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On the Average of the Eccentricities of a Graph

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Abstract. Let $G = (V, E)$ be a simple connected graph of order n with m edges. Also let $e_G(v_i)$ be the eccentricity of a vertex v_i in G. We can assume that $e_G(v_1) \geq e_G(v_2) \geq \cdots \geq e_G(v_{n-1}) \geq e_G(v_n)$. The average eccentricity of a graph G is the mean value of eccentricities of vertices of G,

$$avec(G) = \frac{1}{n}\sum_{i=1}^{n} e_G(v_i)\,.$$

Let $\gamma = \gamma_G$ be the largest positive integer such that

$$e_G(v_{\gamma_G}) \geq avec(G).$$

In this paper, we study the value of γ_G of a graph G. For any tree T of order n, we prove that $2 \leq \gamma_T \leq n-1$ and we characterize the extremal graphs. Moreover, we prove that for any graph G of order n, $2 \leq \gamma_G \leq n$ and we characterize the extremal graphs. Finally some Nordhaus-Gaddum type results are obtained on γ_G of general graphs G.

1. Introduction

We consider finite, simple, undirected, and connected graphs $G = (V(G), E(G))$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $v_i \in V(G)$ is $d_G(v_i)$, i.e., the cardinality of the set of its neighbors, for $i = 1, 2, \ldots, n$. The maximum degree of a graph G is denoted by $\Delta(G)$ and the minimum degree of a graph G is written as $\delta(G)$.

The set of vertices adjacent to $v_i \in V(G)$, denoted by $N_G(v_i)$, refers to the neighborhood of v_i. The *distance* between two vertices $v_i, v_j \in V(G)$, denoted by $d_G(v_i, v_j)$, is defined as the length of a shortest path between v_i and v_j in G. The *eccentricity* $e_G(v_i)$ of a vertex v_i in $V(G)$ is defined to be $e_G(v_i) = \max\left\{d_G(v_i, v_j) | v_j \in V(G)\right\}$. The *radius* of a graph G is denoted by $r(G)$ and defined by $r = r(G) = \min\{e_G(v_i) | v_i \in V(G)\}$. Also, the

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diameter of G, denoted by $d(G)$, is the maximum distance between vertices of a graph G and hence $d = d(G) = \max\{e_G(v_i)|v_i \in V(G)\}$. The center $C(G)$ and the periphery $P(G)$ consist of the vertices of minimum and maximum eccentricity, respectively. Vertices within $C(G)$ and $P(G)$ are called *central* and *peripheral*, respectively. A set $S \subseteq V(G)$ in a graph G is *dominating* if every vertex from $V(G) \setminus S$ has a neighbor in S. A dominating set S in a graph G with $|S| = k$ is called a *k-dominating set* of G. For any graph G, we denote by $\overline{G}$ the *complement* of G.

The average eccentricity of a graph G is the mean value of eccentricities of vertices of G,

$$avec(G) = \frac{1}{n}\sum_{i=1}^{n} e_G(v_i)\,. \tag{1}$$

From the above definition, we have $r(G) \leq avec(G) \leq d(G)$. If $avec(G)$ is equal to $r(G)$ or $d(G)$, then the graph G is called *self-centered*. *Almost self-centered graphs* (*ASC*) were recently introduced in [4] as the graphs with exactly two non-central vertices. Moreover, we say a graph G is *almost-peripheral* ([5]), *AP* for short, if all but one of its vertices lie in the periphery, that is, if $|P(G)| = |V(G)| - 1$ holds. Moreover, very recently *weak almost-peripheral* (WAP for short) graph G is introduced in [10] with $|P(G)| = |V(G)| - 2$. For some recent results on the distance of graphs and related topics can be seen in [3, 6]. The *eccentricity sequence* of a graph G is just a set $\mathcal{E}(G) = \{e_G(v_i) : v_i \in V(G)\}$ of eccentricities of its vertices with their multiplicity listed in a non-increasing order, that is,

$$e_G(v_1) \geq e_G(v_2) \geq \cdots \geq e_G(v_{n-1}) \geq e_G(v_n). \tag{2}$$

If $e_G(v_i)$ appears $l_i \geq 1$ times in $\mathcal{E}(G)$, we write $e_G(v_i)^{(l_i)}$ in it for short. The *disjoint union* of (vertex-disjoint) graphs G_1 and G_2 will be denoted with $G_1 \cup G_2$, while the *join* of G_1 and G_2 will be denoted by $G_1 \oplus G_2$, which is obtained from $G_1 \cup G_2$ by adding an edge between every vertex of G_1 and every vertex of G_2.

Now, for a graph G, we define γ as follows: Let $\gamma = \gamma_G$ be the largest positive integer such that

$$e_G(v_\gamma) \geq avec(G). \tag{3}$$

From the above, we conclude that $1 \leq \gamma \leq n$. A tree containing exactly two non-pendant vertices is called a double-star. A double-star of order n with degree sequence $(p+1, q+1, \underbrace{1, \ldots, 1}_{n-2})$ is denoted by $DS(p, q)\,(p \geq q,\ p + q = n - 2)$. As usual, the path of order n is denoted by P_n, and the star of order n by $K_{1,n-1}$.

The paper is organized as follows. In Section 2, some useful lemmas are listed. In Section 3, we give a lower and an upper bound on γ_G for any tree. In Section 4, we present a lower and upper bound on γ_G for general graphs G and we characterize the graphs G of order n with $\gamma_G = n - 1$ or $n - 2$. In Section 5, some upper bounds with the extremal graphs are determined on $\gamma_G + \gamma_{\overline{G}}$ for any graph G.

2. Some lemmas

In this section, we shall give some results that will be needed in the next sections. Firstly we denote by $\overline{d}$ the diameter of $\overline{G}$ for a graph G.

Lemma 2.1. [11] *Let G be a connected graph whose complement is connected.*
(*i*) *If $d > 3$, then $\overline{d} = 2$.*
(*ii*) *If $d = 3$, then $\overline{G}$ has a spanning subgraph which is a double star.*

We now have the following result:

Lemma 2.2. [2] *Let G be a connected graph of order n. Then* $e_G(v_i) - e_G(v_{i+1}) \le 1$ *for any* i, $i = 1, 2, \ldots, n-1$.

Lemma 2.3. [9] *Let G be a connected graph with diameter d and radius r. For any integer k with* $r < k \le d$, *there exist at least two vertices in G with eccentricity k.*

From Lemma 2.3, the following corollary can be easily obtained.

Corollary 2.4. *Let G be a connected non-self-centered graph with radius r. Then there are at least two vertices in G with eccentricity* $r + 1$.

3. Distribution of eccentricities of trees

If T is a tree of order 3, then $T \cong P_3$ with $\gamma = 2 = n - 1$. So in the following theorem, we assume that $n > 3$. Let T^* be a tree of order n with a vertex $v \in V(T)$ such that $T^* - v = 2\,K_2 \cup (n-5)\,K_1$.

Theorem 3.1. *Let T be a tree of order* $n > 3$. *Then* $2 \le \gamma \le n - 1$. *Moreover, the left equality holds if and only if* $T \cong P_4$ *or* $T \cong T^*$, *and the right equality holds if and only if* $T \cong K_{1,\,n-1}$.

Proof. Let d be the diameter of tree T. Since $n > 3$, we have $d \ge 2$. Let $P_{d+1} : v_{i_1} v_{i_2} \ldots v_{i_d} v_{i_{d+1}}$ be a diametral path in T. Then we have $e_T(v_{i_1}) = e_T(v_{i_{d+1}}) = d$. By (2), we have $e_T(v_1) = e_T(v_2) = d \ge avec(T)$ and hence $\gamma \ge 2$. Since $d(T) \ge 2$, then there exist two vertices v_i and v_j in T such that $e_T(v_i) = r < d = e_T(v_j)$ where r is the radius of T. For any vertex $v_k \in V(T)$, $e_T(v_k) \ge r$, $k \ne i, j$. Therefore $e_T(v_i) = r < avec(T)$ and hence $\gamma \le n - 1$. The first part of the proof is done.

Suppose that $\gamma = 2$. Therefore $e_T(v_1) = e_T(v_2) \ge avec(T) > e_T(v_3)$. Then we have

$$e_T(v_1) = e_T(v_2) \ge \frac{1}{n} \sum_{i=1}^{n} e_T(v_i) > e_T(v_3), \text{that is,}$$

$$(e_T(v_1) - e_T(v_3)) + (e_T(v_2) - e_T(v_3)) > \sum_{i=3}^{n} (e_T(v_3) - e_T(v_i)). \tag{4}$$

First we assume that $e_T(v_3) = e_T(v_n)$. Then we have $e_T(v_1) = e_T(v_2) > e_T(v_3) = e_T(v_4) = \cdots = e_T(v_{n-1}) = e_T(v_n)$. We have $d \ge 2$. For $d = 2$, $T \cong K_{1,n-1}$, a contradiction as $e_T(v_{n-1}) = 2 > 1 = e_T(v_n)$ with $n > 3$. For $d = 3$, $T \cong DS(p, q)\,(p \ge q,\ p + q = n - 2)$ and hence the above inequality holds for P_4 with $e_T(v_1) = e_T(v_2) = 3 > 2 = e_T(v_3) = e_T(v_4)$. Otherwise, $d \ge 4$. There are at least three distinct eccentricities in T and we get a contradiction.

Next we assume that $e_T(v_3) \ne e_T(v_n)$. If $e_T(v_3) > e_T(v_{n-2})$, then by Lemma 2.2,

$$e_T(v_3) > \frac{1}{n} \sum_{i=1}^{n} e_T(v_i) = avec(T) > e_T(v_3), \text{ a contradiction.}$$

Otherwise, $e_T(v_3) = e_T(v_4) = \cdots = e_T(v_{n-2})$. Again, by Lemma 2.2, we have $\big(e_T(v_{n-1}), e_T(v_n)\big)$ is just one of the following triples: $\big(e_T(v_3),\, e_T(v_3) - 1\big), \big(e_T(v_3) - 1,\, e_T(v_3) - 1\big), \big(e_T(v_3) - 1,\, e_T(v_3) - 2\big)$ as $e_T(v_3) \ne e_T(v_n)$. When $\big(e_T(v_{n-1}),\, e_T(v_n)\big) = \big(e_T(v_3) - 1,\, e_T(v_3) - 1\big)$, one can easily see that $avec(T) = e_T(v_3)$ and hence $\gamma > 2$, a contradiction. Moreover, the subcase $\big(e_T(v_{n-1}),\, e_T(v_n)\big) = \big(e_T(v_3) - 1,\, e_T(v_3) - 2\big)$ cannot occur from Corollary 2.4. The remaining case is $\big(e_T(v_{n-1}),\, e_T(v_n)\big) = \big(e_T(v_3),\, e_T(v_3) - 1\big)$. In this case we have $\mathcal{E}(T) = \Big\{(e_T(v_3) + 1)^{(2)},\, e_T(v_3)^{(n-3)},\, (e_T(v_3) - 1)^{(1)}\Big\}$. If $e_T(v_n) = 1$, then $\Delta(T) = n - 1$ and we get a contradiction as $e_T(v_1) = 3$. Otherwise, $e_T(v_n) \ge 2$, that is, $e_T(v_3) \ge 3$. When $e_T(v_3) = 3$, $\mathcal{E}(T) = \{4^{(2)}, 3^{(n-3)}, 2^{(1)}\}$. Hence $G \cong T^*$. When

$e_T(v_3) = 4$, we have $d = 5$ and $n \geq 6$. In this case we have $e_T(v_{n-1}) = 3 \neq e_T(v_3)$, a contradiction. When $e_T(v_3) \geq 5$, we have $d = e_T(v_3) + 1 \geq 6$ and hence we have at least four distinct eccentricities in T, a contradiction.

Suppose that $\gamma = n - 1$. Then we have $e_T(v_1) \geq \cdots \geq e_T(v_{n-1}) \geq avec(T) > r = e_T(v_n)$. Therefore T has one center v_n and hence d is even. If $d = 2$, then $T \cong K_{1,n-1}$. Otherwise, $d \geq 4$. Then

$$avec(T) = \frac{1}{n}\sum_{i=1}^{n} e_T(v_i) > r + 1 = e_T(v_{n-2}) = e_T(v_{n-1}).$$

Thus we have $\gamma \leq n - 3$, a contradiction.

Conversely, one can easily see that $\gamma = 2$ holds for P_4 or for T^*, and $\gamma = n - 1$ holds for $K_{1,n-1}$. □

Theorem 3.2. *Let T be a tree of order $n > 3$. Then $\gamma = n - 2$ if and only if $T \cong DS(p, q)$ $(p \geq q,\ p + q = n - 2)$.*

Proof. Let d be the diameter of tree T. For any tree T of order $n > 3$, $d \geq 2$. For $d = 2$, $T \cong K_{1,n-1}$ with $\gamma = n - 1$. For $d = 3$, $T \cong DS(p, q)$ $(p \geq q,\ p + q = n - 2)$. Thus we have

$$e_T(v_1) = e_T(v_2) = \cdots = e_T(v_{n-2}) \geq avec(T) = \frac{1}{n}\sum_{i=1}^{n} e_T(v_i) > e_T(v_{n-1}) = e_T(v_n)$$

and hence $\gamma = n - 2$. Otherwise, $d \geq 4$. When d is even, that is, T has one central vertex. Then we have $e_T(v_n) = r$ and $e_T(v_{n-1}) = e_T(v_{n-2}) = r + 1 < avec(T)$, and hence $\gamma \leq n - 3$. When d is odd, that is, T has two central vertices. Then we have $e_T(v_n) = e_T(v_{n-1}) = r$ and $e_T(v_{n-2}) = e_T(v_{n-3}) = r + 1 < avec(T)$, and hence $\gamma \leq n - 4$. This completes the proof. □

4. Distribution of eccentricities of general graphs

Let Γ_1 be the class of graphs $H_1 = (V, E)$ such that H_1 is a graph of order n with eccentricity sequence $\{4^{(2)}, 3^{(n-3)}, 2\}$. Denote by Γ_r be the class of graphs $H_r = (V, E)$ such that H_r is a graph of order n with eccentricity sequence $\{(r+2)^{(2)}, (r+1)^{(n-4)}, r^{(2)}\}$, where $r \geq 2$ is an integer. Denote by C_4' the graph obtained by attaching two pendant edges to the non-adjacent vertices in C_4. For $r = 2$, $C_4' \in \Gamma_2$ and $r = 3$, $P_6 \in \Gamma_3$. For $n = 2$ or 3, there is a unique connected graph P_n, for which the eccentricity sequence is $\{1^{(2)}\}$ or $\{2^{(2)}, 1^{(1)}\}$ with $\gamma_{P_n} = 2$. So in the following we always assume that $n > 3$.

Theorem 4.1. *Let G be a graph of order $n > 3$. Then $2 \leq \gamma_G \leq n$. Moreover, the left equality holds if and only if G is almost-self-centered or $G \in \Gamma_1$, and the right equality holds if and only if G is self-centered.*

Proof. For $d = 1$, we have $G \cong K_n$. Then $e_G(v_1) = e_G(v_2) = \cdots = e_G(v_{n-1}) = e_G(v_n) = 1$ and hence $\gamma = n$. Otherwise, $d \geq 2$. Let $P_{d+1} : v_{i_1} v_{i_2} \ldots v_{i_d} v_{i_{d+1}}$ be a diametral path in G. Then we have $e_G(v_{i_1}) = e_G(v_{i_{d+1}}) = d$. By (2), we have $e_G(v_1) = e_G(v_2) \geq avec(G)$ and hence $2 \leq \gamma_G \leq n$. The first part of the proof is done.

Suppose that $\gamma = 2$. Therefore $e_G(v_1) = e_G(v_2) \geq avec(G) > e_G(v_3)$, that is,

$$(e_G(v_1) - e_G(v_3)) + (e_G(v_2) - e_G(v_3)) > \sum_{i=3}^{n} (e_G(v_3) - e_G(v_i)). \tag{5}$$

First we assume that $e_G(v_3) = e_G(v_n)$. Then we have $e_G(v_1) = e_G(v_2) > e_G(v_3) = e_G(v_4) = \cdots = e_G(v_{n-1}) = e_G(v_n)$. Therefore G is almost-self-centered.

Next we assume that $e_G(v_3) \neq e_G(v_n)$. If $e_G(v_3) > e_G(v_{n-2})$, then

$$e_G(v_3) < avec(G) = \frac{1}{n}\sum_{i=1}^{n} e_G(v_i) < e_G(v_3), \text{ a contradiction.}$$

Otherwise, $e_G(v_3) = e_G(v_4) = \cdots = e_G(v_{n-2})$. By Lemma 2.2, we have

$$\Big(e_G(v_{n-1}),\, e_G(v_n)\Big) = \Big(e_G(v_3),\, e_G(v_3) - 1\Big),$$

$$\text{or}\quad \Big(e_G(v_{n-1}),\, e_G(v_n)\Big) = \Big(e_G(v_3) - 1,\, e_G(v_3) - 1\Big),\ \text{or}\ \Big(e_G(v_{n-1}),\, e_G(v_n)\Big) = \Big(e_G(v_3) - 1,\, e_G(v_3) - 2\Big).$$

When $\Big(e_G(v_{n-1}), e_G(v_n)\Big) = \Big(e_G(v_3) - 1, e_G(v_3) - 1\Big)$ or $\Big(e_G(v_3) - 1, e_G(v_3) - 2\Big)$, we have $avec(G) \leq e_G(v_3)$ with $\gamma > 2$, a contradiction. It follows that $\Big(e_G(v_{n-1}), e_G(v_n)\Big) = \Big(e_G(v_3), e_G(v_3) - 1\Big)$. Thus $\mathcal{E}(G) = \{(e_G(v_3) + 1)^{(2)}, e_G(v_3)^{(n-3)}, (e_G(v_3) - 1)^{(1)}\}$. If $e_G(v_n) = 1$, then $\Delta(G) = n - 1$ and we get a contradiction as $e_G(v_1) = 3$. Otherwise, $e_G(v_n) \geq 2$, that is, $e_G(v_3) \geq 3$.

Case (i): $d = 4$. We have three distinct eccentricities $\{4, 3, 2\}$ in G. Since $e_G(v_1) = e_G(v_2) = 4 > 3 = e_G(v_3) = \cdots = e_G(v_{n-1}) > 2 = e_G(v_n)$, we have a diametral path $P_5 : v_{i_1} v_{i_2} v_{i_3} v_{i_4} v_{i_5}$ in G and $e_G(v_{i_1}) = e_G(v_{i_5}) = 4$, $e_G(v_{i_2}) = e_G(v_{i_4}) = 3$, $e_G(v_{i_3}) = 2$. Then all other vertices have same eccentricity 3. Then $G \in \Gamma_1$.

Case (ii): $d \geq 5$. Three distinct eccentricities are $\{r + 2, r + 1, r\}$ in G with $r \geq 3$. If $d \geq 6$, then there are at least four distinct eccentricities in G, a contradiction. Otherwise, $d = 5$. In this case 3 appears twice in $\mathcal{E}(G)$, contradicting the structure of $\mathcal{E}(G)$ shown above.

Suppose that $\gamma = n$. If $e_G(v_1) = e_G(v_n)$, then $e_G(v_i) = avec(G)$ for $i = 1, 2, \ldots, n$. Therefore G is self-centered. Otherwise, $e_G(v_1) \neq e_G(v_n)$. Thus we have $e_G(v_n) < avec(G)$ and hence $\gamma < n$, a contradiction.

Conversely, one can see easily that the left equality holds for almost-self-centered graph or for graphs in Γ_1, and the right equality holds for self-centered graph. □

Remark 4.2. *If G is a self-centered graph, then $\overline{G}$ is not necessarily a self-centered graph. For $n \geq 5$, $\overline{P}_n$ is self-centered graph as $e_{\overline{P}_n}(v_i) = 2$, but P_n is not self-centered.*

Theorem 4.3. *Let G be a graph of order $n > 3$. Then $\gamma = n - 1$ if and only if G is almost-peripheral.*

Proof. Since $\gamma = n - 1$, we have

$$e_G(v_{n-1}) \geq \frac{1}{n} \sum_{i=1}^{n} e_G(v_i) > e_G(v_n). \tag{6}$$

By Lemma 2.2, we have $e_G(v_{n-1}) = e_G(v_n) + 1$. By (2), we have $e_G(v_1) = e_G(v_2)$. If $e_G(v_1) = e_G(v_{n-1}) + 1$, then $avec(G) > e_G(v_{n-1})$, a contradiction as $\gamma = n - 1$. Otherwise, $e_G(v_1) = e_G(v_2) = \cdots = e_G(v_{n-1}) = e_G(v_n) + 1$. So G is almost-peripheral.

Clearly, we have $\gamma = n - 1$ if G is almost-peripheral. □

Theorem 4.4. *Let G be a graph of order $n > 3$. Then $\gamma_G = n - 2$ if and only if G is weak almost-peripheral or $G \in \Gamma_r$ with $r \in \{2, 3\}$.*

Proof. Since $\gamma_G = n - 2$, we have

$$e_G(v_{n-2}) \geq \frac{1}{n} \sum_{i=1}^{n} e_G(v_i) > e_G(v_{n-1}). \tag{7}$$

By Lemma 2.2, we have $e_G(v_{n-2}) = e_G(v_{n-1}) + 1$. By (2), we have $e_G(v_1) = e_G(v_2)$. Since $\gamma = n - 2$, we claim that $e_G(v_1) = e_G(v_{n-2}) + 1$ or $e_G(v_1) = e_G(v_{n-2})$. Otherwise, we have $e_G(v_1) \geq e_G(v_{n-2}) + 2$. Assume that $e_G(v_{n-2}) = a$. Then, by Lemma 2.3, we have $e_G(v_n) = e_G(v_{n-1}) = a - 1$, $e_G(v_{n-2}) = e_G(v_{n-3}) = a$ and $e_G(v_1) = e_G(v_2) \geq a + 2$. Therefore, $n \geq 8$ and $avec(G) = \frac{1}{n} \sum_{i=1}^{n} e_G(v_i) \geq a + \frac{1}{2}$. Thus we have $\gamma_G = n - 5$ as a contradiction.

Case (i): $e_G(v_1) = e_G(v_{n-2})$. If $e_G(v_n) = e_G(v_{n-1})$, then $e_G(v_1) = e_G(v_2) = \cdots = e_G(v_{n-2}) = e_G(v_{n-1}) + 1 = e_G(v_n) + 1$ and hence G is weak almost-peripheral. Otherwise, $e_G(v_1) = e_G(v_2) = \cdots = e_G(v_{n-2}) = e_G(v_{n-1}) + 1 = e_G(v_n) + 2$.

In this subcase, we have $e_G(v_n) = r$ and $|\mathcal{E}(G)| = 3$. Now there is only one vertex v_{n-1} in G with $e_G(v_{n-1}) = r+1$. This is a contradiction from Corollary 2.4.

Case (ii): $e_G(v_1) = e_G(v_{n-2})+1$. In this case we have two possibilities: (a) $e_G(v_1)-1 = e_G(v_2)-1 = e_G(v_3) = \cdots = e_G(v_{n-2}) = e_G(v_{n-1})+1 = e_G(v_n)+1$, (b) $e_G(v_1)-1 = e_G(v_2)-1 = e_G(v_3)-1 = e_G(v_4) = \cdots = e_G(v_{n-2}) = e_G(v_{n-1})+1 = e_G(v_n)+2$. By Corollary 2.4, the subcase (b) cannot occur. Now we characterize the graphs satisfying the subcase (a). Assume that $e_G(v_n) = e_G(v_{n-1}) = r$. Then $e_G(v_1) = e_G(v_2) = r+2$, $e_G(v_3) = \cdots = e_G(v_{n-2}) = r+1$. Note that $r \geq 2$. By the definition of Γ_r, we have $G \in \Gamma_r$.

Clearly, it can be easily checked that $\gamma = n-2$ if G is weak almost-peripheral or $G \in \Gamma_r$ with $r \in \{2, 3\}$. □

In the following theorem we present the existence of graph G with $\gamma_G = k$ for any positive integer k.

Theorem 4.5. *Let $n > 3$ and k be an integer with $2 \leq k \leq n$. Then there exists a graph G with $\gamma_G = k$.*

Proof. From Theorems 4.1 and 4.3, it suffices to consider the case when $k \in [3, n-2]$ with $n > 3$.

For any $k \in [3, n-2]$, let $G = K_{n-k} \oplus \overline{K_k}$. Then $\mathcal{E}(G) = \{1^{(n-k)}, 2^{(k)}\}$. By definition, we have $\gamma_G = k$, finishing the proof of this theorem. □

5. Nordhaus-Gaddum type results

For a graph G, the chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of G in such a way that no two adjacent vertices are assigned the same color. In 1956, Nordhaus and Gaddum [8] gave the lower and the upper bounds involving the chromatic number $\chi(G)$ of a graph G and its complement $\overline{G}$ as follows: $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$. A graph G is *strong self-centered* if both G and its complement $\overline{G}$ are self-centered. For example, the cycle C_n is strong self-centered.

Motivated by the above result, we now obtain analogous conclusions for $\gamma_G + \gamma_{\overline{G}}$.

Theorem 5.1. *Let G be a connected graph of order n with connected complement $\overline{G}$. If $d \geq 4$, then*

$$\gamma_G + \gamma_{\overline{G}} \leq 2n \tag{8}$$

with the equality holding if and only if G is a strong self-centered graph.

Proof. By Lemma 2.1 (i), we have $\overline{d} = 2$. Then $\gamma_{\overline{G}} = n$. If not, we have $\gamma_{\overline{G}} < n$. Then $\overline{G}$ has at least one vertex with degree $n-1$, which implies that G contains at least one isolated vertex. This is a contradiction to the fact that G is connected. By Theorem 4.1, $\gamma_G \leq n$. Hence $\gamma_G + \gamma_{\overline{G}} \leq 2n$.

By Theorem 4.1, again, we deduce that $\gamma_G + \gamma_{\overline{G}} = 2n$ if and only if G is a strong self-centered graph. □

Lemma 5.2. *Let G be a graph with exactly two eccentricities 2, 3. If $v_i \in V(G)$ with $e_G(v_i) = 3$, then $e_{\overline{G}}(v_i) = 2$.*

Proof. The set $V(G) \setminus v_i$ can be partitioned into: $V(G) \setminus v_i = N_G(v_i) \bigcup Ecc_2(v_i) \bigcup Ecc_3(v_i)$ where $Ecc_j(v_i)$ is the set of vertices in G with the distance j to v_i with $j \in \{2, 3\}$. And $N_{\overline{G}}(v_i) = Ecc_2(v_i) \bigcup Ecc_3(v_i)$. Thus we have $d_{\overline{G}}(v_i, v_k) = 2$ for any vertex $v_k \in N_G(v_i)$, since v_k is adjacent to each vertex in $Ecc_3(v_i)$ in $\overline{G}$. So this claim holds immediately. □

Theorem 5.3. *Let G be a connected graph of order n with connected complement $\overline{G}$. If $d = 3$, then*

$$\gamma_G + \gamma_{\overline{G}} \leq \begin{cases} 2n & \text{if } \overline{d} = 2, \\ \\ n & \text{if } \overline{d} = 3. \end{cases} \tag{9}$$

The first equality holds if and only if G is a strong self-centered graph. The second equality holds if and only if, for any central vertex in G, there is another central vertex as its neighbor such that they form a 2-dominating set of G.

Proof. By Lemma 2.1 (*ii*), we have $2 \le \overline{d} \le 3$. If $\overline{d} = 2$, from a similar reasoning as that in the proof of Theorem 5.1, $\overline{G}$ must be a self-centered graph. Clearly, $\gamma_{\overline{G}} = n$. Then, in view of Theorem 4.1, the first inequality holds. Moreover, the equality holds if and only if G is a strong self-centered graph.

For any graph with $d = \overline{d} = 3$, let k be the number of vertices in G of eccentricity 3. Then the number of vertices of eccentricity 2 in G is exactly $n - k$ as both G and $\overline{G}$ are connected. Moreover, by Lemma 5.2, the number of vertices of eccentricity 2 in $\overline{G}$ are at least k. Then the total number of vertices of eccentricity 2 in G and $\overline{G}$ is at least n. Hence $\gamma_G + \gamma_{\overline{G}} \le n$ as there are only two types of eccentricities in G and $\overline{G}$.

Now we determine the graphs for which the second equality holds. Let G be a graph of order n with $d = \overline{d} = 3$ and $\gamma_G + \gamma_{\overline{G}} = n$. For $t \in \{2, 3\}$ we denote by n_t and $\overline{n}_t$ the numbers of vertices with eccentricity t in G and $\overline{G}$, respectively. By Lemma 5.2, considering that $\gamma_G + \gamma_{\overline{G}} = n$, we have $\overline{n}_2 = n_3$ and $\overline{n}_3 = n_2$. Thus it suffices to prove the following claim.

Claim 1. Any vertex in G with eccentricity 2 has eccentricity 3 in $\overline{G}$.

If, for any central vertex v_i in G, there is another central vertex v_j adjacent to v_i such that $\{v_i, v_j\}$ forms a 2-dominating set of G, then $d_{\overline{G}}(v_i, v_j) = 3$. Otherwise, considering that $v_iv_j \in E(G)$, we have $d_{\overline{G}}(v_i, v_j) = 2$, that is, there exists a vertex $v_k \in V(G)$ with $v_iv_k, v_kv_j \in E(\overline{G})$. Now we have $v_k \in V(G) \setminus (N_G(v_i) \bigcup N_G(v_j))$, contradicting to the fact that $\{v_i, v_j\}$ is a 2-dominating set of G. So $e_{\overline{G}}(v_i) = 3$. By the arbitrary choice of central vertex v_i, Claim 1 holds clearly.

Conversely, now Claim 1 holds for G. Then, for any central vertex in G, there is another central vertex as its neighbor such that they form a 2-dominating set of G. Otherwise, there exists a vertex v_i in G with $e_G(v_i) = 2$ such that $\{v_i, v_j\}$ cannot be a 2-dominating set of G for any central neighbor v_j of v_i. Then there is a vertex $v_k \in V(G)$ with $v_kv_i \notin E(G)$, $v_kv_j \notin E(G)$. Moreover, $v_kv_i, v_kv_j \in E(\overline{G})$. Thus $d_{\overline{G}}(v_i, v_j) = 2$. If there is a neighbor v_m of v_i with $e_G(v_m) = 3$, by Lemma 5.2, we have $e_{\overline{G}}(v_m) = 2$. Therefore $d_{\overline{G}}(v_i, v_m) = 2$. In conclusion, $e_{\overline{G}}(v_i) = 2$, which contradicts to Claim 1. This completes the proof of this theorem. □

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