE	5.969	22.701	0.091	-	-	-	-
	(0.827)	(51.134)	0.187	-	-	-	-
EE	-	-	1.013	-	-	7.788	-
	-	-	(0.087)	-	-	(1.497)	-
N	-	-	-	2.621	1.009	-	-
	-	-	-	(0.101)	(0.071)	-	-

Table 5: The statistics $\hat{\ell}_{max}$, AIC, CAIC, BIC, HQIC, A^*, W^* and K-S for the first data set.

Distribution	$\hat{\ell}_{max}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P-value (K-S)
W-E{LL}	57.168	118.335	119.041	120.327	118.724	0.451	0.079	0.175	0.514
Ga-E{LL}	58.399	122.799	124.299	125.786	123.382	0.143	0.022	0.106	0.961
GEE	62.345	130.690	132.190	133.677	131.273	0.883	0.152	0.233	0.193
BE	62.296	130.592	132.092	133.580	131.176	0.882	0.152	0.230	0.207
EE	62.384	128.769	129.475	130.761	129.158	0.896	0.154	0.233	0.194

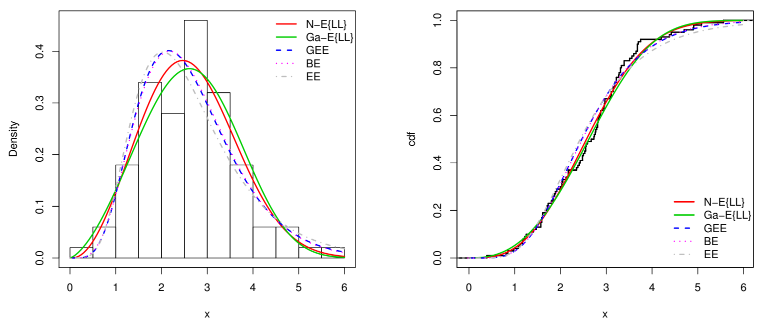
Table 6: The statistics $\hat{\ell}_{max}$, AIC, CAIC, BIC, HQIC, A^*, W^* and K-S for the second data set.

Distribution	$\hat{\ell}_{max}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P-value (K-S)
N-E{L}	141.360	288.719	288.969	296.535	291.882	0.409	0.067	0.061	0.853
Ga-E{LL}	142.239	290.480	290.730	298.295	293.643	0.497	0.066	0.068	0.744
GEE	143.741	293.482	293.732	301.297	296.645	0.833	0.162	0.096	0.311
BE	143.257	292.514	292.764	300.329	295.677	0.761	0.149	0.094	0.345
EE	146.182	296.365	296.488	301.575	298.473	1.186	0.227	0.108	0.197



(a) (b)

Figure 3: Plots of the estimated pdfs and cdfs for the W-E{LL}, Ga-E{LL}, GEE, BE and EE models.



(a) (b)

Figure 4: Plots of the estimated pdfs and cdfs for the N-E{L}, Ga-E{LL}, GEE, BE and EE models.

The results in Tables 5 and 6 show that the W-E{LL} and the Ga-E{LL} models provide the best fit for the failure time data while the N-E{L} and the Ga-E{LL} models provide the best fit for the breaking stress of carbon fibres data. The histograms of both data sets and the estimated pdfs and cdfs for the fitted models are displayed in Figures 3 and 4. The figures support the results from Tables 5 and 6.

7. Concluding remarks

Recently, there has been a great interest among statisticians and applied researchers in constructing flexible distributions in order to facilitate better modeling for complex data sets. Consequently, a significant progress has been made toward the generalization of some well-known lifetime models. In this context, we define and study a new class of generalized exponential family, the *T-exponential*$\{Y\}$ family. Some submodels of *T-exponential*$\{Y\}$ family are studied in some detail. These models show the great flexibility of *T-exponential*$\{Y\}$ family in terms of the shapes of the density and hazard rate functions.

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