Filomat 32:4 (2018), 1447–1453
https://doi.org/10.2298/FIL1804447L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete Moment Convergence for Sung's Type Weighted Sums of ρ^*-Mixing Random Variables

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Abstract. In this paper, the authors study a complete moment convergence result for Sung's type weighted sums of ρ^*-mixing random variables. This result extends and improves the corresponding theorem of Sung [S.H. Sung, Complete convergence for weighted sums of ρ^*-mixing random variables, Discrete Dyn. Nat. Soc. 2010 (2010), Article ID 630608, 13 pages].

1. Introduction and Main Result

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ an array of real numbers. The limiting behaviors for weighted sums $\sum_{i=1}^n a_{ni}X_i$ have been studied by many authors. We refer to Bai and Cheng [1], Chen and Gan [6], Chen *et al.* [9], Cuzick [11], Sung [18, 19], Wu [26], and Zhang [28], and so on. Since many useful linear statistics, such as least squares estimators, nonparametric regression function estimators and jackknife estimators, are of the form of the weighted sums, so it is interesting and meaningful to study the limiting behaviors for them.

Recently, Sung [19] obtained a complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables (we call Sung's type weighted sums).

Theorem A. Let $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$ and $E|X|^p < \infty$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers with

$$\sup_{n\geq 1} n^{-1} \sum_{i=1}^n |a_{ni}|^q < \infty \tag{1.1}$$

for some $q > p$. Then

$$\sum_{n=1}^\infty n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^j a_{ni}X_i\right| > \varepsilon n^\alpha\right) < \infty,\ \forall\ \varepsilon > 0. \tag{1.2}$$

2010 *Mathematics Subject Classification*. Primary 60F15
Keywords. ρ^*-mixing random variables; Complete convergence; Complete moment convergence; Sung's type weighted sum.
Received: 02 February 2016; Accepted: 10 September 2016
Communicated by Miljana Jovanović
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Conversely, if (1.2) holds for any array $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > p$, then $E|X|^p < \infty$.

Set $a_{ni} = 1$ for all $1 \le i \le n$ and $n \ge 1$. Then (1.1) holds for any $q > 0$ and therefore the weighted sums include the partial sums. Set $a_{ni} = 1$ if $1 \le i \le n-1$ and $a_{nn} = n^{1/q}$ for some $q > 0$. Then (1.1) holds, meanwhile (1.1) does not hold for any $q' > q$, and obviously the weights are unbounded in this case. So the weights satisfying (1.1) are very general. But very few authors continue to study the kind of weighted sums except Zhang [28] who obtained Theorem A for END random variables.

In this paper, we will continue to discuss the complete moment convergence for Sung's type weighted sums of ρ^*-mixing random variables, which is more exact than Theorem A.

Firstly, we introduce some concepts.

Definition 1.1. (1) A sequence $\{Y_n, n \ge 1\}$ of random variables is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P\{|Y_n - \theta| > \varepsilon\} < \infty, \ \forall\, \varepsilon > 0.$$

(2) A sequence $\{Y_n, n \ge 1\}$ of random variables is said to converge completely to a constant θ in the mean of q-th moment for some $q > 0$, if

$$\sum_{n=1}^{\infty} E\{|Y_n - \theta| - \varepsilon\}_+^q < \infty, \ \forall\, \varepsilon > 0,$$

where and in the following, x_+ means $\max\{0, x\}$.

The concept of complete convergence was introduced by Hsu and Robbins [13] and the one of complete moment convergence is due to Chow [10]. It is easy to show that the complete moment convergence implies the corresponding complete convergence. The complete convergence and complete moment convergence have attracted many authors. We refer to Bai and Su [2], Baum and Katz [3], Deng *et al.* [12], Li and Spătaru [14], Katz [15], Rosalsky *et al.* [17], Wang *et al.* [22], Wang and Hu [23], Wang and Su [25], and their references.

Definition 1.2. Let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. For any $S \subset N = \{1, 2, \cdots\}$, define $\mathcal{F}_S = \sigma(X_i, i \in S)$. Given two σ-algebra $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}$, put

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\left\{ \frac{EXY - EXEY}{\sqrt{E(X-EX)^2 E(Y-EY)^2}} : X \in L_2(\mathcal{A}), Y \in L_2(\mathcal{B}) \right\}.$$

Define the ρ^*-mixing coefficients by

$$\rho_n^* = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : S, T \subset N \text{ with } \mathrm{dist}(S, T) \ge n\},$$

where $\mathrm{dist}(S, T) = \inf\{|s - t| : s \in S, t \in T\}$. Obviously, $0 \le \rho_{n+1}^* \le \rho_n^* \le \rho_0^* = 1$. Then the sequence $\{X_n, n \ge 1\}$ is called ρ^*-mixing if there exists $k \in N$ such that $\rho_k^* < 1$.

A number of limit results for ρ^*-mixing sequence of random variables have been established by many authors. We refer to Bradley [4] for the central limit theorem, Bryc and Smolenski [5], Peligrad and Gut [16], and Utev and Peligrad [21] for the moment inequalities, Chen and Liu [8] (see Remak 1 on page 289) for the complete moment convergence, and Sung [19], Wang *et al.* [24], and Wu *et al.* [27] for the complete convergence of weighted sums.

Now we state the main result. Some auxiliary lemmas and the proof of the main result will be detailed in the next section.

Theorem 1.1. Let $p > 1/\alpha$, $1/2 < \alpha \leq 1$ and $0 < v < p$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$ and $E|X|^p < \infty$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers with (1.1) for some $q > p$. Then

$$\sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_{+}^{v} < \infty, \quad \forall\, \varepsilon > 0. \tag{1.3}$$

Conversely, if (1.3) holds for any array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfying (1.1) for some $q > p$, then $EX = 0$ and $E|X|^p < \infty$.

Remark 1.1. Theorem A and Theorem 1.1 do not discuss the very interesting case of $p = 1/\alpha$. We guess that Theorem A and Theorem 1.1 are also true when $p\alpha = 1$. But, we can not prove them by using the method of the proof of Theorem A or Theorem 1.1.

Remark 1.2. For the case $v \geq p$, it is still unknown whether Theorem 1.1 holds or not under the corresponding moment conditions of Lemma 2.2.

Remark 1.3. Sung [20] gave a generalized method to prove the complete moment convergence. But Theorem 1.1 can not follow from the results in Sung [20].

Throughout this paper, C always stands for a positive constant which may differ from one place to another.

2. Lemmas and Proofs

To prove the main result, we need the following lemmas. The first one is due to Utev and Peligrad [21].

Lemma 2.1. Let $r \geq 2$, $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $E|X_n|^r < \infty$ for every $n \geq 1$. Then for all $n \geq 1$,

$$E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^r \leq C_r\left\{\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{r/2}\right\},$$

where $C_r > 0$ depends only on r and the ρ^*-mixing coefficients.

Lemma 2.2. Let $p > 1/\alpha$, $1/2 < \alpha \leq 1$ and $v > 0$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$ and

$$\begin{cases} E|X|^p < \infty, & \text{if } v < p, \\ E|X|^p \log(1 + |X|) < \infty, & \text{if } v = p, \\ E|X|^v < \infty, & \text{if } v > p. \end{cases}$$

Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers with $|a_{ni}| \leq 1$ for $1 \leq i \leq n$ and $n \geq 1$. Then (1.3) holds.

Proof. The proof is similar to that of Chen and Liu [8]. So we omit the detail. □

Checking the arguments of (2.15)-(2.17) and (2.21)-(2.23) in Sung [19] carefully, we have the following two lemmas.

Lemma 2.3. Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let Y be a random variable with $E|Y|^p < \infty$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers with

$$\sup_{n\ge1} n^{-1} \sum_{i=1}^{n} |a_{ni}|^q \le 1 \tag{2.1}$$

for some $q > p$ and $a_{ni} = 0$ or $|a_{ni}| > 1$. Then there exists a positive constant C_0 without depending on Y such that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|a_{ni}Y| > n^\alpha) \le C_0 E|Y|^p.$$

Lemma 2.4. Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let Y be a random variable with $E|Y|^p < \infty$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers with (2.1) for some $q > p$ and $a_{ni} = 0$ or $|a_{ni}| > 1$. Then there exists a positive constant C_1 without depending on Y such that

$$\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} \sum_{i=1}^{n} E|a_{ni}Y|^r I(|a_{ni}Y| \le n^\alpha) \le C_1 E|Y|^p,$$

where $r > \max\{2(p\alpha-1)/(2\alpha-1), q\}$ if $p \ge 2$ and $r = 2$ if $p < 2$.

Lemma 2.5. Let $p > 1/\alpha$, $1/2 < \alpha \le 1$ and $0 < v < p$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$ and $E|X|^p < \infty$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers with (2.1) for some $q > p$ and $a_{ni} = 0$ or $|a_{ni}| > 1$. Then

$$\sum_{n=1}^{\infty} n^{p\alpha-2} \int_1^\infty P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^\alpha x^{1/v}\right) dx < \infty. \tag{2.2}$$

Proof. Set $Y_{ni}(x) = X_i I(|a_{ni}X_i| \le n^\alpha x^{1/v})$ for $1 \le i \le n$ and $n \ge 1$. It is easy to show that

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^\alpha x^{1/v}\right) \le P\left(\max_{1\le i\le n}|a_{ni}X_i| > n^\alpha x^{1/v}\right) + P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}(x)\right| > n^\alpha x^{1/v}\right).$$

Therefore, in order to (2.2) it is enough to prove that

$$I_1 = \sum_{n=1}^{\infty} n^{p\alpha-2} \int_1^\infty P(\max_{1\le i\le n}|a_{ni}X_i| > n^\alpha x^{1/v})dx < \infty$$

and

$$I_2 = \sum_{n=1}^{\infty} n^{p\alpha-2} \int_1^\infty P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}(x)\right| > n^\alpha x^{1/v}\right) dx < \infty.$$

We first prove that $I_1 < \infty$. Taking $Y = X/x^{1/v}$ in Lemma 2.3, we have

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{p\alpha-2} \int_1^\infty \sum_{i=1}^{n} P(|a_{ni}X| > n^\alpha x^{1/v})dx \\
&= \int_1^\infty \left(\sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|a_{ni}X| > n^\alpha x^{1/v})\right) dx \\
&\le C \int_1^\infty E|X/x^{1/v}|^p dx = CE|X|^p \int_1^\infty x^{-p/v} dx < \infty.
\end{aligned}$$

Now we prove that $I_2 < \infty$. Note that by $EX = 0$, $E|X|^p < \infty$, (2.1) and Hölder's inequality,

$$\begin{aligned}\sup_{x\ge 1} n^{-\alpha}x^{-1/v}\max_{1\le j\le n}|\sum_{i=1}^{j} a_{ni}EY_{ni}(x)| &\le \sup_{x\ge 1} n^{-\alpha}x^{-1/v}\sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| > n^{\alpha}x^{1/v}) \\ &= \sup_{x\ge 1} n^{-\alpha}x^{-1/v}\sum_{i=1}^{n} E[(|a_{ni}X|^p\cdot|a_{ni}X|^{1-p})I(|a_{ni}X| > n^{\alpha}x^{1/v})] \\ &\le E|X|^p\cdot\sup_{x\ge 1} n^{-\alpha p}x^{-p/v}\sum_{i=1}^{n}|a_{ni}|^p \\ &\le E|X|^p\cdot n^{1-p\alpha}\to 0\end{aligned}$$

as $n\to\infty$. Therefore, to prove $I_2 < \infty$, it is enough to prove that

$$I_2^* = \sum_{n=1}^{\infty} n^{p\alpha-2}\int_1^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}[Y_{ni}(x) - EY_{ni}(x)]\right| > n^{\alpha}x^{1/v}/2\right) < \infty.$$

By Markov's inequality and Lemma 2.1, we have that for any $r \ge 2$,

$$\begin{aligned}I_2^* &\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\int_1^{\infty} x^{-r/v}(\sum_{i=1}^{n}|a_{ni}|^2E|Y_{ni}(x)|^2)^{r/2}dx + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\int_1^{\infty} x^{-r/v}\sum_{i=1}^{n}|a_{ni}|^rE|Y_{ni}(x)|^r dx \\ &= CI_{21}^* + CI_{22}^*.\end{aligned}$$

If $p \ge 2$, we choose r such that $r > \max\{2(p\alpha-1)/(2\alpha-1), q\}$. Then $E|X|^2 < \infty$ and hence we have

$$I_{21}^* \le (E|X|^2)^{r/2}\sum_{n=1}^{\infty} n^{p\alpha+r/2-r\alpha-2}\int_1^{\infty} x^{-r/v}dx < \infty.$$

Taking $Y = X/x^{1/v}$ in Lemma 2.4, we also have

$$\begin{aligned}I_{22}^* &= \int_1^{\infty}\left(\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n}|a_{ni}|^rE|Y_{nk}(x)/x^{1/v}|^r\right)dx \\ &\le C\int_1^{\infty} E|X/x^{1/v}|^p dx \\ &= CE|X|^p\int_1^{\infty} x^{-p/v}dx < \infty.\end{aligned}$$

If $p < 2$, we choose $r = 2$. In this case, $I_{21}^* = I_{22}^*$. By Lemma 2.4 again, $I_{21}^* = I_{22}^* < \infty$. So we complete the proof. □

Proof of Theorem 1.1. Sufficiency. Without loss of generality, we can assume that $\sum_{i=1}^{n}|a_{ni}|^q \le n$ for all $n \ge 1$. Set $a'_{ni} = a_{ni}I(|a_{ni}| \le 1)$ and $a''_{ni} = a_{ni}I(|a_{ni}| > 1)$ for $1 \le i \le n$ and $n \ge 1$. Then $a_{ni} = a'_{ni} + a''_{ni}$. By the monotonicity of x_+ and the elementary inequality $(|a| + |b| - 2\varepsilon)_+ \le (|a| - \varepsilon)_+ + (|b| - \varepsilon)_+$, we have

$$\begin{aligned}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - 2\varepsilon n^{\alpha}\right\}_+ &\le \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a'_{ni}X_i\right| + \max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - 2\varepsilon n^{\alpha}\right\}_+ \\ &\le \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a'_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+ + \left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+ .\end{aligned}$$

Hence, by the C_r-inequality and Lemma 2.2, to prove (1.3), it is enough to prove that

$$\sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+^{v} < \infty, \ \forall\, \varepsilon > 0.$$

Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+^{v} \\
&= \sum_{n=1}^{\infty} n^{p\alpha-2}\int_0^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha} > n^{\alpha}x^{1/v}\right)dx \\
&= \sum_{n=1}^{\infty} n^{p\alpha-2}\int_0^{1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha} > n^{\alpha}x^{1/v}\right)dx + \sum_{n=1}^{\infty} n^{p\alpha-2}\int_1^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| - \varepsilon n^{\alpha} > n^{\alpha}x^{1/v}\right)dx \\
&\le \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| > \varepsilon n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{p\alpha-2}\int_1^{\infty} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a''_{ni}X_i\right| > n^{\alpha}x^{1/v}\right)dx.
\end{aligned}$$

Hence, we have the desired result by Theorem A and Lemma 2.5.

Necessity. Note that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+^{v} \\
&\ge \sum_{n=1}^{\infty} n^{(p-v)\alpha-2}\int_0^{\varepsilon^v n^{v\alpha}} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > x^{1/v}\right)dx \\
&= \varepsilon^{v}\sum_{n=1}^{\infty} n^{(p-v)\alpha-2}\int_0^{n^{v\alpha}} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > \varepsilon x^{1/v}\right)dx \qquad (2.3)\\
&\ge \varepsilon^{v}\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > 2\varepsilon n^{\alpha}\right).
\end{aligned}$$

Thus, we have $E|X|^p < \infty$ by Theorem A. It remains to show that $EX = 0$. Set $a_{ni} = 1$ for $1 \le i \le n$ and $n \ge 1$. Then $\{a_{ni}\}$ satisfies (1.1). We have by (2.3) that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \ \forall\, \varepsilon > 0. \qquad (2.4)$$

Since $E|X|^p < \infty$, we also have by the sufficiency and (2.3) that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} (X_i - EX_i)\right| > \varepsilon n^{\alpha}\right) < \infty, \ \forall\, \varepsilon > 0. \qquad (2.5)$$

Combining (2.4) and (2.5) gives $EX = 0$. □

Acknowledgments

The authors would like to thank the referee for the helpful comments. The research of Wei Li is supported by National Natural Science Foundation of China (No. 61374067). The research of Pingyan Chen is supported by the National Natural Science Foundation of China (No. 11271161). The research of Soo Hak Sung is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014R1A1A2058041).

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