Filomat 32:4 (2018), 1413–1427
https://doi.org/10.2298/FIL1804413E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Di-Extremities and Totally Bounded Di-Uniformities

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Abstract. In our previous studies, we have defined a counterpart, called a di-extremity, to the classical notion proximity in the complement-free setting of a texture. In this article, we will investigate relationship between totally bounded di-uniformities and di-extremities. We will also characterize fuzzy proximities in the sense of Artico-Hutton as complemented di-extremities on Hutton textures.

1. Introduction

In classical topology the notion of open set is usually taken as primitive with that of closed set being auxiliary. However, since the closed sets are easily obtained as the complements of open sets, they often play an important, sometimes dominating role in topological arguments. A similar situation holds for topologies on lattices where the role of set complement is played by an order reversing involution. It is the case, however, that there may be an order reversing involution available, or that the presence of such an involution is otherwise irrelevant to the topic under consideration. To deal with such cases it is natural to consider a topological structure considering of a prior unrelated families of open sets and of closed sets. This was the approach adapted from the beginning for topological structures called fuzzy structures originally introduced as a point-based representation for fuzzy sets. Then these topological structures were called dichotomous topologies, or ditopologies for short. They consist of a family τ of open sets and a generally unrelated family κ of closed sets. Hence, both the open and the closed sets are regarded as primitive concepts for a ditopology and the open and the closed sets have the same role in the ditopology as a topological structure.

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different. Indeed, a bitopology consists of two separate topological structures (complete with their open and closed sets) whose interrelations we wish to study, whereas a ditopology represents a single topological structure.

Ditopological spaces [2, 3] were introduced by L.M. Brown as a natural extension of the work of the first author on the representation of lattice valued topologies by bitopologies [10]. However, in place of the full lattice of subsets of some base set S, attention is now focused on a suitable subfamily of subsets, called a texturing of S, and within this context bitopologies are replaced by dichotomous topologies, or ditopologies for short. Fuzzy sets [21] can be represented as textures [5] and a texture provides a complement-free

2010 *Mathematics Subject Classification*. Primary 54E05; Secondary 03E20, 06D10, 54A05, 54E15
Keywords. Texture spaces, fuzzy proximity, di-extremity, di-uniformity, totally bounded
Received: 10 May 2017; Revised: 23 November 2017; Accepted: 29 January 2018
Communicated by Ljubiša D.R. Kočinac
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framework for generalizing topology related structures such as uniformities and metrics. For motivation and background on textures the reader is referred to [4, 6–8, 15].

Di-uniform texture spaces and totally bounded di-uniformities were introduced by Özçağ and Brown in [14, 16]. Later, they also gave a point-free representation of direlational uniformities in [17] as well as they characterized Hutton uniformities [11] as di-uniformities on Hutton textures.

Proximity and quasi proximity constitute an important and intensely investigated area in the field of classical and fuzzy topological spaces, because they possess rich topological properties as well as they characterize totally bounded uniform spaces. With this motivation, di-extremial texture spaces were introduced in [19] and [20] as a counterpart to the classical notion of proximity in the complement-free setting of a texture. As it is shown in [19], there is a bijection between the quasi-proximities [13] on a set X and di-extremities on the discrete texture $(X, \mathcal{P}(X))$. Moreover the proximities on X are also characterized in terms of the complemented di-extremities on the discrete texture $(X, \mathcal{P}(X), \pi_X)$. Thus di-extremities are generalizations of classical quasi-proximities and proximities. It is also shown that every di-uniformity and every dimetric induce a compatible di-extremity.

The layout of the paper is as follows. In Section 2, we will recall some definitions about texture spaces, ditopological texture spaces, di-extremities and difunctional uniformities briefly. In Section 3, we characterize fuzzy proximities in the sense of Artico-Hutton [1] as complemented di-extremities on Hutton textures. In Section 4, we will show that every di-extremity has a compatible totally bounded di-uniformity. Thus we show that a ditopology is completely biregular if and only if it has a compatible di-extremity. At the end of this section, we point out that the category of di-extremities with extremial bicontinuous difunctions are isomorphic to a full, reflexive subcategory of difunctional uniform spaces with uniform bicontinuous difunctions.

2. Preliminaries

We recall various concepts and properties from [4, 6–8, 15] under the following subtitle.

Texture and Ditopological Texture Spaces: Let S be a set. A texturing $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point separating, complete, completely distributive lattice with respect to inclusion which contains $S, \emptyset$ and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S, \mathcal{S})$ is then called a texture space or shortly a texture.

In general, a texturing of S need not be closed under set complementation, but it may be that there exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is said to be a complemented texture.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p − sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the *q − sets*

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

Recall that $M \in \mathcal{S}$ is called a molecule if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The sets $P_s, s \in S$ are molecules, and the texture $(S, \mathcal{S})$ is called "simple" if these are the only molecules in $\mathcal{S}$. For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Theorem 2.1. ([6]) *In any texture $(S, \mathcal{S})$, the following statements hold:*

1. $s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.
2. $A^\flat = \{s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
3. *For* $A_j \in \mathcal{S}$, $j \in J$ *we have* $(\bigvee_{j \in J} A_j)^\flat = \bigcup_{j \in J} A_j^\flat$.

4. *A is the smallest element of* $\mathcal{S}$ *containing* $A^\flat$ *for all* $A \in \mathcal{S}$.
5. *For* $A, B \in \mathcal{S}$, *if* $A \not\subseteq B$ *then there exists* $s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$.
6. $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.
7. $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

Let $\mathbb{L}$ be a fuzzy lattice, in other words a Hutton algebra, i.e. a completely distributive, complete lattice with an order reversing involution $'$ and L denote the set of molecules in $\mathbb{L}$ and $\mathcal{L} = \{\varphi(a)|a \in \mathbb{L}\}$ where $\varphi(a) = \{m \in L|m \leq a\}$ for $a \in \mathbb{L}$. Then:

Theorem 2.2. ([5]) *With the above notations,* $(L, \mathcal{L})$ *is a simple texture with complement* $\lambda(\varphi(a)) = \varphi(a')$, $a \in \mathbb{L}$ *and* $\varphi : \mathbb{L} \to \mathcal{L}$ *is a lattice isomorphism which preserves complementation.*

Conversely, every complemented simple texture may be obtained in this way from a suitable fuzzy lattice.

Example 2.3. (1) If $\mathcal{P}(X)$ is the powerset of a set X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{J} = \{[0, r), [0, r] \,|r \in \mathbb{I}\}$ gives the unit interval texture $(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$. The mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0, r] = [0, 1 - r)$, $\iota[0, r) = [0, 1 - r]$ is a complementation on this texture.

(3) The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$, $\lambda((0, r]) = (0, 1 - r]$. For $r \in L$, $P_r = (0, r] = Q_r$. This texture corresponds to fuzzy lattice $(\mathbb{I} = [0, 1],')$ in the sense of Theorem 2.2.

(4) Let $X \neq \emptyset$, W be the set of "fuzzy points" of $\mathbb{I}^X$, i.e. the functions

$$x_m(z) = \begin{cases} m, & z = x \\ 0, & \text{otherwise} \end{cases}$$

for $x \in X$ and $m \in L = (0, 1]$, where as before L is the set of molecules of $\mathbb{I}$. By representing x_m by the pair (x, m), it can be written that $W = X \times L$. Then $(W, \mathcal{W}, \omega)$ is the texture corresponds to fuzzy lattice $\mathbb{I}^X$ in the sense of Theorem 2.2. where $\mathcal{W} = \{\varphi(f) \mid f \in \mathbb{I}^X\}$, $\varphi(f) = \{(x, m) \in W \mid x_m \leq f\} = \{(x, m) \in W \mid m \leq f(x)\}$ and $\omega(\varphi(f)) = \varphi(f')$.

(5) $\mathcal{S} = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, S\}$ is a simple texturing of $S = \{a, b, c\}$. Clearly, $P_a = \{a, b\}$, $P_b = \{b\}$, $P_c = \{b, c\}$. It is not possible to define a complementation on $(S, \mathcal{S})$.

(6) If $(S, \mathcal{S}), (V, \mathcal{V})$ are textures, the product texturing $\mathcal{S} \otimes \mathcal{V}$ of $S \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{V}$, and $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is called the product of $(S, \mathcal{S})$ and $(V, \mathcal{V})$. For $s \in S, v \in V$, $P_{(s,v)} = P_s \times P_v$ and $Q_{(s,v)} = (Q_s \times V) \cup (S \times Q_v)$.

A dichotomous topology, or shortly a ditopology, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ satisfies

(T_1) $S, \emptyset \in \tau$,

(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,

(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$,

(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,

(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no priori relation between the open and closed sets.

Let $(S, \mathcal{S}, \sigma)$ be a complemented texture and (τ, κ) a ditopology on this texture. Then if τ and κ are related by $\kappa = \sigma[\tau]$, we say that (τ, κ) is a *complemented ditoplogy* on $(S, \mathcal{S}, \sigma)$.

Di-Extreme Ditopological Texture Spaces: A di-extremity [19] and [20] is a counterpart to the classical notion of proximity in the complement-free setting of a texture. Let us recall the definition.

Definition 2.4. Let $(S,\mathcal{S})$ be a texture. $\overline{\delta}$, $\underline{\delta}$ two binary relations on $\mathcal{S}$. Then $\delta = (\overline{\delta},\underline{\delta})$ is called a *di-extremity* on $(S,\mathcal{S})$ if δ satisfies the following conditions:

(E1) $A\overline{\delta}B$ implies $A \neq \emptyset$, $B \neq S$,
(E2) $(A \cup B)\overline{\delta}C$ if and only if $A\overline{\delta}C$ or $B\overline{\delta}C$,
(E3) $A\overline{\delta}(B \cap C)$ if and only if $A\overline{\delta}B$ or $A\overline{\delta}C$,
(E4) If $A\overline{\not\delta}B$, there exist $E \in \mathcal{S}$ such that $A\overline{\not\delta}E$ and $E\overline{\not\delta}B$,
(E5) $A\overline{\not\delta}B$ implies $A \subseteq B$,
(DE) $A\underline{\delta}B \Longleftrightarrow B\overline{\delta}A$,
(CE1) $A\underline{\delta}B$ implies $A \neq S$, $B \neq \emptyset$,
(CE2) $A\underline{\delta}(B \cup C)$ if and only if $A\underline{\delta}B$ or $A\underline{\delta}C$,
(CE3) $(A \cap B)\underline{\delta}C$ if and only if $A\underline{\delta}C$ or $B\underline{\delta}C$,
(CE4) If $A\underline{\not\delta}B$, there exists $E \in \mathcal{S}$ such that $A\underline{\not\delta}E$ and $E\underline{\not\delta}B$,
(CE5) $A\underline{\not\delta}B$ implies $B \subseteq A$.

In this case, it is said that $\overline{\delta}$ is the *extremity*, $\underline{\delta}$ is the *co-extremity* of δ and $(S,\mathcal{S},\delta)$ is known as *di-extremial* texture space.

When giving examples it will clearly suffice to give only $\overline{\delta}$ satisfying the extremity conditions, since (DE) may then be used to define $\underline{\delta'}$ which will automatically satisfy the co-extremity conditions. This is also the case for dimetrics [14] and difunctional uniformities [17]. Only when one removes the symmmetry condition (DE) to produce a quasi di-extremity, it is absolutely necessary to consider both parts.

Let $\delta = (\overline{\delta},\underline{\delta})$ be a di-extremity on a complemented texture $(S,\mathcal{S},\sigma)$. Define $\delta' = \sigma(\delta) = (\overline{\delta'},\underline{\delta'})$ where for all $A, B \in \mathcal{S}$, $A\overline{\delta'}B \Longleftrightarrow \sigma(A)\underline{\delta}\sigma(B)$ and $A\underline{\delta'}B \Longleftrightarrow \sigma(A)\overline{\delta}\sigma(B)$. Then δ' is a di-extremity on $(S,\mathcal{S},\sigma)$. The di-extremity δ' is said to be *complement* of δ. A di-extremity δ is said to be *complemented* if $\delta = \delta'$.

For every $A \in \mathcal{S}$, the interior of A, $int(A) = \bigcap\{Q_s \mid P_s\overline{\delta}A, s \in S\}$ and the closure of A, $Cl(A) = \bigvee\{P_s \mid Q_s\underline{\delta}A, s \in S\}$. Thus every di-extremity δ induces a ditopology $(\tau(\delta),\kappa(\delta))$. In the case, where δ is complemented the induced ditopology $(\tau(\delta),\kappa(\delta))$ is also complemented.

Proposition 2.5. ([19]) *Let $\delta = (\overline{\delta},\underline{\delta})$ be a di-extremity on $(S,\mathcal{S})$. Then:*

1. $A\overline{\delta}B, A \subseteq C, D \subseteq B \Longrightarrow C\overline{\delta}D$.
2. *If there exists $s \in S$ such that $A\overline{\delta}Q_s$ and $P_s\overline{\delta}B$, then $A\overline{\delta}B$.*
3. $A\underline{\delta}B, C \subseteq A, B \subseteq D \Longrightarrow C\underline{\delta}D$.
4. *If there exists $s \in S$ such that $A\underline{\delta}P_s$ and $Q_s\underline{\delta}B$, then $A\underline{\delta}B$.*
5. $\bigcup_{i=1}^{n} A_i\overline{\not\delta}\bigcap_{j=1}^{m} B_j$ *if and only if for all* $i = 1,\ldots,n$, $j = 1,\ldots,m$, $A_i\overline{\not\delta}B_j$.

Proof. We give only the proof of 5., the other proofs can be seen similarly. Suppose that for all $i = 1,\ldots,n$, $j = 1,\ldots,m$ we have $A_i\overline{\not\delta}B_j$. Then by (E2), for all j, $\bigcup_{i=1}^{n} A_i\overline{\not\delta}B_j$. By (E3), $\bigcup_{i=1}^{n} A_i\overline{\not\delta}\bigcap_{j=1}^{m} B_j$. For the converse, suppose that $\bigcup_{i=1}^{n} A_i\overline{\not\delta}\bigcap_{j=1}^{m} B_j$ holds. Set $A = \bigcup_{i=1}^{n} A_i$. Then by (E3), for all j, $A\overline{\not\delta}\bigcap_{j=1}^{m} B_j$. Now similarly, if we use (E2), we get $A_i\overline{\not\delta}B_j$ for all $i = 1,\ldots,n$, $j = 1,\ldots,m$. □

Sometimes proximity concept is also described as a strongly inclusion relation $\ll$ such as in [13],[1]. Although implication is trivial, we believe this alternative approach is worth to be mentioned because it may provide a smooth transition between classical proximities, fuzzy proximities and di-extremities.

Let $(S,\mathcal{S},\delta)$ be a di-extremial texture space. Define $\ll$, $\gg$ as two binary relations on $\mathcal{S}$ such that $A \ll B \Longleftrightarrow A\overline{\not\delta}B$ and $A \gg B \Longleftrightarrow A\underline{\not\delta}B$ for every $A, B \in \mathcal{S}$. Then it is easy to show that the relations $\ll$, $\gg$ verify the following conditions:

(Q1) $S \ll S$,
(Q2) $A \ll B$ implies $A \subseteq B$,
(Q3) $B \ll C$, $A \subseteq B$, $C \subseteq D$ implies $A \ll D$,
(Q4) $A_i \ll B_j$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$ if and only if $\bigcup_{i=1}^{n} A_i \ll \bigcap_{j=1}^{m} B_j$,

(Q5) $A \ll B$ implies there is a C such that $A \ll C \ll B$,
(Q6) $A \ll B \Longleftrightarrow B \gg A$.
Vice versa, given the binary relations $\ll$ and $\gg$ on $\mathcal{S}$ which satisfies the properties (Q1) - (Q6) above, one obtains a di-extremity $\delta = (\overline{\delta}, \underline{\delta})$ putting $A \overline{\not\delta} B \Longleftrightarrow A \ll B$ and $A \underline{\not\delta} B \Longleftrightarrow A \gg B$. In this case, $(\ll, \gg)$ is also called a *di-extremity* on $(S, \mathcal{S})$. We say that B is a $\overline{\delta}$*-neighborhood* of A if and only if $A \ll B$ and C is a $\underline{\delta}$*-neighborhood* of D if and only if $C \gg D$. For the details about di-extremities, the reader is referred to [19] and [20], and the details of proximity and quasi-proximity spaces can be seen from [9, 12, 13, 18, 19].

Di-Uniform Texture Spaces: Di-uniform texture spaces were introduced in [14] and later, a point-free representation of direlational uniformities, called difunctional uniformities were given in [17]. We omit the details and recall some fundamental definitions and results about difunctional uniformities.

Definition 2.6. ([17]) Let $(S, \mathcal{S})$ be a texture.

1. We denote by $\mathcal{F}^S_{\mathcal{RR}}$ (or simply $\mathcal{F}_{\mathcal{RR}}$ when there is no confusion) the family of functions $\varphi : \mathcal{S} \to \mathcal{S}$ satisfying
   (a) $A \subseteq \varphi(A)$, for all $A \in \mathcal{S}$, and
   (b) $\varphi(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} \varphi(A_j)$ for all $A_j \in \mathcal{S}$, $j \in J$.
2. We denote by $\mathcal{F}^S_{\mathcal{RCR}}$ (or simply $\mathcal{F}_{\mathcal{RCR}}$) the family of functions $\psi : \mathcal{S} \to \mathcal{S}$ satisfying
   (a) $\psi(A) \subseteq A$, for all $A \in \mathcal{S}$, and
   (b) $\psi(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \psi(A_j)$ for all $A_j \in \mathcal{S}$, $j \in J$.
3. We will denote by $\mathcal{F}^S_{\mathcal{RDR}} = \mathcal{F}^S_{\mathcal{RR}} \times \mathcal{F}^S_{\mathcal{RCR}}$ (or simply $\mathcal{F}_{\mathcal{RDR}}$).

Definition 2.7. Let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{F}_{\mathcal{RDR}}$. If $\varphi_1 \leq \varphi_2$ and $\psi_2 \leq \psi_1$ then $\mathcal{F}_{\mathcal{RDR}}$ is ordered by $(\varphi_1, \psi_1) \leq (\varphi_2, \psi_2)$.

Definition 2.8. ([17]) For $\varphi \in \mathcal{F}_{\mathcal{RR}}$, the right adjoint of φ is defined by $\varphi^{\leftarrow}(B) = \bigvee\{A \in \mathcal{S} \mid \varphi(A) \subseteq B\}$, for all $B \in \mathcal{S}$. Dually for $\psi \in \mathcal{F}_{\mathcal{RCR}}$, the left adjoint of ψ is defined by $\psi^{\leftarrow}(B) = \bigcap\{A \in \mathcal{S} \mid B \subseteq \psi(A)\}$, for all $B \in \mathcal{S}$.

Definition 2.9. ([17]) Let (f, F): $(S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction and $(\varphi, \psi) \in \mathcal{F}^T_{\mathcal{RDR}}$. With the equalities $(f, F)^{\leftarrow}(\varphi(A)) = F^{\leftarrow}(\varphi(f^{\to} A))$, $A \in \mathcal{S}$ and $(f, F)^{\leftarrow}(\psi(A)) = f^{\leftarrow}(\psi(F^{\to} A))$, $A \in \mathcal{S}$, we will define $(f, F)^{-1}(\varphi, \psi) = ((f, F)^{-1}(\varphi), (f, F)^{-1}(\psi)) \in \mathcal{F}^S_{\mathcal{RDR}}$.

In this article, we will use the alternative characterization of difunctional uniformity which was given in [17].

Definition 2.10. ([17]) Let $(S, \mathcal{S})$ be a texture. The subfamilies $\overline{\mathcal{U}} \subseteq \mathcal{F}_{\mathcal{RR}}$ and $\underline{\mathcal{U}} \subseteq \mathcal{F}_{\mathcal{RCR}}$ satisfy the following conditions:
(UF1) $\varphi \in \overline{\mathcal{U}}, \varphi_1 \in \mathcal{F}_{\mathcal{RR}}$ with $\varphi \leq \varphi_1 \Longrightarrow \varphi_1 \in \overline{\mathcal{U}}$,
(UF2) $\varphi_1, \varphi_2 \in \overline{\mathcal{U}} \Longrightarrow \varphi_1 \wedge \varphi_2 \in \overline{\mathcal{U}}$,
(UF3) $\varphi \in \overline{\mathcal{U}} \Longrightarrow \exists \varphi_1 \in \overline{\mathcal{U}}$ with ${\varphi_1}^2 \leq \varphi$,
(SYM) $\varphi \in \overline{\mathcal{U}} \Longleftrightarrow \varphi^{\leftarrow} \in \underline{\mathcal{U}}$,
(CUF1) $\psi \in \underline{\mathcal{U}}, \psi_1 \in \mathcal{F}_{\mathcal{RCR}}$ with $\psi_1 \leq \psi \Longrightarrow \psi_1 \in \underline{\mathcal{U}}$,
(CUF2) $\psi_1, \psi_2 \in \underline{\mathcal{U}} \Longrightarrow \psi_1 \vee \psi_2 \in \underline{\mathcal{U}}$,
(CUF3) $\psi \in \underline{\mathcal{U}} \Longrightarrow \exists \psi_1 \in \underline{\mathcal{U}}$ with $\psi \leq {\psi_1}^2$,
Then $\mathcal{U} = \overline{\mathcal{U}} \times \underline{\mathcal{U}}$ is called a *difunctional uniformity* on the texture $(S, \mathcal{S})$.

Definition 2.11. ([17]) Let $(S, \mathcal{S}, \mathcal{U})$ and $(T, \mathcal{T}, \mathcal{V})$ be difunctional uniform texture spaces and $(f, F) : (S, \mathcal{S}, \mathcal{U}) \to (T, \mathcal{T}, \mathcal{V})$ be a difunction. Then (f, F) is called $\mathcal{U} - \mathcal{V}$ *uniformly bicontinuous* if the implication $(\varphi, \psi) \in \mathcal{V} \Longrightarrow (f, F)^{-1}(\varphi, \psi) \in \mathcal{U}$ is satisfied.

Corollary 2.12. ([17]) *Let $(S, \mathcal{S}, \mathcal{U})$ and $(T, \mathcal{T}, \mathcal{V})$ be difunctional uniform texture spaces and $(f, F) : (S, \mathcal{S}, \mathcal{U}) \to (T, \mathcal{T}, \mathcal{V})$ be a difunction. Then the following are equivalent.*

1. *(f, F) is $\mathcal{U} - \mathcal{V}$ uniformly bicontinuous.*
2. $\varphi \in \overline{\mathcal{V}} \Longrightarrow (f, F)^{-1}(\varphi) \in \overline{\mathcal{U}}$.
3. $\psi \in \underline{\mathcal{V}} \Longrightarrow (f, F)^{-1}(\psi) \in \underline{\mathcal{U}}$.

3. Di-Extremities and Fuzzy Proximities

In this section we will investigate the relation between fuzzy proximities in the sense of Artico-Hutton [1] and di-extremities on Hutton Textures.

Definition 3.1. ([1]) Let $\mathbb{L}$ be a Hutton algebra, that is, a completely distributive, complete lattice with an order reversing involution $'$. If a binary relation η satisfies the following conditions:
(FP1) $0 \not\eta\, 1$,
(FP2) $a \vee b\eta c \iff a\eta c$ or $a\eta c$,
(FP3) $a\eta b \vee c \iff a\eta b$ or $b\eta c$,
(FP4) $a \not\eta\, b \Longrightarrow$ there exists $e \in \mathbb{L}$ such that $a \not\eta\, e$ and $e' \not\eta\, b$,
(FP5) $a \not\eta\, b \Longrightarrow a \leq b'$
for all $a, b, c \in \mathbb{L}$.

Then it is called a *fuzzy quasi-proximity in the sense of Artico-Hutton*. We will mention it as the fuzzy quasi-proximity or f. quasi-proximity shortly. If η satisfies the condition (*FPS*) "$a\eta b \iff b\eta a$" as well as the above properties (FP1)-(FP5) then it becomes f. proximity. If for all $a \in \mathbb{L}$, the interior of a is defined by $int(a) = \bigvee\{b \mid b \not\eta\, a'\}$, then this interior operator $Int : \mathbb{L} \to \mathbb{L}$ satisfies the interior operator properties and hence induce a f. topology $\tau(\eta)$. That is, each f. quasi-proximity η generates a f. topology.

Let $(\mathbb{L}_1, \eta_1)$, $(\mathbb{L}_2, \eta_2)$ be f. quasi proximities and let $\theta : \mathbb{L}_1 \to \mathbb{L}_2$ be a mapping which preserves arbitrary meets and joins. Then θ is called a *proximity mapping or proximial continuous mapping* if $c\not\eta_2 d$ implies $\theta^{\leftarrow}(c)\not\eta_1\theta^{\leftarrow}(d)$, for all $c, d \in \mathbb{L}_2$.

We should note that f. proximity in [1] is defined as a binary relation on $\mathbb{L}^X$. However most of the results can still be carried if one use simply $\mathbb{L}$ instead of $\mathbb{L}^X$. Since $\mathbb{L}^X$ is also Hutton algebra, it will be a special case for this section.

Every Hutton algebra $\mathbb{L}$ is associated with Hutton texture $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})$ as in [5]. Here $M_{\mathbb{L}}$ is set of molecules of $\mathbb{L}$, $\mathcal{M}_{\mathbb{L}} = \{\hat{a} \mid a \in \mathbb{L}\}$ where $\hat{a} = \{m \in \mathcal{M}_{\mathbb{L}} \mid m \leq a\}$ and $\mu_{\mathbb{L}}(\hat{a}) = \hat{a'}$. The mapping $a \to \hat{a}$ is a Hutton algebra isomorphism between $(\mathbb{L}, ')$ and $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})$. For $\phi : \mathbb{L} \to \mathbb{L}$, the mapping $\hat{\phi} : \mathcal{M}_{\mathbb{L}} \to \mathcal{M}_{\mathbb{L}}$ is defined by $\hat{\phi}(\hat{a}) = \widehat{\phi(a)}$.

Theorem 3.2. ([20]) *Let η be a f. quasi proximity on the Hutton algebra $(\mathbb{L}, ')$ and define $\hat{a}\overline{\delta_\eta}\hat{b} \iff a\eta b'$, $\hat{a}\underline{\delta_\eta}\hat{b} \iff b'\eta a$. Then $\delta_\eta = (\overline{\delta_\eta}, \underline{\delta_\eta})$ is a di-extremity on $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})$ and it is called* the di-extremity corresponding to *η and it is denoted by $\overline{\delta_\eta}$ or $\delta(\eta)$. Conversely, let δ be a di-extremity on Hutton texture $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})$ and define $a\eta b \iff \hat{a}\overline{\delta}\mu_{\mathbb{L}}(\hat{b})$. Then η is a f. proximity on $(\mathbb{L}, ')$. Furthermore in both cases, we have $int(\hat{a}) = \widehat{inta}$.*

Thus, we see that di-extremities on Hutton textures correspond exactly to the f.quasi-proximities on Hutton algebras. Moreover if η is an f. proximity on $(\mathbb{L}, ')$, then δ_η is complemented and conversely, every complemented di-extremity on a Hutton texture can be obtained in this way. We note that the difference between quasi-proximities and proximities in the classical and fuzzy description is a question of symmetry, but this question becomes a matter of complementation in di-extremity case. This is also the case for di-uniformities [16].

For the Hutton algebras $(\mathbb{L}_1, '_1)$, $(\mathbb{L}_2, '_2)$, we know from Proposition 4.1 in [7] that if a mapping $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ preserves arbitrary meets and joins then we have a difunction $(f^\theta, F^\theta) : (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mu_{\mathbb{L}_1}) \to (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mu_{\mathbb{L}_2})$ satisfying $f^{\theta^{\leftarrow}}(\hat{b}) = \theta(b) = F^{\theta^{\leftarrow}}(\hat{b})$ for all $b \in \mathbb{L}_2$. Moreover the difunction (f^θ, F^θ) is complemented if and only if θ preserves involutions. Conversely, if $(f, F) : (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mu_{\mathbb{L}_1}) \to (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mu_{\mathbb{L}_2})$ is any (complemented) difunction then $\theta_{(f,F)} : (\mathbb{L}_2, '_2) \to (\mathbb{L}_1, '_1)$ defined by $\widehat{\theta_{(f,F)}} = f^{\leftarrow}(\hat{b}) = F^{\leftarrow}(\hat{b})$ preserves (involutions) meets and joins. Moreover $\theta = \theta_{(f_\theta, F_\theta)}$ and $(f, F) = (f^\theta_{(f,F)}, F^\theta_{(f,F)})$. Thus, the functor $\mathfrak{T}$ defined by

$$\mathfrak{T}((\mathbb{L}_1, '_1) \xrightarrow{\theta} (\mathbb{L}_2, '_2)) = (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mu_{\mathbb{L}_1}) \xrightarrow{(f^\theta, F^\theta)} (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mu_{\mathbb{L}_2})$$

is an isomorphism between the categories $\mathbf{HutAlg^{op}}$ and $\mathbf{cdfSTex}$ [7].

Definition 3.3. Let $(\mathbb{L}_1, '_1, \eta_1), (\mathbb{L}_2, '_2, \eta_2)$ be f.quasi-proximity spaces and $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ be a function which preserves arbitrary meets and joins. If η satisfy the property; $c \not\eta_2 d \Longrightarrow \theta(c) \not\eta_1 \theta(d)$ for all $c, d \in \mathbb{L}_2$, then θ is called *quasi-proximial continuous* **HutAlg^{op}** *morphism*.

In this study, the category whose objects are fuzzy quasi-proximity spaces defined on Hutton algebras $(\mathbb{L}_1, '_1), (\mathbb{L}_2, '_2)$ and whose morphisms are proximal continuous **HutAlg^{op}**-morphisms will be denoted by **HutAlgQFP^{op}**, and the category whose objects are di-extremities defined on Hutton textures $(M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mu_{\mathbb{L}_1})$, $(M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mu_{\mathbb{L}_2})$ and whose morphisms are complemented extremial bicontinuous difunctions will be denoted by **cdfSEx**.

As pointed out above, there exists one to one correspondence between fuzzy quasi-proximity spaces on Hutton algebras and di-extremities on Hutton textures that means, there exists one to one correspondence between the objects of these two categories. One can see from Lemma 3.12 of [19] and thanks to the fact that $f^{\theta\leftarrow}(\widehat{b}) = \theta(b) = F^{\theta\leftarrow}(\widehat{b})$ for each $b \in \mathbb{L}_2$, there exists one to one correspondence between morphisms of these two categories. As a conclusion, the functor $\mathfrak{T}$ is an isomorphism between these two categories and hence they are isomorphic.

We have a similar result between the category whose objects are fuzzy proximity spaces defined on Hutton algebras and whose morphisms are proximal continuous **HutAlg^{op}**-morphisms will be denoted by **HutAlgFP^{op}** and the category whose objects are complemented di-extremities defined on Hutton textures and whose morphisms are complemented extremial bicontinuous difunctions **cdfScEx**. As a result, (complemented) di-extremities defined on Hutton texture characterize the fuzzy quasi-proximity (fuzzy proximity) defined on Hutton algebras.

Finally, note that if we take $L = \{0, 1\}$, then the definition of fuzzy quasi-proximity relation coincides with the definition of quasi relation in the sense of Efremovic [9]. Therefore, we can carry the results to over classical one when we take $L = \{0, 1\}$. This process can be done directly, so we will left the details for the classical case here.

4. Di-Extremities and Totally Bounded Di-Uniformities

In the fuzzy set theory, there is a one-to-one correspondence between f. proximities and totally bounded uniformities. The category of proximities with proximally continuous functions is isomorphic to a full, reflexive subcategory of the category of Hutton uniformities with uniform continuous functions [1]. It is natural to ask whether a similar result is possible for di-extremities and totally bounded di-uniformities or not. The answer is affirmative as we will show in this section.

The following definition is difunctional uniform space version of the Definition 4.7 in [19].

Definition 4.1. Let $(S, \mathcal{S}, \mathcal{U})$ be a difunctional uniform space. Define $A\overline{\delta}B \Longleftrightarrow \varphi(A) \not\subseteq B$ for all $\varphi \in \overline{\mathcal{U}}$ and $\underline{\delta} = \overline{\delta}^{-1}$. Then $\delta = (\overline{\delta}, \underline{\delta})$ is called di-extremity induced by $\mathcal{U}$ and it is denoted by $\delta_{\mathcal{U}}$ or $\delta(\mathcal{U})$.

The subbases and bases of a difunctional uniformity are mentioned briefly in [14, 17] and the details are omitted because they are analogous to their counterparts of direlational and classical uniformity. We will mention some of these omitted results here since they are used in this section.

Lemma 4.2. *Let $(S, \mathcal{S}, \mathcal{U})$ be a difunctional uniform space, $\mathcal{U}_{\mathcal{B}}$ is a base for $\mathcal{U}$ and $\mathcal{U}_{\mathcal{S}\mathcal{B}}$ is a subbase for $\mathcal{U}$. Then the following statements are equivalent for all $A, B \in \mathcal{S}$:*

(1) $A\overline{\delta}_{\mathcal{U}}B$.

(2) $\varphi(A) \not\subseteq B$ for all $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$.

(3) $\varphi(A) \not\subseteq B$ for all $\varphi \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$.

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (1) Let $\varphi^* \in \mathcal{U}$. Then there exists $\varphi_1, \varphi_2, \ldots, \varphi_n \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$ such that $\wedge_{k=1}^{n} \varphi_k(A) \not\subseteq B$ and so $\varphi^*(A) \not\subseteq B$. □

Totally bounded diuniformities were introduced in [16]. The following definition is a difunctional uniformity version of totally boundedness.

Definition 4.3. Let $(S, \mathcal{S}, \mathcal{U})$ be a difunctional uniform space. $\mathcal{U}$ is said to be *totally bounded* if and only if for each $(\varphi, \psi) \in \mathcal{U}$, there exists $s_1, s_2, \ldots, s_n \in S$ such that $\{(\varphi(P_{s_k}), \psi(Q_{s_k})) \mid k = 1, \ldots n\}$ is a dicover of S.

Lemma 4.4. *Let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{F}_{\mathcal{R}\mathcal{D}\mathcal{R}}$ and $(\varphi_1, \psi_1) \leq (\varphi_2, \psi_2)$. If there exists $s_1, s_2, \ldots, s_n \in S$ such that $\{(\varphi_1(A_k), \psi_1(B_k)) \mid k = 1, \ldots n\}$ is a dicover, then $\{(\varphi_2(A_k), \psi_2(B_k)) \mid k = 1, \ldots n\}$ is also a dicover of S.*

Proof. Let $\{(\varphi_1(A_k), \psi_1(B_k)) \mid k = 1, \ldots n\}$ be a dicover, that is, for every partition I_1, I_2 of $I = \{1, 2, \ldots n\}$, we have $\bigcap_{i \in I_1} \psi_1(B_i) \subseteq \bigvee_{j \in I_2} \varphi_1(A_j)$. Since $\psi_2(B_i) \subseteq \psi_1(B_i)$ and $\varphi_1(A_i) \subseteq \varphi_2(A_i)$ for all $i \in I_1$, $j \in I_2$, we get $\bigcap_{i \in I_1} \psi_2(A_i) \subseteq \bigvee_{j \in I_2} \varphi_2(B_j)$ for every partition I_1, I_2 of I. □

Lemma 4.5. *Let $(S, \mathcal{S})$ be a texture and assume that $\overline{\mathcal{U}}_{\mathcal{B}} \subseteq \mathcal{F}_{\mathcal{R}\mathcal{R}}$, $\underline{\mathcal{U}}_{\mathcal{B}} \subseteq \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$ satisfy the following conditions:*

(UB1) For all $\varphi_1, \varphi_2 \in \overline{\mathcal{U}}_{\mathcal{B}}$, there exists $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \varphi_1 \wedge \varphi_2$,

(UB2) For all $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$, there exists $\varphi_1 \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that ${\varphi_1}^2 \leq \varphi$,

(UB3) $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}} \Longrightarrow \exists \psi \in \underline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi^{\leftarrow} \leq \psi$,

(UB4) $\psi \in \underline{\mathcal{U}}_{\mathcal{B}} \Longrightarrow \exists \varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \psi^{\leftarrow}$.

Then $\mathcal{U}_{\mathcal{B}} = \{(\varphi, \psi) \mid \varphi \in \overline{\mathcal{U}}_{\mathcal{B}}, \psi \in \underline{\mathcal{U}}_{\mathcal{B}}\}$ is a base for a difunctional uniformity on $(S, \mathcal{S})$.

Proof. We will show that $\overline{\mathcal{U}} = \{\varphi^* \in \mathcal{F}_{\mathcal{R}\mathcal{R}} \mid \exists \varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \varphi^*\}$ and $\underline{\mathcal{U}} = \{\psi^* \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}} \mid \exists \psi \in \underline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi^* \leq \psi\}$ satisfy the conditions (UF1), (UF2), (UF3) and (SYM) of Definition 2.10.

(UF1): Clear.

(UF2): Let ${\varphi_1}^*, {\varphi_2}^* \in \overline{\mathcal{U}}$. Then there exists $\varphi_1, \varphi_2 \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi_1 \leq {\varphi_1}^*$, $\varphi_2 \leq {\varphi_2}^*$. By (UB1), there exists $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \varphi_1 \wedge \varphi_2$. Clearly $\varphi \leq {\varphi_1}^* \wedge {\varphi_2}^*$ and hence ${\varphi_1}^* \wedge {\varphi_2}^* \in \overline{\mathcal{U}}$.

(UF3): Let $\varphi^* \in \overline{\mathcal{U}}$. Then there exists $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \varphi^*$. By (UB2), there exists $\varphi_1 \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that ${\varphi_1}^2 \leq \varphi \leq \varphi^*$. Since $\overline{\mathcal{U}}_{\mathcal{B}} \subseteq \overline{\mathcal{U}}$, (UF3) is satisfied.

(SYM): Let $\varphi^* \in \overline{\mathcal{U}}$. Then there exists $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi \leq \varphi^*$. By (UB3), there exists $\psi \in \underline{\mathcal{U}}_{\mathcal{B}}$ such that $\varphi^{\leftarrow} \leq \psi$. On the other hand, $\varphi^{*\leftarrow} \leq \varphi^{\leftarrow}$ since $\varphi \leq \varphi^*$. Hence we get $\varphi^{*\leftarrow} \leq \psi$ and hence $\varphi^{*\leftarrow} \in \underline{\mathcal{U}}$. The converse can be shown in a similar way. □

Before giving explicit construction of a totally bounded di-uniformity compatible with a given di-extremity, first let us give the following definitions. Let δ be a di-extremity on the texture $(S, \mathcal{S})$. For each $A \bar{\not\delta} B$ and $D \underline{\not\delta} C$, define $\varphi_{AB}, \psi_{DC} : \mathcal{S} \to \mathcal{S}$ by

$$\varphi_{AB}(Z) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ B & \text{if } Z \subseteq A, Z \neq \emptyset \\ S & \text{if } Z \not\subseteq A, \end{cases}$$

$$\psi_{DC}(Z) = \begin{cases} S & \text{if } Z = S \\ C & \text{if } D \subseteq Z, Z \neq S \\ \emptyset & \text{if } D \not\subseteq Z, \end{cases}$$

Lemma 4.6. *If φ_{AB} and ψ_{DC} satisfy the conditions of Definition 2.6, then we have $\varphi_{AB} \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$ and $\psi_{DC} \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$.*

Proof. Let $\varphi = \varphi_{AB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$. We know that $A \subseteq B$ since $A \bar{\not\delta} B$. From this fact and by the definition of φ_{AB} it is clear that for all $Z \in \mathcal{S}$, $Z \subseteq \varphi(Z)$. Now let $Z_j \in \mathcal{S}$, $j \in J$. To show that φ preserves supremum, consider three possibilities. Firstly, if $\bigvee_j Z_j = \emptyset$ then clearly $\varphi(\bigvee_j Z_j) = \bigvee_j \varphi(Z_j) = \emptyset$. Secondly, if $\bigvee_j Z_j \subseteq A$ then for all j, $Z_j \subseteq A$ and so $\varphi(Z_j) = B$. Hence $\varphi(\bigvee_j Z_j) = \bigvee_j \varphi(Z_j) = B$. Lastly, if $\bigvee_j Z_j \not\subseteq A$ then there exists $j \in J$ such that $Z_j \not\subseteq A$. For this j, $\varphi(Z_j) = S$. Therefore $\varphi(\bigvee_j Z_j) = \bigvee_j \varphi(Z_j) = S$.

Let $\psi = \psi_{DC} \in \underline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$. Since $C \subseteq D$, it is clear $\psi(Z) \subseteq Z$, for all $Z \in \mathcal{S}$. By considering three cases as $\bigcap_j Z_j = S$, $D \subseteq \bigcap_j Z_j$ and $D \not\subseteq \bigcap_j Z_j$, it can be easily shown that ψ preserves intersection. □

Now, let us constitute the subbase and base of difunctional uniform space which we need:

Define $\overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}} = \{\varphi_{AB} \mid A\overline{\not\delta}B\}$, $\underline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}} = \{\psi_{DC} \mid D\underline{\not\delta}C\}$, $\mathcal{U}^{\delta}_{\mathcal{S}\mathcal{B}} = \{(\varphi,\psi) \mid \varphi \in \overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}, \psi \in \underline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}\}$, $\overline{\mathcal{U}}^{\delta}_{\mathcal{B}} = \{\bigwedge_{k=1}^{n} \varphi_{A_kB_k} \mid \forall k = 1,\dots n, \varphi_{A_kB_k} \in \overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}\}$, $\underline{\mathcal{U}}^{\delta}_{\mathcal{B}} = \{\bigvee_{k=1}^{n} \psi_{D_kC_k} \mid \forall k = 1,\dots n, \psi_{D_kC_k} \in \underline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}\}$ and $\mathcal{U}^{\delta}_{\mathcal{B}} = \{(\varphi,\psi) \mid \varphi \in \overline{\mathcal{U}}^{\delta}_{\mathcal{B}}, \psi \in \underline{\mathcal{U}}^{\delta}_{\mathcal{B}}\}$.

Before starting to show that $\mathcal{U}^{\delta}_{\mathcal{B}}$ is a base for difunctional uniform space, we want to give the following two lemmas.

Lemma 4.7. *For all $\varphi_{AB} \in \overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}, \psi_{DC} \in \underline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}, \varphi^{\leftarrow}_{AB} = \psi_{BA}$ and $\psi^{\leftarrow}_{DC} = \varphi_{CD}$.*

Proof. Note that $\varphi^{\leftarrow}_{AB}(Z) = \bigvee\{L \in \mathcal{S} \mid \varphi_{AB}(L) \subseteq Z\}$. We will show that $\varphi^{\leftarrow}_{AB}(Z) = \psi_{BA}(Z)$ for all Z. If Z=S, then $\varphi(L) \subseteq Z$ and $\varphi^{\leftarrow}_{AB}(Z) = S = \psi_{BA}(Z)$ for all L,. If $B \subseteq Z \neq S$, then $\varphi_{AB}(L) \subseteq Z$ if and only if $L \subseteq A$. Thus $\varphi^{\leftarrow}_{AB}(Z) = A = \psi_{BA}(Z)$. Lastly, if $B \not\subseteq Z$ then $\varphi_{AB}(L) \subseteq Z$ if and only if $L = \emptyset$. Hence $\varphi^{\leftarrow}_{AB}(Z) = \emptyset = \psi_{BA}(Z)$. The other claim can be shown in a similar way. □

Lemma 4.8. *Let $(\varphi,\psi) \in \mathcal{U}^{\delta}_{\mathcal{B}}$. Then there exists $(\varphi_{AB},\psi_{DC}) \in \mathcal{U}^{\delta}_{\mathcal{S}\mathcal{B}}$ such that $(\varphi_{AB},\psi_{DC}) \leq (\varphi,\psi)$.*

Proof. Let $(\varphi,\psi) \in \mathcal{U}^{\delta}_{\mathcal{B}}$. Then there exists $k = 1,\dots,n, l = 1,\dots,m$, $\varphi_{A_kB_k} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}, \psi_{D_lC_l} \in \underline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$ such that $\varphi = \bigwedge_{k=1}^{n} \varphi_{A_kB_k}$ and $\psi = \bigvee_{l=1}^{m} \psi_{D_lC_l}$.

Firstly, set $A = \bigvee_{k=1}^{n} A_k$ and $B = \bigcap_{k=1}^{n} B_k$. By Proposition 2.5(5), $A\overline{\not\delta}B$ so $\varphi_{AB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}} \subseteq \overline{\mathcal{U}}_{\mathcal{B}}$. If $Z = \emptyset$, then clearly $\varphi_{AB}(Z) \subseteq \varphi(Z)$. If $Z \not\subseteq A$, then $Z \not\subseteq A_k$ for all k, and so $\varphi_{A_kB_k}(Z) = S$. Thus $\varphi_{AB}(Z) = S \subseteq \varphi(Z)$ and so $\varphi(Z) = S$. If $Z \subseteq A$, then $\varphi_{AB}(Z) = B$. On the other hand, $\varphi_{A_kB_k}(Z) = B_k$ or $\varphi_{A_kB_k}(Z) = S$ for each k. Then $\varphi_{AB}(Z) = B = \bigcap_{k=1}^{n} B_k \subseteq \varphi(Z)$. Hence $\varphi_{AB} \leq \varphi$.

Secondly, by setting $C = \bigvee_{k=1}^{n} C_k$ and $D = \bigcap_{k=1}^{n} D_k$, it can be shown that $\psi \leq \psi_{DC}$, in a similar manner above. □

Now we are ready to give explicit construction of a totally bounded di-uniformity compatible with a given di-extremity.

Theorem 4.9. *Let δ be a di-extremity on the texture $(S,\mathcal{S})$. Then $\mathcal{U}^{\delta}_{\mathcal{B}}$ is a base for a difunctional uniformity $\mathcal{U}^{\delta}$ on $(S,\mathcal{S})$. Moreover $\mathcal{U}^{\delta}$ is compatible with δ.*

Proof. We will show that $\mathcal{U}^{\delta}_{\mathcal{B}}$ satisfy the conditions of Lemma 4.5 and hence it is a base for a di-uniformity on $(S,\mathcal{S})$.

(UB1): Let $\varphi_1, \varphi_2 \in \overline{\mathcal{U}}^{\delta}_{\mathcal{B}}$. Then there are $\varphi_{A_kB_k}, \varphi_{C_lD_l} \in \mathcal{U}^{\delta}_{\mathcal{S}\mathcal{B}}$ such that $\varphi_1 = \bigwedge_{k=1}^{n} \varphi_{A_kB_k}$ and $\varphi_2 = \bigwedge_{l=1}^{m} \varphi_{C_kD_k}$, where $A_k\overline{\not\delta}B_k$, $C_l\overline{\not\delta}D_l$ for all $k = 1,\dots,n, l = 1,\dots,m$. Now set $A_{n+l} = C_l$ and $B_{n+l} = D_l$ for all $l = 1,\dots,m$ and set $A = \bigvee_{k=1}^{n+m} A_k$ and $B = \bigcap_{k=1}^{n+m} B_k$. By Proposition 2.5(5), $A\overline{\not\delta}B$ so $\varphi_{AB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}} \subseteq \overline{\mathcal{U}}_{\mathcal{B}}$. We will show that $\varphi_{AB} \leq \varphi_1 \wedge \varphi_2$. If $Z = \emptyset$, then clearly $\varphi_{AB}(Z) \subseteq \varphi_1(Z) \cap \varphi_2(Z)$. If $Z \not\subseteq A$ then we have $Z \not\subseteq A_k$ for all k, and so $\varphi_{A_kB_k}(Z) = S$. Thus $\varphi_{AB}(Z) = S \subseteq \varphi_1(Z) \cap \varphi_2(Z) = S$. If $Z \subseteq A$, then $\varphi_{AB}(Z) = B$. On the other hand, $\varphi_{A_kB_k}(Z) = B_k$ or $\varphi_{A_kB_k}(Z) = S$ for each k. Nevertheless, $\varphi_{AB}(Z) = \bigcap_{k=1}^{n+m} B_k \subseteq \varphi_1(Z) \cap \varphi_2(Z)$.

(UB2): Let $\varphi \in \overline{\mathcal{U}}^{\delta}_{\mathcal{B}}$. Then there are $\varphi_{A_kB_k}, \varphi_{C_lD_l} \in \mathcal{U}^{\delta}_{\mathcal{S}\mathcal{B}}$ such that $\varphi = \bigwedge_{k=1}^{n} \varphi_{A_kB_k}$ where $A_k\overline{\not\delta}B_k$ for all $k = 1,\dots,n$. Now set $A = \bigvee_{k=1}^{n} A_k$ and $B = \bigcap_{k=1}^{n} B_k$. It can be easily shown that $\varphi_{AB} \leq \varphi$. By applying (E4) to $A\overline{\not\delta}B$, we get $E \in \mathcal{S}$ such that $A\overline{\not\delta}E$ and $E\overline{\not\delta}B$. Thus $\varphi_{AE}, \varphi_{EB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}$. Now set $\varphi_1 = \varphi_{AE} \wedge \varphi_{EB} \in \overline{\mathcal{U}}_{\mathcal{B}}$ and show that ${\varphi_1}^2 \leq \varphi$. Also, we know that $A \subseteq E \subseteq B$. So there are two possibilities; it may be either $E = A$ or $E \neq A$, that is, $E \not\subseteq A$. For first case,

$$\varphi_1(Z) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ A & \text{if } Z \subseteq A, Z \neq \emptyset \\ S & \text{if } Z \not\subseteq A. \end{cases}$$

And for the second case,

$$\varphi_1(Z) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ E & \text{if } Z \subseteq A, Z \neq \emptyset \\ B & \text{if } Z \not\subseteq A, Z \subseteq E \\ S & \text{if } Z \not\subseteq E. \end{cases}$$

Thus for the first case,

$$\varphi_1(\varphi_1(Z)) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ A & \text{if } Z \subseteq A, Z \neq \emptyset \\ S & \text{if } Z \not\subseteq A. \end{cases}$$

And for the second case, we have to consider either $E \neq B$ or $E = B$.
If $E \neq B$ then

$$\varphi_1(\varphi_1(Z)) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ B & \text{if } Z \subseteq A, Z \neq \emptyset \\ S & \text{if } Z \not\subseteq A. \end{cases}$$

and if $E = B$ then

$$\varphi_1(\varphi_1(Z)) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ B & \text{if } Z \subseteq E, Z \neq \emptyset \\ S & \text{if } Z \not\subseteq E. \end{cases}$$

For each case we see that ${\varphi_1}^2 \leq \varphi_{AB}$.

(UB3): Let $\varphi \in \overline{\mathcal{U}}_{\mathcal{B}}$. Then by Lemma 4.8 there exists $\varphi_{AB} \in \overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}$ such that $\varphi_{AB} \leq \varphi$. By Lemma 4.7, $\varphi^{\leftarrow}_{AB} = \psi_{BA} \in \underline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}} \subseteq \overline{\mathcal{U}}^{\delta}_{\mathcal{B}}$. On the other hand, $\varphi^{\leftarrow} \leq \varphi^{\leftarrow}_{AB}$ since $\varphi_{AB} \leq \varphi$. Therefore (UB3) is satisfied.

(UB4) can be shown in a similar manner.

As a result, $\mathcal{U}^{\delta}_{\mathcal{B}}$ produces a difunctional uniformity $\mathcal{U}^{\delta}$ on $(S, \mathcal{S})$. Finally, let us show that $\mathcal{U}^{\delta}$ is compatible with δ. By Lemma 4.2, it is enough to consider the elements of $\overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}$. Suppose $A \bar{\not\delta} B$. Since $\varphi_{AB}(A) = B$, clearly $A \bar{\not\delta}_{\mathcal{U}} B$. Now let $A \bar{\not\delta}_{\mathcal{U}} B$ and suppose that $A \bar{\delta} B$. Since $A \bar{\not\delta}_{\mathcal{U}} B$, there exists $\varphi \in \overline{\mathcal{U}}^{\delta}_{\mathcal{B}}$ and thus $\varphi_{CD} \in \overline{\mathcal{U}}^{\delta}_{\mathcal{S}\mathcal{B}}$ such that $\varphi_{CD} \leq \varphi$, $\varphi(A) \subseteq B$ and $C \bar{\not\delta} D$. Finally, $\varphi_{CD}(A) \subseteq \varphi(A) \subseteq B$. There are three possibilities:

Case 1: $A = \emptyset$ is not possible since $A \bar{\delta} B$.

Case 2: If $\emptyset \neq A \subseteq C$, then $\varphi_{CD}(A) = D \subseteq B$. Thus, we get $C \bar{\delta} D$ since $A \subseteq C, D \subseteq B$ and $A \bar{\delta} B$. This contradicts with the fact that $C \bar{\not\delta} D$.

Case 3: If $A \not\subseteq C$, then $\varphi_{CD}(A) = S$ and we get $B = S$. But this also contradicts with $A \bar{\delta} B$.

Therefore $A \bar{\not\delta} B$ and the proof is completed. □

Definition 4.10. The difunctional uniformity $\mathcal{U}^{\delta}$ obtained in the previous theorem is called *di-uniformity induced by* δ.

One of the main targets of this work is given in the following theorem.

Theorem 4.11. *Let δ be a di-extremity on the texture $(S, \mathcal{S})$. Then the di-uniformity $\mathcal{U}^{\delta}$ induced by δ is totally bounded.*

Proof. By Corollary 4.4 and Lemma 4.8 it is enough to consider only the elements of $\mathcal{U}_{\mathcal{S}\mathcal{B}}$. Let $(\varphi, \psi) = (\varphi_{AB}, \psi_{DC}) \in \mathcal{U}_{\mathcal{S}\mathcal{B}}$ and we claim that there are $s_1, s_2, ..., s_n \in S$ such that $\{(\varphi_{AB}(P_{s_k}), \psi_{DC}(Q_{s_k})) \mid k = 1, \dots n\}$ is a dicover of S. We know that $D \not\delta C$ and therefore $A \subseteq C$, $C \subseteq D$. There are three possibilities:

Case 1: If $C \subseteq B$ and $B \neq S$, then there exists $s_1 \in S$ such that $P_{s_1} \not\subseteq B$, $Q_{s_1} \neq S$. For this s_1, $\varphi(P_{s_1}) = S, \psi(Q_{s_1}) = C$ or $\psi(Q_{s_1}) = \emptyset$. Now, take any s_2 such that $D \not\subseteq Q_{s_2}$. For this s_2, $\varphi(P_{s_2}) = S$ or $\varphi(P_{s_2}) = B$ and $\psi(Q_{s_2}) = \emptyset$. It can be easily verified that $\{(\varphi(P_{s_1}), \psi(Q_{s_1})), (\varphi(P_{s_2}), \psi(Q_{s_2}))\}$ is a dicover.

Case 2: If $C \subseteq B$ and $B = S$, then take any $s \in S$ such that $D \not\subseteq Q_s$. For this s, $\varphi(P_s) = S, \psi(Q_s) = \emptyset$. It is clear that $\{(\varphi(P_s), \psi(Q_s))\}$ is a dicover.

Case 3: If $C \not\subseteq B$, then there exists $s \in S$ such that $C \not\subseteq Q_s$ and $P_s \not\subseteq B$. Since $\varphi(P_s) = S$ and $\psi(Q_s) = \emptyset$, clearly $\{(\varphi(P_s), \psi(Q_s))\}$ is a dicover. □

Corollary 4.12. *Let δ be a di-extremity on the texture $(S, \mathcal{S})$. Then the di-uniformity $\mathcal{U}^{\delta}$ induced by δ is the smallest di-uniformity compatible with δ.*

Proof. Let $\mathcal{U}$ be another difunctional uniformity compatible with δ. By the property (SYM), it is enough to show that $\overline{\mathcal{U}}^{\delta} \subseteq \overline{\mathcal{U}}$. Take a subbase element φ_{AB} of $\overline{\mathcal{U}}^{\delta}$. Since $A\overline{\not\delta}B$ and $\mathcal{U}$ is compatible with δ, there exists $\varphi \in \overline{\mathcal{U}}$ such that $\varphi(A) \subseteq B$. We will show that $\varphi \leq \varphi_{AB}$. It is easy to observe that $\varphi(\emptyset) = \emptyset$ and φ preserves supremum since it is an increasing function. If $Z = \emptyset$, then $\varphi(Z) = \emptyset \subseteq \varphi_{AB}(Z) = \emptyset$. If $Z \subseteq A$, then $\varphi(Z) \subseteq \varphi(A) \subseteq B = \varphi_{AB}(Z)$. If $Z \not\subseteq A$, then $\varphi(Z) \subseteq S = \varphi_{AB}(Z)$. Hence $\varphi \leq \varphi_{AB}$ and $\varphi_{AB} \in \overline{\mathcal{U}}$. Therefore, $\overline{\mathcal{U}}^{\delta} \subseteq \overline{\mathcal{U}}$ since all subbase elements of $\overline{\mathcal{U}}^{\delta}$ belong to $\overline{\mathcal{U}}$. □

Corollary 4.13. *Let δ be a di-extremity on the texture $(S, \mathcal{S})$. Then the di-uniformity $\mathcal{U}^{\delta}$ induced by δ is the only totally bounded di-uniformity compatible with δ.*

Proof. Let $\mathcal{W}$ be a totally bounded difunctional uniformity compatible with δ. For the proof, it is needed to show that $\mathcal{W} \subseteq \mathcal{U}^{\delta}$, by Corollary 4.12. So, let $(\varphi, \psi) \in \mathcal{W}$. Then there exists $(\varphi_*, \psi_*) \in \mathcal{W}$ such that $(\varphi_*, \psi_*) = (\varphi_*, \psi_*)^{-1} = (\varphi_*^{\leftarrow}, \psi_*^{\leftarrow})$ and $(\varphi_*, \psi_*)^3 \leq (\varphi, \psi)$. Since $\mathcal{W}$ is a totally bounded diuniformity, there are $s_1, s_2, \ldots, s_n \in S$ such that $\mathcal{D} = \{(\varphi_*(P_{s_k}), \psi_*(Q_{s_k})) \mid k = 1, \ldots n\}$ is a dicover of S.

First observation: Since $\mathcal{D}$ is a dicover of S, we have the following;

$$\bigvee_{k=1}^{n} \varphi_*(P_{s_k}) = \varphi_* \bigvee_{k=1}^{n} (P_{s_k}) = S.$$

Second observation: There exists a $k \in \{1, 2, 3, ..., n\}$ such that $\varphi_*^3(P_{s_k}) \subseteq \varphi_*^3(Z)$ for all $Z \in \mathcal{S}$. On the contrary, suppose that $\varphi_*^3(P_{s_k}) \not\subseteq \varphi_*^3(Z)$ for all k. In this case, it is obtained that $S = \bigvee_{k=1}^{n} \varphi_*(P_{s_k}) \subseteq \bigvee_{k=1}^{n} \varphi_*^3(P_{s_k}) \not\subseteq \varphi_*^3(Z)$ which is a contradiction.

Third observation: Since $\mathcal{D}$ is a dicover of S, we have $\cap_{i \neq k}(\psi_*(Q_{s_i})) \subseteq (\varphi_*(Q_{s_k}))$. Then, by considering $\varphi_* = \psi_*^{\leftarrow}$, and applying sequentially two times φ_* to the both of above inclusions, it is obtained that $\cap_{i \neq k}(\psi_*(Q_{s_i})) = \psi_*(\cap_{i \neq k} Q_{s_i}) \subseteq \varphi_*{}^3(P_{s_k})$.

Set $A_k = \cap_{i \neq k} Q_{s_i}$ and $B_k = \varphi_*^3(P_{s_k})$. Then, we have $A_k \overline{\not\delta} B_k$ since $\mathcal{U}$ is compatible with δ and $\varphi_* \in \overline{\mathcal{W}}$. Thus $A_k \subseteq B_k$ and $\varphi_{A_k B_k} \in \overline{\mathcal{U}}^{\delta}$.

Now, let us see $\bigwedge_{i=1}^{n} \varphi_{A_i B_i} \leq \varphi_*^3$. By the second observation, there exists a $k \in \{1, 2, 3, ..., n\}$ such that $B_k = \varphi_*^3(P_{s_k}) \subseteq \varphi_*^3(Z)$. There are three cases: Firstly, if $Z = \emptyset$ then it is clear that $\bigcap_{i=1}^{n} \varphi_{A_i B_i}(Z) \subseteq \varphi_*^3 Z$. Secondly, if $Z \neq \emptyset$ and $Z \subseteq A_k$ then $\varphi_{A_k B_k}(Z) = B_k \subseteq \varphi_*^3(Z)$. Therefore $\bigcap_{i=1}^{n} \varphi_{A_i B_i}(Z) \subseteq \varphi_{A_k B_k}(Z) = B_k \subseteq \varphi_*^3(Z)$. Thirdly, if $Z \not\subseteq A_k$ then there exits a $j \neq k$ such that $Z \not\subseteq Q_{s_j}$ and $P_{s_j} \not\subseteq A_k$, that is $P_{s_j} \subseteq Z \longrightarrow B_j = \varphi_*{}^3(P_{s_j}) \subseteq \varphi_*{}^3(Z) \longrightarrow \bigwedge_{i=1}^{n} \varphi_{A_i B_i}(Z) \subseteq \varphi_{A_j B_j}(Z) = B_j \subseteq \varphi_*{}^3(Z)$.

Thus $\bigwedge_{i=1}^{n} \varphi_{A_i B_i}(Z) \subseteq \varphi_*{}^3(Z)$, and $\varphi_* \leq \varphi \in \overline{\mathcal{U}}$ therefore, we have $\overline{\mathcal{W}} \subseteq \overline{\mathcal{U}}^{\delta}$. This means that $\mathcal{W} \subseteq \mathcal{U}^{\delta}$. □

Corollary 4.14. *If $(S, \mathcal{S}, \delta_1), (S, \mathcal{S}, \delta_2)$ are di-extremial texture spaces and $\delta_1 < \delta_2$, then $\mathcal{U}^{\delta_1} \subseteq \mathcal{U}^{\delta_2}$.*

Proof. It is enough to show that $\overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}^{\delta_1} \subseteq \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}^{\delta_2}$. So, if take a $\varphi_{AB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}^{\delta_1}$, then we have $A\overline{\not\delta_1}B$ by definition, and hence $\varphi_{AB} \in \overline{\mathcal{U}}_{\mathcal{S}\mathcal{B}}^{\delta_2}$ since $\delta_1 < \delta_2$ and finally $A\overline{\not\delta_2}B$. □

Theorem 4.15. *Let $(S, \mathcal{S}, \mathcal{U}), (T, \mathcal{T}, \mathcal{V})$ be difunctional uniform texture spaces, $\mathcal{V}$ totally bounded and $(f, F) : (S, \mathcal{S}, \mathcal{U}) \longrightarrow (T, \mathcal{T}, \mathcal{V})$ a difunction. Then (f, F) is $\mathcal{U} - \mathcal{V}$ uniformly bicontinuous if and only if (f, F) is extremial bicontinuous with respect to the induced di-extremities.*

Proof. By adjusting the proof of Theorem 4.9 in [19], one can easily show that if (f, F) is uniformly bicontinuous, then it is also extremial bicontinuous with respect to the induced di-extremities. To show the converse, take $\mathcal{W}$ as the totally bounded difunctional uniformity induced by $\delta_{\mathcal{U}}$. We will first show that if $(f, F) : (S, \mathcal{S}, \delta_{\mathcal{U}}) \longrightarrow (T, \mathcal{T}, \delta_{\mathcal{V}})$ is extremial bicontinuous, then $(f, F) : (S, \mathcal{S}, \mathcal{W}) \longrightarrow (T, \mathcal{T}, \mathcal{V})$ is $\mathcal{W} - \mathcal{V}$ uniformly bicontinuous. To do this, it is enough to show that for all $\varphi \in \overline{\mathcal{V}}$, $(f,F)^{-1}(\varphi) \in \mathcal{W}$, by Corollary 2.8. Take a subbase element φ_{CD} of $\overline{\mathcal{V}}$. Set $A = f^{\leftarrow}C$ and $B = f^{\leftarrow}D$. Then $A\bar{\not\delta}_{\mathcal{V}}B$ since (f,F) is extremial bicontinuous and thus, φ_{AB} is a subbase element of $\overline{\mathcal{W}}$.

We claim that $\varphi_{AB} \leq (f,F)^{-1}(\varphi_{CD})$. To make the notation easier, denote $\varphi^* = (f,F)^{-1}(\varphi_{CD})$.
Thus, note that

$$\varphi^*(Z) = \begin{cases} \emptyset & \text{if } f^{\rightarrow}Z = \emptyset \\ f^{\leftarrow}D & \text{if } f^{\rightarrow}Z \subseteq C \\ S & \text{if } f^{\rightarrow}Z \not\subseteq C \end{cases}$$

and

$$\varphi_{AB}(Z) = \begin{cases} \emptyset & \text{if } Z = \emptyset \\ f^{\leftarrow}D & \text{if } f^{\rightarrow}Z \subseteq A = f^{\leftarrow}C \\ S & \text{if } Z \not\subseteq A = f^{\leftarrow}C \end{cases}$$

In this case, there are three possibilities:

Case 1: If $Z = \emptyset$, then $\varphi_{AB}(Z) = \emptyset \subseteq \varphi^*(Z) = \emptyset$

Case 2: If $Z \subseteq A = f^{\leftarrow}C$, then $f^{\rightarrow}Z \subseteq f^{\rightarrow}A = f^{\rightarrow}f^{\leftarrow}C \subseteq C$. Thus $\varphi_{AB}(Z) = f^{\leftarrow}D \subseteq \varphi^*(Z) = f^{\leftarrow}D$.

Case 3: If $Z \not\subseteq A$ then $f^{\rightarrow}Z \not\subseteq C$. Otherwise, if $f^{\rightarrow}Z \subseteq C$, then $Z \subseteq f^{\leftarrow}f^{\rightarrow}Z \subseteq f^{\leftarrow}C = A$ and we get a contradiction. Thus $\varphi_{AB}(Z) = S \subseteq \varphi^*(Z) = S$ and we get $\varphi_{AB} \leq \varphi^*$. Hence (f,F) is $\mathcal{W} - \mathcal{V}$ uniformly bicontinuous. By Corollary 4.11, we know that $\mathcal{W} \subseteq \mathcal{U}$. Therefore (f,F) is also $\mathcal{U} - \mathcal{V}$ uniformly bicontinuous and the proof is completed. □

We know that every completely biregular ditopology has a compatible di-uniformity [14] and thus it has a compatible di-extremity. Hence the ditopology induced by a di-extremity is also completely biregular and we obtain the following result.

Corollary 4.16. *(τ,κ) is completely biregular if and only if it has a compatible di-extremity.*

The following proposition shows that one can get a compatible di-extremity from a completely biregular ditopology, directly.

Proposition 4.17. *Let $(S,\mathcal{S},\tau,\kappa)$ be a completely biregular ditopological texture space. Define "$A\bar{\delta}B \iff$ there exists a bicontinuous difunction $(f,F) : (S,\mathcal{S}) \longrightarrow (\mathbb{I},\mathcal{J})$ such that $A \subseteq f^{\rightarrow}P_0$ and $F^{\rightarrow}Q_1 \subseteq B$". Then $\delta = (\bar{\delta}, \bar{\delta}^{-1})$ is a di-extremity on $(S,\mathcal{S})$ and it is compatible with (τ,κ).*

Proof. Let us verify the conditions of Definition 2.4: (E1) Suppose $A = \emptyset$. We will show that $A\bar{\not\delta}B$ for each $B \in \mathcal{S}$. Since $\emptyset \in \kappa$ and (τ,κ) is completely coregular, there exists a bicontinuous difunction $(f_s,F_s) : (S,\mathcal{S}) \longrightarrow (\mathbb{I},\mathcal{J})$ such that $\emptyset \subseteq f_s{}^{\leftarrow}P_0$ and $F_s{}^{\leftarrow}Q_1 \subseteq Q_s$. Since this is valid for every $s \in S$ and particularly this is true for every $P_s \not\subseteq B$. Now if we set $(f,F) = \sqcap_{P_s \not\subseteq B}(f_s,F_s)$, then (f,F) is also bicontinuous difunction and $\emptyset \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq Q_s$. Hence $\emptyset\bar{\not\delta}B$. In a similar way, one can show that if $B = S$ then $A\bar{\not\delta}B$ for each $A \in \mathcal{S}$.

(E2), (E3) are clear by the definition.

(E4) Suppose $A\bar{\not\delta}B$. By the definition, there exists $(f,F) : (S,\mathcal{S}) \longrightarrow (S_i,\mathcal{S}_i)$ such that $A \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq B$. On the other hand, define the point functions $\phi_1,\phi_2 : \mathbb{I} \longrightarrow \mathbb{I}$ as

$$\phi_1(y) = \begin{cases} 2y & \text{if } 0 \leq y \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq y. \end{cases}$$

and

$$\phi_2(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{2} \\ 2y-1 & \text{if } \frac{1}{2} \leq y. \end{cases}$$

It is easy to see that both ϕ_1 and ϕ_2 satisfy the condition of the Lemma 3.8 in [17], and so there exist corresponding difunctions $(g_1, G_1), (g_2, G_2) : (\mathbb{I}, \mathcal{J}) \longrightarrow (\mathbb{I}, \mathcal{J})$, respectively, such that $g_1^{\leftarrow}(Z) = G_1^{\leftarrow}(Z) = \phi_1^{\leftarrow}(Z)$ and $g_2^{\leftarrow}(Z) = G_2^{\leftarrow}(Z) = \phi_2^{\leftarrow}(Z)$ for all $Z \in \mathcal{J}$. It can be easily verified that both (g_1, G_1) and (g_2, G_2) are bicontinuous.

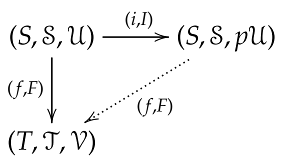
Now set $(f_1, F_1) = (g_1, G_1) \circ (f, F)$, $(f_2, F_2) = (g_2, G_2) \circ (f, F)$ and $E = f^{\leftarrow}Q_{\frac{1}{2}}$. Since $(g_1, G_1), (g_2, G_2)$ and (f, F) are bicontinuous, (f_1, F_1) and (f_2, F_2) are also bicontinuous. Moreover, we see that $A \subseteq f^{\leftarrow}(P_0) = f^{\leftarrow}(\phi_1(P_0)) = f^{\leftarrow}(g_1^{\leftarrow}(P_0)) = f_1(P_0)$ and $F_1^{\leftarrow}(Q_1) = F_1^{\leftarrow}(G_1^{\leftarrow}(Q_1)) = F^{\leftarrow}(\phi_1^{-1}([0, 1)) = F^{\leftarrow}[0, \frac{1}{2}) = F^{\leftarrow}Q_{\frac{1}{2}} = E$. Similarly, $E \subseteq f_2^{\leftarrow}P_0$ and $F_2^{\leftarrow}Q_1 \subseteq B$. Thus we see that $A\overline{\not\delta}E$ and $E\overline{\not\delta}B$.

E5) Let $A\overline{\not\delta}B$. Then there exists $(f, F) : (S, \mathcal{S}) \longrightarrow (S_i, \mathcal{S}_i)$ such that $A \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq B$. We see that $f^{\leftarrow}P_0 \subseteq F^{\leftarrow}Q_1$ since $P_0 = \{0\} \subseteq Q_0 = [0, 1)$. Thus $A \subseteq B$. □

To conclude this section, we present some categorical notes. We denote by **dfUnif** the category of difunctional uniformities and uniformly bicontinuous difunctions , by **dfTbUnif** the category of totally bounded difunctional uniformities and uniformly bicontinuous difunctions, and by **dfDiex** the category of di-extremities and extremial bicontinuous difunctions.

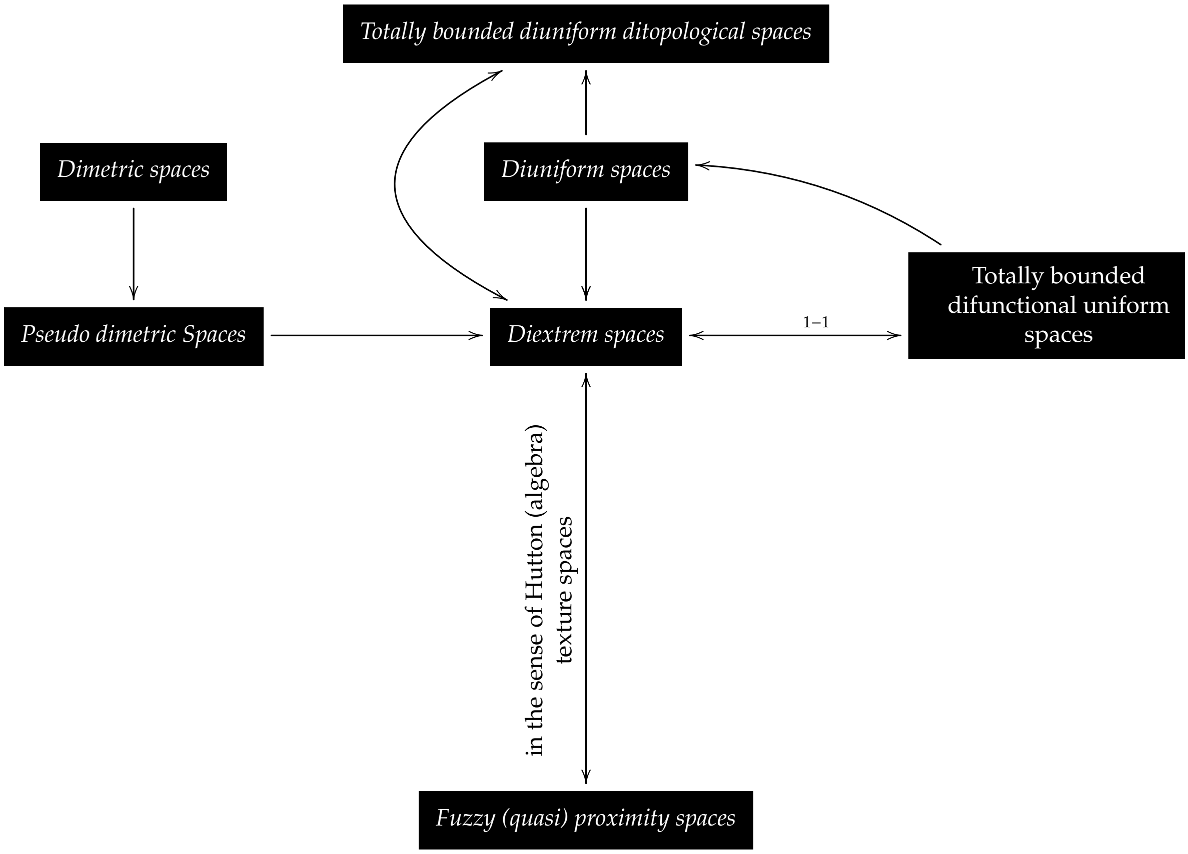
Let $(S, \mathcal{S}, \mathcal{U})$ be a difunctional uniform texture space. Now consider di-extremity $\delta_{\mathcal{U}}$ induced by $\mathcal{U}$. By Theorem 4.10, we have a totally bounded $\mathcal{U}_{\delta_{\mathcal{U}}}$ compatible with $\delta_{\mathcal{U}}$. Let us denote $\mathcal{U}_{\delta_{\mathcal{U}}}$ by $p\mathcal{U}$ for short. If $\mathcal{U}$ is totally bounded, then $\mathcal{U} = p\mathcal{U}$ by Corollary 4.12. Thus, we see that there is a bijection between the objects of **dfTbUnif** and of **dfDiex**. Now let $(f, F) : (S, \mathcal{S}, \delta_1) \longrightarrow (T, \mathcal{T}, \delta_2)$ be extremial bicontinuous difunction. Then by Theorem 4.13, $(f, F) : (S, \mathcal{S}, \mathcal{U}_{\delta_1}) \longrightarrow (T, \mathcal{T}, \mathcal{U}_{\delta_2})$ is uniformly bicontinuous difunction. In the light of these facts, it is easy to show that **dfTbUnif** and **dfDiex** are isomorphic categories.

Now let us show that there is a reflection from **dfUnif** onto **dfTbUnif**. Let $(S, \mathcal{S}, \mathcal{U})$ be a difunctional uniform texture space. Then clearly $(i, I) : (S, \mathcal{S}, \mathcal{U}) \longrightarrow (S, \mathcal{S}, p\mathcal{U})$ is uniformly bicontinuous. For all $(T, \mathcal{T}, \mathcal{V}) \in Ob(\textbf{dfUnif})$ and $(f, F) : (S, \mathcal{S}, \mathcal{U}) \longrightarrow (T, \mathcal{T}, \mathcal{V}) \in hom(\textbf{dfUnif})$, the following diagram is commutative.



Thus, we see that the correspondence $(S, \mathcal{S}, \mathcal{U}) \longrightarrow (S, \mathcal{S}, p\mathcal{U})$ is a reflection. As a result, **dfDiex** is isomorphic to a full subcategory of **dfUnif** as expected.

The results obtained in this study and [19] can be summarized by the following diagram. In this diagram, the interrelations between di-extremity and dimetric, pseudo-dimetric, di-uniform spaces have been given in [19], the interrelations between di-extremity and fuzzy (quasi) proximity, totally bounded diuniform ditopological spaces, totally bounded difunctional uniform spaces are investigated in this study.



Acknowledgement

The authors gratefully thank to the referee for carefully reading our manuscript and for giving such constructive comments which substantially helped improving the quality of the paper.
The material in this work forms a part of second named author's Ph.D. Thesis, currently being written under the supervision of the first author.

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