Filomat 32:4 (2018), 1429–1437
https://doi.org/10.2298/FIL1804429T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Another Approach of Morrey Estimate for Linear Elliptic Equations with Partially BMO Coefficients in a Half Space

**Hong Tian[a], Shenzhou Zheng[a]**

[a]*Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China.*

**Abstract.** Making use of an elementary approach instead of the weighted $L^p$ estimate with a special weight, we prove global Morrey estimates of the weak derivatives to the Dirichlet problems of linear elliptic equations with small partially BMO coefficients in a half space. Here, the leading coefficients $a^{ij}(x)$ are assumed to be merely measurable in one variable, and have small BMO in the remaining spatial variables.

## 1. Introduction

The main purpose of this paper is to use a direct approach to attain global Morrey estimates of the weak derivative for the Dirichlet problems regrading linear elliptic equations in divergence form with small partially BMO coefficients over the half spaces. Rather than the weighted $L^p$ estimate with a special weight to the weak derivatives (cf. [2, 27]), we are here devoted to its estimate in the framework of Morrey spaces in accordance with the $L^p$ estimates via a direct argument first introduced by Lieberman [20]. For technical simplicity, we only consider their Dirichlet problems over a half space. In fact, its conclusion can be extended to the corresponding problems defined in the Reifenberg flat domain due to the $L^p$ theory of linear elliptic equations with small partially BMO coefficients by Byun and Wang in [4]. By $\mathbb{R}^d$ denotes $d$-dimensional Euclidean space for $d \geq 2$. Let us write $\mathbb{R}^d_+ = \{x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid x_1 > 0\}$ to be open upper half space, and its boundary of $\mathbb{R}^d_+$ by $\partial\mathbb{R}^d_+ = \{x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid x_1 = 0\}$. We consider the following Dirichlet problems of linear elliptic equations:

$$\begin{cases} \mathcal{L}u := \sum_{i,j=1}^d D_{x_j}(a^{ij}(x)D_{x_i}u) = \sum_{j=1}^d D_{x_j}f^j(x) & \text{in} \quad \mathbb{R}^d_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^d_+. \end{cases} \tag{1}$$

As usual, we suppose the coefficients $a^{ij}(x)$ to be uniform boundedness and ellipticity, which means that there exist $0 < \nu \leq \Lambda < \infty$ such that

$$\nu|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 35J15; Secondary 35B65, 35D30
*Keywords*. divergence linear elliptic equations; Dirichlet problems; partially BMO coefficients; Morrey spaces.
Received: 10 May 2017; Revised: 21 November 2017; Accepted: 26 November 2017
Communicated by Marko Nedeljkov
This paper was supported by the Natural Science Foundation of China grant No.11371050.
*Email addresses:* 14118404@bjtu.edu.cn (Hong Tian), shzhzheng@bjtu.edu.cn. (Shenzhou Zheng)

for any $x \in \mathbb{R}^d_+$ and $\xi \in \mathbb{R}^d$. As we know, the existence and uniqueness of weak solution in $H^1_0(\Omega)$ for the zero boundary value problems (1) with (2) in a bounded domain is a direct result of the classical Lax-Milgram theorem; moveover, there exists a constant $c > 0$ such that

$$\|Du\|_{L^2(\Omega)} \leq c\|F\|_{L^2(\Omega)}. \tag{3}$$

Various regularities to elliptic and parabolic problems with the minimal regular assumptions are extremely popular researches in recent decades. In particular, the Calderón-Zygmund estimates of weak derivatives to divergence elliptic and parabolic equations with discontinuous coefficients. This is due to a subtle link with applications to stochastic processes [16], linearly elastic laminates [8], composite material [21], etc. Generally speaking, it does not exist solvability in the Sobolev spaces $u \in W^{1,p}_0(\Omega)$ with $p > 2$ to the Dirichlet problems (1) merely with boundedness and ellipticity (2) even if the domain is smooth. In fact, if the coefficients $a^{ij}$ are only measurable, then there could not exist a unique solution to the above-mentioned equations even in a very generalized sense due to the famous counterexample by Nadirashvili. Furthermore, Ural'tseva [28] in 1967 constructed an example of an equation in $\mathbb{R}^d$ ($d \geq 3$) with the coefficients depending only on the first two coordinates so that we reached that there is no unique solvability in $W^{2,p}$ for any $p > 1$. This reminds us of the significance to treat particular cases for the leading discontinuous coefficients in order to obtain the solvability in Lebesque or Morrey spaces for the problems (1). Therefore, this makes necessary to impose some suitable minimal regular assumptions on coefficients and geometric restriction on domains.

We would like to mention that Sarason [24] in 1975 introduced the classes of so-called VMO functions (Vanishing Mean Oscillations) which is not shared by general bounded measurable functions and BMO functions. Since then, there have been a lot of research activities on the Calderón-Zygmund theory to elliptic and parabolic problems, and it developed various different arguments to deal with divergence or non-divergence elliptic or parabolic PDEs with VMO leading coefficients. For example, a celebrated approach for the $L^p$ boundedness of singular integral operators and Coifman-Rochberg-Weiss commutators by Chiarenza-Frasca-Longo [6, 7], a geometrical technique by way of modified Vitali covering by Byun-Wang [3], an unified approach based on pointwise estimates of Fefferman sharp functions by Dong-Kim-Krylov [15, 16] and the Large-M-inequality principle introduced by Acerbi-Mingione [1]. Regarding the Calderón-Zygmund theory, we would like to point out that a recent distinguishing feature of small partially BMO coefficients originated from Kim-Krylov's work [14], which means that they are allowed to be very irregular with respect to one spatial direction, and the remaining $(d-1)$ variables are controlled in terms of small BMO discontinuous one. It is also worth noticing that the equations with continuous, VMO and small BMO coefficients are the special setting with partially VMO ones. Indeed, this is a kind of minimal regular requirement on the leading coefficients in accordance with the famous counterexample by Nadirashvili. For more generalizations and extensions involving VMO or small partially BMO coefficents can be found in Dong, Kim and Krylov [10, 11, 17], Byun, Palagachev and Wang [2, 4], Ragusa and Tachikawa [22, 23], Leonardi and Stará [18, 19], Guliyev, Omarova, Ragusa and Scapellato [13, 25, 26].

On the other hand, in recent years many analogous estimates with Morrey spaces replacing Lebesgue spaces have been considered. Morrey estimates for elliptic and parabolic problems have usually been attained by way of refining weighted $L^p$ estimates by taking a special weight function. For instance, in order to show Morrey regularity of the weak derivative for the zero Dirichlet boundary problem of divergence linear elliptic equations, Byun-Palagachev [2] established the global weighted estimate in $L^p_w(\Omega)$ with Muckenhoupt weight $w \in A_p$. Also, Tang [27] proved the weighted $L^p$ solvability with $w \in A_p$ for divergence and nondivergence parabolic equations with small partially BMO coefficients by making use of Dong-Kim-Krylov's argument based on the pointwise estimates of $A_p$ weighted sharp functions. Then, with the weighted $L^p_w(\Omega)$ estimates, one may obtain the Morrey estimates of the derivatives of its weak solution by way of taking a special weight, also see Di Fazio-Ragusa's work [9].

Our motivation of this paper is to show that the Morrey estimates can be derived from the $L^p$ estimates by a direct method, which means that once we have suitable forms for the $L^p$ estimates then Morrey estimates is almost identical by Campanato's argument. In fact, it was also done for elliptic and parabolic equations with VMO coefficients by Lieberman in [20]. More precisely, our main consideration is to derive global Morrey estimates of the weak derivatives to the Dirichlet problem (1) with partially VMO coefficients in accordance with the $L^p$ estimates from Dong-Kim's papers [10]. We here employ an elementary approach in [20] rather than using the usual weighted $L^p$ estimates.

Before stating our main theorem let us recall some notations and basic facts. A type point in $\mathbb{R}^d$ will be denoted by $x = (x_1, \cdots, x_d) = (x_1, x')$ with $x' = (x_2, \cdots, x_d) \in \mathbb{R}^{d-1}$. Set

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}, \quad B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |y' - x'| < r\}$$

and

$$Q_r(x) = B'_r(x') \times (x_d - r, x_d + r).$$

For convenience, in the context we write $B_r = B_r(0)$, $B'_r = B'_r(0')$ and denote by $|B_r|$, $|B'_r|$, $|Q_r|$ the volume of $B_r$, $B'_r$, $Q_r$, respectively. We denote an average of $f$ over the ball $B_r$ by

$$(f)_{B_r} = \fint_{B_r} f(x)\, dx = \frac{1}{|B_r|} \int_{B_r} f(x)\, dx,$$

and $(d-1)$-dimensional average only with respect to $x'$ by

$$(f)_{B'_r}(x_1) = \fint_{B'_r} f(x_1, x')\, dx' = \frac{1}{|B'_r|} \int_{B'_r} f(x_1, x')\, dx'.$$

Now we are in a position to impose a partially BMO regular assumption on the leading coefficients $A(x) = \left(a^{ij}(x)\right)$ in the neighborhood of an interior point and boundary point, respectively.

**Assumption 1.1.** *Let $\delta \in (0,1)$ be a constant specified later. There is a constant $r_0 \in (0,1]$ such that for any $x_0 \in \mathbb{R}^d_+$ and any $r \in (0, r_0]$ so that $B_r(x_0) \subset \mathbb{R}^d_+$, one has the coefficient matrices $A(x)$ satisfying*

$$\fint_{B_r(x_0)} \left| A(x_1, x') - (A)_{B'_r(x'_0)}(x_1) \right|^2 dx \le \delta^2;$$

*while $x_0 \in \partial\mathbb{R}^d_+$ and any $r \in (0, r_0]$, one has the coefficient matrices $A(x)$ satisfying*

$$\fint_{B^+_r(x_0)} \left| A(x_1, x') - (A)_{B'_r(x'_0)}(x_1) \right|^2 dx \le \delta^2,$$

*where $B^+_r(x_0) = \{ (x_1, x') \in B_r(x_0) \mid x_1 > 0 \}$.*

**Remark 1.2.** *From the above assumption, it is clear that there is no regular assumption on $A(x)$ with respect to $x_1$ variable, and so there maybe a big jumping of $A(x)$ along the $x_1$ variable while $A(x)$ is a small BMO along the $x'$ variable. We here refer the reader to [11] for a similar assumption to $A(x)$ in a half space.*

We focus on the estimates in Morrey spaces to the weak derivatives of the Dirichlet problems (1) by way of an elementary argument. Therefore, let us recall the definition of the Morrey spaces $L^{p,\lambda}$ in a half space $\mathbb{R}^d_+$ as follows.

**Definition 1.3.** *For $p \ge 1$ and $0 \le \lambda < d$, we say that $u(x) \in L^p_{loc}(\mathbb{R}^d_+)$ belongs to the Morrey space $L^{p,\lambda}(\mathbb{R}^d_+)$, iff*

$$L^{p,\lambda}(\mathbb{R}^d_+) := \left\{ u(x) \in L^p_{loc(\mathbb{R}^d_+)} \,\Big|\, \|u\|_{L^{p,\lambda}(\mathbb{R}^d_+)} := \sup_{x \in \mathbb{R}^d_+;\, \rho > 0} \Big( \frac{1}{\rho^\lambda} \int_{U_\rho(x)} |u(y)|^p dy \Big)^{\frac{1}{p}} < \infty \right\}$$

*where $U_\rho(x) = B_\rho(x) \cap \mathbb{R}^d_+$. Moreover, $L^{p,\lambda}(\mathbb{R}^d_+)$ is a Banach space with the norm $\|u\|_{L^{p,\lambda}(\mathbb{R}^d_+)}$.*

Finally, we are ready to present the main result of this paper.

**Theorem 1.4.** *For $1 < p < \infty$ and $0 \le \lambda < d$, let $u \in W_0^{1,2}(\mathbb{R}_+^d)$ be any weak solution of the Dirichlet problems (1). Assume that there exists a positive constant $\delta = \delta(d, p, \lambda, \nu, \Lambda)$ such that for coefficients $a^{ij}(x)$ satisfying **Assumption 1.1**,and $\boldsymbol{f} = (f^1, f^2, \cdots, f^d) \in L^{p,\lambda}(\mathbb{R}_+^d)$, then we have $Du \in L^{p,\lambda}(\mathbb{R}_+^d)$ with the estimate*

$$\|Du\|_{L^{p,\lambda}(\mathbb{R}_+^d)} \le N \|f\|_{L^{p,\lambda}(\mathbb{R}_+^d)}, \tag{4}$$

*where constant $N > 0$ depends only on $d, p, \lambda, \nu, \Lambda, \delta$ and $r_0$.*

An ingredient of our proof is based on Dong and Kim's $L^p$ estimates of the Dirichlet problems (1) for elliptic equations with partially BMO coefficients. We also exploited it by using Lieberman's elementary approach. As he pointed out that the direct argument has the advantage that it is easily applied to any equations with non-VMO coefficients which have $L^p$ estimates.

The remainder of this paper is organized as follows. Section 2 is devoted to establishing a global Morrey estimate to the Dirichlet problems (1) for linear elliptic equations with partially small BMO coefficients in a half space.

## 2. Proof of main result

Throughout this paper, we by $N(d, p, \cdots)$ denote a constant depending only on the prescribed quantities $d, p, \cdots$. Let us start this section with a well-known $L^p$ solvability theory to linear elliptic operator $\mathcal{L}$ with partially BMO coefficients from Dong and Kim's papers [10, 11] with the $L^p$ estimates in a particular form, which the constant $N$ is independent of $R$.

**Lemma 2.1.** *Assume that there exists a positive constant $\delta = \delta(d, p, \lambda, \nu, \Lambda)$ such that for coefficients $a^{ij}(x)$ satisfying **Assumption 1.1** and $\boldsymbol{f} \in L^p(B_R)$ with $1 < p < \infty$. Then there is a unique solution $u \in W_0^{1,p}(B_R)$ of*

$$\mathcal{L}u := \sum_{i,j=1}^{d} D_{x_j}(a^{ij}(x) D_{x_i} u) = div\, \boldsymbol{f}(x) \quad x \in B_R;$$

*with the estimate*

$$\|Du\|_{L^p(B_R)} \le N \|f\|_{L^p(B_R)}, \tag{1}$$

*where the constant $N > 0$ depends only on $d, p, \nu, \Lambda, \delta$ and $r_0$.*

**Proof.** The estimate (1) with $N$ depending also on $R$ is Theorem 8.6 in [11]. For convenience, we give brief proof just as Lieberman showed. In fact, we make a scale argument by

$$\tilde{a}^{ij}(x) = a^{ij}(Rx), \quad \tilde{u}(x) = u(Rx)/R, \quad \tilde{\mathbf{f}}(x) = \mathbf{f}(Rx),$$

then we check that $\tilde{a}^{ij}(x)$ still satisfies **Assumption 1.1** with the dilated scale $1/R$ for $\tilde{u} \in W_0^{1,p}(B_1)$ and

$$\tilde{\mathcal{L}}\tilde{u} = \sum_{i,j=1}^{d} D_{x_j}(\tilde{a}^{ij}(x) D_{x_i} \tilde{u}) = \mathrm{div}\, \tilde{\mathbf{f}}(x) \quad x \in B_1$$

with the estimate

$$\|D\tilde{u}\|_{L^p(B_1)} \le N(d, p, \nu, \Lambda, \delta, r_0) \|\tilde{\mathbf{f}}\|_{L^p(B_1)}.$$

Hence,re-scaling it by $u$ and $\mathbf{f}$ immediately yields (1). □

Next, we go back to focus on an interior Morrey estimate and the boundary Morrey estimate of the gradients of weak solution to the problems (1), respectively.

*2.1. Interior estimates*

We begin this subsection with recalling a locally estimate of homogeneous linear elliptic equations

$$\mathcal{L}w = \sum_{ij} D_{x_j}\Big(a^{ij}(x)D_{x_i}w(x)\Big) = 0 \qquad \forall x \in B_R \subset\subset \mathbb{R}^d_+, \tag{2}$$

whose proof can be found from Dong-Kim's paper [10], see Corollary 3 in Section 8.

**Lemma 2.2.** *For $1 < p < q < \infty$, there exists a positive constant $\delta = \delta(d, p, q, \nu, \Lambda)$ such that the leading coefficients $a^{ij}(x)$ satisfy **Assumption 1.1**. If $w \in W^{1,p}(B_R)$ satisfies (2), then $w \in W^{1,q}(B_{R/2})$ with the estimate*

$$\Big(|Dw|^q\Big)^{\frac{1}{q}}_{B_{R/2}} \leq N\Big(|Dw|^p\Big)^{\frac{1}{p}}_{B_R}, \tag{3}$$

*where $N = N(d, p, q, \nu, \Lambda, \delta, r_0)$.*

Furthermore, we can rewrite the above conclusion in a more useful form as follows.

**Lemma 2.3.** *Under the same assumptions as Lemma 2.2 above, we have $w \in W^{1,q}(B_{R/2})$ with the estimate*

$$\|Dw\|_{L^q(B_{R/2})} \leq NR^{\frac{d}{q}-\frac{d}{p}}\|Dw\|_{L^p(B_R)}, \tag{4}$$

*where $N = N(d, p, q, \omega_d, \nu, \Lambda, \delta, r_0)$ and $\omega_d$ is the volume of unit ball in $\mathbb{R}^d$.*

**Proof.** Observe that

$$\Big(|Dw|^p\Big)^{\frac{1}{p}}_{B_R} = \Big(\frac{1}{|B_R|}\Big)^{\frac{1}{p}}\|Dw\|_{L^p(B_R)} \qquad \text{and} \qquad \Big(|Dw|^q\Big)^{\frac{1}{q}}_{B_{R/2}} = \Big(\frac{1}{|B_{R/2}|}\Big)^{\frac{1}{q}}\|Dw\|_{L^q(B_{R/2})}.$$

By Lemma 2.2 it yields

$$\Big(\frac{1}{|B_{R/2}|}\Big)^{\frac{1}{q}}\|Dw\|_{L^q(B_{R/2})} \leq N_1\Big(\frac{1}{|B_R|}\Big)^{\frac{1}{p}}\|Dw\|_{L^p(B_R)},$$

which implies that

$$\|Dw\|_{L^q(B_{R/2})} \leq N_1\frac{|B_{R/2}|^{\frac{1}{q}}}{|B_R|^{\frac{1}{p}}}\|Dw\|_{L^p(B_R)} \leq NR^{\frac{d}{q}-\frac{d}{p}}\|Dw\|_{L^p(B_R)},$$

where $N = N(d, p, q, \omega_d, \nu, \Lambda, \delta, r_0)$. $\square$

To this aim, we need the following iterating Lemma, see [12].

**Lemma 2.4.** *Let $\Phi(\rho)$ be a non-negative and non-decreasing function on $(0, R)$. Suppose that*

$$\Phi(\rho) \leq A\left[\Big(\frac{\rho}{R}\Big)^{\alpha} + \epsilon\right]\Phi(R) + BR^{\beta} \quad \forall\, 0 < \rho < R, \tag{5}$$

*with non-negative constants $A$, $B$, $\alpha$ and $\beta$ such that $\alpha > \beta$. Then there exist two constants $\epsilon_0 = \epsilon_0(A, \alpha, \beta)$ and $N = N(A, \alpha, \beta)$ such that for any $0 < \epsilon < \epsilon_0$ we have*

$$\Phi(\rho) \leq N\left[\Big(\frac{\rho}{R}\Big)^{\beta}\Phi(R) + B\rho^{\beta}\right], \tag{6}$$

*for any $0 < \rho < R$.*

To attain an interior Morrey estimate to the weak derivatives of solutions for the Dirichlet problems (1), let us locally observe the following linear elliptic equations

$$\mathcal{L}u := \sum_{i,j=1}^{d} D_{x_j}(a^{ij}(x)D_{x_i}u) = \operatorname{div}\mathbf{f}(x) \qquad \forall x \in B_R \subset\subset \mathbb{R}^d_+, \tag{7}$$

with $\mathbf{f}(x) \in L^{q,\lambda}(B_R)$.

**Theorem 2.5.** *(Interior Morrey estimate) For $1 < p < \infty$ and $0 \le \lambda < d$, there exists a positive constant $\delta = \delta(d, p, \lambda, \nu, \Lambda)$ such that the coefficients $a^{ij}(x)$ satisfying* ***Assumption 1.1*** *and $f \in L^{p,\lambda}(B_R)$. Then any weak solution $u \in W^{1,2}(B_R)$ of (7) satisfies $Du \in L^{p,\lambda}(B_{R/2})$ with the estimate*

$$\|Du\|_{L^{p,\lambda}(B_{R/2})} \le N\Big(R^{-\frac{\lambda}{p}}\|Du\|_{L^p(B_R)} + \|f\|_{L^{p,\lambda}(B_R)}\Big), \tag{8}$$

*where $N = N(d, p, \lambda, \nu, \Lambda, \delta, r_0)$.*

**Proof.** Suppose that $u(x) \in W^{1,2}(B_R)$ is a weak solution of linear elliptic equations (7), and $w \in W^{1,2}(B_R)$ is a weak solution of the following homogeneous Dirichlet problems

$$\begin{cases} \mathcal{L}w = 0 & \text{in } B_R, \\ w = u & \text{on } \partial B_R. \end{cases}$$

Let $\bar{u} = u - w$. Then $\bar{u} \in W_0^{1,2}(B_R)$ is a unique solution of linear elliptic equations $\mathcal{L}\bar{u} = \operatorname{div}\mathbf{f}(x)$. Thanks to the $L^p$ theory of any weak solution $\bar{u} \in W_0^{1,2}(B_R)$ in Lemma 2.1, it yields

$$\|D\bar{u}\|_{L^p(B_{\sigma R})} \le \|D\bar{u}\|_{L^p(B_R)} \le N\|\mathbf{f}\|_{L^p(B_R)} = NR^{\frac{\lambda}{p}}\|\mathbf{f}\|_{L^{p,\lambda}(B_R)} \tag{9}$$

for any $\sigma \in (0, \frac{1}{2})$. In addition, Lemma 2.3 deduces

$$\|Dw\|_{L^q(B_{R/2})} \le NR^{\frac{d}{q}-\frac{d}{p}}\|Dw\|_{L^p(B_R)}. \tag{10}$$

Therefore, by Hölder inequality it follows that

$$\begin{aligned} \|Dw\|_{L^p(B_{\sigma R})} &\le N(\sigma R)^{\frac{d}{p}-\frac{d}{q}}\|Dw\|_{L^q(B_{\sigma R})} \\ &\le N(\sigma R)^{\frac{d}{p}-\frac{d}{q}}\|Dw\|_{L^q(B_{R/2})} \\ &\le N(\sigma R)^{\frac{d}{p}-\frac{d}{q}}R^{\frac{d}{q}-\frac{d}{p}}\|Dw\|_{L^p(B_R)} \\ &= N\sigma^{\frac{d}{p}-\frac{d}{q}}\|Dw\|_{L^p(B_R)} \end{aligned} \tag{11}$$

for the $\sigma$ mentioned in (9).

Since $u = \bar{u} + w$, then by (9) and (11) we deduce that

$$\begin{aligned} \|Du\|_{L^p(B_{\sigma R})} &\le \|Dw\|_{L^p(B_{\sigma R})} + \|D\bar{u}\|_{L^p(B_{\sigma R})} \\ &\le N\Big(\sigma^{\frac{d}{p}-\frac{d}{q}}\|Dw\|_{L^p(B_R)} + R^{\frac{\lambda}{p}}\|\mathbf{f}\|_{L^{p,\lambda}(B_R)}\Big) \\ &\le N\left(\sigma^{\frac{d}{p}-\frac{d}{q}}\Big(\|Du\|_{L^p(B_R)} + \|D\bar{u}\|_{L^p(B_R)}\Big) + R^{\frac{\lambda}{p}}\|\mathbf{f}\|_{L^{p,\lambda}(B_R)}\right) \\ &\le N\Big(\sigma^{\frac{d}{p}-\frac{d}{q}}\|Du\|_{L^p(B_R)} + R^{\frac{\lambda}{p}}\|\mathbf{f}\|_{L^{p,\lambda}(B_R}\Big), \end{aligned}$$

which implies

$$\|Du\|_{L^p(B_r)} \le N\Big(\frac{r}{R}\Big)^{\frac{d}{p}-\frac{d}{q}}\|Du\|_{L^p(B_R)} + NR^{\frac{\lambda}{p}}\|\mathbf{f}\|_{L^{p,\lambda}(B_R)}. \tag{12}$$

We now take $q > \frac{p}{1-(\frac{\lambda}{d})}$ so that $\frac{\lambda}{p} < \frac{d}{p} - \frac{d}{q}$. The iteration argument in Lemma 2.4 yields

$$\|Du\|_{L^p(B_r)} \leq Nr^{\frac{\lambda}{p}}\Big(R^{-\frac{\lambda}{p}}\|Du\|_{L^p(B_R)} + \|\mathbf{f}\|_{L^{p,\lambda}(B_R)}\Big),$$

which implies (8). □

*2.2. Boundary estimates*

With Lemma 2.2 in hand, by using an argument of odd extension with respect to $x'$-hyperplane, one has easily the following conclusion over a half ball, see Theorem 2 in [10].

**Lemma 2.6.** *Let $1 < p < q < \infty$, there exists a positive constant $\delta = \delta(d, p, q, \nu, \Lambda, R_0)$ such that the coefficients $a^{ij}(x)$ satisfying **Assumption 1.1**. If $w \in W^{1,p}(B_R^+)$ is any weak solution of*

$$\begin{cases} \sum_{i,j=1}^d D_{x_j}\Big(a^{ij}(x)D_{x_i}w\Big) = 0 & in \quad B_R^+, \\ w = 0 & on \quad B_R \cap \{x_1 = 0\}. \end{cases} \tag{13}$$

*Then $w \in W^{1,q}(B_{R/2}^+)$ with the estimate*

$$(|Dw|^q)_{B_{R/2}^+}^{\frac{1}{q}} \leq N(|Dw|^p)_{B_R^+}^{\frac{1}{p}}, \tag{14}$$

*where $N = N(d, p, q, \nu, \Lambda, r_0, \delta)$.*

By using the same argument as Lemma 2.3, we can also rewrite the above conclusion in a more convenient version.

**Lemma 2.7.** *Under the same assumptions as Lemma 2.6 above, we have $w \in W^{1,q}(B_{R/2}^+)$ with the estimate*

$$\|Dw\|_{L^q(B_{R/2}^+)} \leq NR^{\frac{d}{q}-\frac{d}{p}}\|Dw\|_{L^p(B_R^+)}, \tag{15}$$

*where $N = N(d, p, q, \nu, \Lambda, \gamma, \delta)$.*

According to Lemma 2.1 with an argument of odd extension and Lemma 2.7, a local version of the Morrey estimate on the boundary point is immediate.

**Theorem 2.8.** *(Boundary Morrey estimate) For $1 < p < \infty$, $0 \leq \lambda < d$, there exists a positive constant $\delta = \delta(d, p, \lambda, \nu, \Lambda)$ such that the coefficients $a^{ij}(x)$ satisfying **Assumption 1.1** and $\boldsymbol{f} \in L^{p,\lambda}(B_R^+)$, then for any weak solution $u \in W^{1,2}(B_R^+)$ of*

$$\begin{cases} \mathcal{L}u := \sum_{i,j=1}^d D_{x_j}\Big(a^{ij}(x)D_{x_i}u\Big) = div\,\boldsymbol{f}(x) & in \quad B_R^+, \\ u = 0 & on \quad B_R \cap \{x_1 = 0\}; \end{cases} \tag{16}$$

*we have $Du \in L^{p,\lambda}(B_{R/2}^+)$ with the estimate*

$$\|Du\|_{L^{p,\lambda}(B_{R/2}^+)} \leq N\Big(R^{-\frac{\lambda}{p}}\|Du\|_{L^p(B_R^+)} + \|\boldsymbol{f}\|_{L^{p,\lambda}(B_R^+)}\Big), \tag{17}$$

*where $N = N(d, p, \lambda, \nu, \Lambda, \delta, r_0)$.*

*2.3. Proof of Theorem 1.4*

With an interior Morrey estimate in Theorem 2.5 and the boundary Morrey estimate in Theorem 2.8 in hand, now we could complete the proof of our main result.

**Proof of Theorem 1.4** Let $R > 0$, $B_R(x_0) \subset\subset \mathbb{R}^d_+$. We note that $\mathcal{L}u = \mathrm{div}\mathbf{f}(x)$ a.e. $x \in B_R(x_0)$, by interior Morrey estimates (8) we get

$$\|Du\|_{L^{p,\lambda}(B_{R/2}(x_0))} \le N\Big(R^{-\frac{\lambda}{p}}\|Du\|_{L^p(\mathbb{R}^d_+)} + \|\mathbf{f}\|_{L^{p,\lambda}(\mathbb{R}^d_+)}\Big). \tag{18}$$

By using the $L^p$ estimate of the gradient of any weak solution for the Dirichlet problems (1) over above half-space, and the embedding $L^{p,\lambda}(\mathbb{R}^d_+) \subset L^p(\mathbb{R}^d_+)$ for $0 \le \lambda < d$, it yields

$$\|Du|_{L^p(\mathbb{R}^d_+)} \le N\,\|\mathbf{f}(x)\|_{L^p(\mathbb{R}^d_+)} \le N\,\|\mathbf{f}(x)\|_{L^{p,\lambda}(\mathbb{R}^d_+)},$$

where constant $N$ is independent of $R$. Putting it into (18) deduces

$$\|Du\|_{L^{p,\lambda}(B_{R/2}(x_0))} \le N(R^{-\frac{\lambda}{p}} + 1)\,\|\mathbf{f}\|_{L^{p,\lambda}(\mathbb{R}^d_+)}. \tag{19}$$

We also handle Morrey estimates on the boundary setting in the same way, and get

$$\|Du\|_{L^{p,\lambda}(B^+_{R/2}(x_0))} \le N(R^{-\frac{\lambda}{p}} + 1)\,\|\mathbf{f}\|_{L^{p,\lambda}(\mathbb{R}^d_+)}. \tag{20}$$

Putting estimates (19) and (20) together, then we take $R \to +\infty$, which implies (4). □

**Remark 2.9.** *Similar to Lieberman's argument in [20], we consider the following nonzero boundary data* $\varphi \in W^1_{p,\lambda}(\mathbb{R}^d_+)$ *of the Dirichlet problem or the conormal derivative problem to Equations (1)*

$$\begin{cases} \mathcal{L}u = div\,f(x) & x \in \mathbb{R}^d_+, \\ u = \varphi(x)\ or\ a^{ij}(x)D_{x_i}u \cdot n^j = \varphi(x) & x \in \partial\mathbb{R}^d_+; \end{cases} \tag{21}$$

*then we have*

$$\|Du\|_{L^{p,\lambda}(\mathbb{R}^d_+)} \le N\Big(\|f\|_{L^{p,\lambda}(\mathbb{R}^d_+)} + \|\nabla\varphi\|_{L^{p,\lambda}(\mathbb{R}^d_+)}\Big),$$

*with* $N = N(d, p, \lambda, \nu, \Lambda, \delta, r_0)$.

## 3. Conclusions

In this paper, we provide another proof to global Morrey estimate for the Dirichlet problems of linear elliptic equations with small partially BMO coefficients in a half space. Instead of the weighted $L^p$ estimate with a special weight, we mainly realize the global Morrey estimates by a direct argument based on the given $L^p$ estimate of their derivatives. This is a flexible approach, as Lieberman indicates in his paper [20], which has the advantage that it is easily applied to any equations with non-VMO coefficients which have $L^p$ estimates.

## References

[1] E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, Duke Mathematical Journal 136 (2) (2007) 285–320.
[2] S. S. Byun, D. Palagachev, Weighted $L^p$-estimates for elliptic equations with measurable coefficients in nonsmooth domains, Potential Analysis 41 (2014) 51–79.
[3] S. S. Byun, L. H. Wang, Elliptic equations with BMO coefficients in Reifenberg domains, Communications on Pure and Applied Mathematics 57 (10) (2004) 1283–1310.

[4] S. S. Byun, L. H. Wang, Elliptic equations with measurable coefficients in Reifenberg domains, Advances in Mathematics 225 (5) (2010) 2648–2673.

[5] L. A. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, Communications on Pure and Applied Mathematics 51 (1) (1998) 1–21.

[6] F. Chiarenza, M. Frasca, P. Longo, Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients, Ricerche di Matematica 40 (1991) 149–168.

[7] F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$ solvability of the Dirichlet problem for nonlinear elliptic equations with VMO coefficients, Transactions of the American Mathematical Society 336(2)(1993) 841–853.

[8] M. Chipot, D. Kinderlehrer, G. Vergara-Caffarelli, Smoothness of linear laminates, Archive for Rational Mechanics and Analysis 96 (1986) 81–96.

[9] G. Di Fazio, M. A. Ragusa, Interior estimate in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, Journal of Functional Analysis 112(2)(1993) 241–256.

[10] H. J. Dong, D. Kim, Elliptic equations in divergence form with partially BMO coefficients, Archive for Rational Mechanics and Analysis 196 (2010) 25–70.

[11] H. J. Dong, D. Kim, Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains, Journal of Functional Analysis 261 (2011) 3279–3327

[12] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Annals of Mathematics Studies, Princeton University Press, Vol.105, 1983.

[13] V. S. Guliyev, M. N. Omarova, M. A. Ragusa, A. Scapellato, Commutators and generalized local Morrey spaces, Journal of Mathematical Analysis and Applications 457 (2)(2018) 1388–1402.

[14] D. Kim, N. V. Krylov, Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others, SIAM Journal on Mathematical Analysis 39(2007) 489–506.

[15] N. V. Krylov, Parabolic and elliptic equations with VMO coefficients, Communications in Partial Differential Equations 32 (1-3)(2007) 453–475.

[16] N. V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, American Mathematical Society, Providence, RI, 2008.

[17] N. V. Krylov, Second-order elliptic equations with variably partially VMO coefficients, Journal of Functional Analysis 257 (2009) 1695–1712.

[18] S. Leonardi, Gradient estimates below duality exponent for a class of linear elliptic systems, Nonlinear Differential Equations and Applications 18 (3)(2011) 237–254.

[19] S. Leonardi, J. Stará, Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO coefficients and $L^{1,\lambda}$ data, Complex Variables and Elliptic Equations 56 (12)(2011) 1085–1098.

[20] G. M. Lieberman, A mostly elementary proof of Morrey space estimates for elliptic and parabolic equations with VMO coefficients, Journal of Functional Analysis 201(2003) 457–479.

[21] Y. Y. Li, L. Nirenberg, Estimates for elliptic systems from composite material, Communications on Pure and Applied Mathematics 56 (2003) 892–925.

[22] M. A. Ragusa, A. Tachikawa, On continuity of minimizers for certain quadratic growth functionals, Journal of the Mathematical Society of Japan 57 (3)(2005) 691–700.

[23] M. A. Ragusa, A. Tachikawa, On interior regularity of minimizers of $p(x)$-energy functionals, Nonlinear Analysis 93 (2013) 162–167.

[24] D. Sarason, Functions of vanishing mean oscillation, Trans. Transactions of the American Mathematical Society 207(1975) 391–405.

[25] A. Scapellato, On some qualitative results for the solution to a Dirichlet problem in Local Generalized Morrey Spaces, AIP Conference Proceedings 1789 (2017) Article Number: UNSP 020138, doi: 10.1063/1.4972730.

[26] A. Scapellato, Some properties of integral operators on generalized Morrey spaces, AIP Conference Proceedings 1863 (2017) Article number 510004, doi: 10.1063/1.4992662.

[27] L. Tang, Weighted $L_p$ solvability for parabolic equations with partially BMO coefficients and its applications, Journal of the Australian Mathematical Society 96(2014) 396–428.

[28] N. N. Ural'tseva, On impossibility of $W^{2,p}$ estimates for multidimensional elliptic equations with discontinuous coefficients, Zap Nauchn Seminarov LOMI im Steklova, t. 5, "Nauka," Leningrad, 1967.