Filomat 32:4 (2018), 1193–1198
https://doi.org/10.2298/FIL1804193K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Simplified Constructions of almost Peripheral Graphs and Improved Embeddings into them

Sandi Klavžar[a,b,c], Kishori P. Narayankar[d], Srishti P. Narayankar[d], D. Shubhalakshmi[d]

[a]*Faculty of Mathematics and Physics, University of Ljubljana*
[b]*Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia*
[c]*Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[d]*Department of Mathematics, Mangalore University, Mangalagangothri, Mangalore-574199, India*

Abstract. The center and the periphery of a graph are the sets of vertices with minimum and maximum eccentricity, respectively. A graph is called almost peripheral (AP) if all its vertices but one lie in the periphery. The r-AP index $\mathrm{AP}_r(G)$ of a graph G is the smallest number of vertices needed to add to G to obtain an r-AP graph in which G lies as an induced subgraph. In this paper new, simplified constructions of AP graphs are presented. It is proved that if $r \geq 2$ and $n \geq 2$, then $AP_r(K_n) \leq 4r - 3$. Moreover, if G is not complete and has at least three vertices, then $AP_r(G) \leq 4r - 4$. In this way the previously best know bound $AP_r(G) \leq 4r - 2$ is improved.

1. Introduction

Graphs considered in this paper are finite and contain no loops or multiple edges. The distance $d_G(u,v)$ between vertices u and v of a graph G is the shortest path distance. The *eccentricity* $\mathrm{ecc}_G(u)$ of vertex u is $\max\{d_G(u,v) :\ v \in V(G)\}$. The *radius* $\mathrm{rad}(G)$ of G and the *diameter* $\mathrm{diam}(G)$ of G are the minimum and the maximum eccentricity of the vertices of G, respectively. The *center* $C(G)$ and the *periphery* $P(G)$ are the sets of the vertices of G of minimum and maximum eccentricity, respectively.

Central and peripheral vertices of graphs are of great importance in location theory and in investigations of (large) networks. Consequently, different classes of graphs and networks in which the center and the periphery have a special structure were introduced. These classes include self-centered graphs (alias eccentric graphs) [1, 3, 4, 15], their generalization to graphs whose center is a k-distance dominating set [5], and almost self-centered graphs [2, 8, 10]. The latter graphs (as well as almost peripheral graphs) turned out to be extremal graphs for a newly introduced measure of non-self-centrality introduced and studied in [17]. Eccentricity in graphs has also been studied from many additional aspects, cf. [6, 11, 13, 14, 16]. Finally, different derived graphs have been proposed based on the eccentricity such as radial graphs [7] or the recently introduced graphs with a bit unfortunate name "eccentric graphs" (which are not eccentric graphs in the above sense) [12].

2010 *Mathematics Subject Classification*. 05C12; 90B80
Keywords. eccentricity; diameter; almost peripheral graph; induced embedding
Received: 12 December 2016; Accepted: 27 April 2018
Communicated by Francesco Belardo
Research of SK supported by the Slovenian Research Agency (research core funding Nos. N1-0043 and P1-0297). KN and DS thank the Department of Science and Technology (SERB), Government of India, for support through SB/EMEQ-119/2013.
Email addresses: sandi.klavzar@fmf.uni-lj.si (Sandi Klavžar), kishori_pn@yahoo.co.in (Kishori P. Narayankar), shubhaksd@gmail.com (D. Shubhalakshmi)

In this paper we are interested in almost peripheral graphs that were introduced in [9] and in part motivated by location problems in which it is required that most of the resources do not lie in the center. A graph G is called *almost-peripheral*, AP for short, if all but one of its vertices lie in the periphery, that is, if $|P(G)| = |V(G)| - 1$ holds. If G is an AP graph with $\mathrm{rad}(G) = r$ then we will say that G is an *r-AP graph*.

We proceed as follows. In the next section we first give a new construction that from a given r-AP graph and an arbitrary graph produces a new r-AP graphs. Then we present, for any integer $r \geq 1$, an r-AP graph of order $4r - 1$. The present construction is significantly simpler than a related construction given in [10]. Then, in Section 3, we prove that the complete graph K_n can be embedded as a subgraph into an r-AP graph H of order $n + 4r - 3$, and that an arbitrary graph of order $n \geq 3$ that is not complete can be embedded as an induced subgraph into an r-AP graph of order $n + 4r - 4$. This improves the best earlier such embeddings from [10], where the host graph is of order $n+4r-2$. We conclude the paper with a couple of open problems.

Before we start, let us recall some additional concepts and notations needed. If x is a vertex of G, then its closed neighborhood is denoted with $N[x]$. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for any $u, v \in V(H)$. The *vertex deleted d-cube Q_d^-*, $d \geq 1$, is obtained from the d-cube Q_d by removing one of its vertices.

2. New constructions of AP graphs

If G and H are disjoint graphs and $S \subseteq V(G)$, then let $G \oplus_S H$ denote the graph obtained from the disjoint union of G and H by adding a join between S and $V(H)$, that is, adding an edge xy for each $x \in S$ and $y \in V(H)$. In [9, Theorem 2.3] it was proved that if u is the center vertex of an r-AP graph G, $r \geq 1$, then $G \oplus_{\{u\}} H$ is an r-AP graph for any graph H. We now prove a variation of this result that yields a much larger class of AP graphs.

Theorem 2.1. *If u is a peripheral vertex of an r-AP graph G, $r \geq 1$, and H is a graph, then $G \oplus_{N[u]} H$ is an r-AP graph.*

Proof. Let u be an arbitrary vertex of G that is not the center vertex, and set $K = G \oplus_{N[u]} H$. Let x and x' be arbitrary vertices of K and consider the following cases.

Suppose first that $x, x' \in V(G)$. Let P be a shortest x, x'-path. If P lies completely in G, then clearly $d_K(x, x') = d_G(x, x')$. Otherwise P contains a vertex y of H. Let z be the last vertex of P that is still in G and let z' be the first vertex of P after y that lies in G. Then $z, z' \in N[u]$. Since P is a shortest path, we necessarily have $d_G(z, z') = 2$ and consequently P contains the subpath $z - y - z'$. Replacing this subpath of P with $z - u - z'$, a shortest x, x'-path in G is obtained that has the same length as P. In conclusion,

$$d_K(x, x') = d_G(x, x'), \quad x, x' \in V(G). \tag{1}$$

Assume next that $x \in V(G)$, $x \neq u$, and $x' \in V(H)$. Let Q be a shortest x, u-path in G. If u' is the neighbor of u on Q ((it is possible that $u' = x$), then clearly $u' \in N[u]$. Since $u'x' \in E(K)$ we conclude that

$$d_K(x, x') = d_G(x, u), \quad x \in V(G), x \neq u, x' \in V(H). \tag{2}$$

We also clearly have

$$d_K(x, x') \leq 2, \quad x, x' \in V(H). \tag{3}$$

Since u is adjacent to every vertex of H, we infer from (1) that $\mathrm{ecc}_K(u) = \mathrm{ecc}_G(u) = r + 1$. In addition, from (1) and (2) we get that $\mathrm{ecc}_K(x) = \mathrm{ecc}_G(x)$ holds for every vertex $x \in V(G)$, $x \neq u$. In particular, if z is the center vertex of G, then $\mathrm{ecc}_K(z) = \mathrm{ecc}_G(z) = r$. Finally, from (2) and (3) we obtain that $\mathrm{ecc}_K(x) = \mathrm{ecc}_G(u) = r + 1$ holds for every vertex $x \in V(H)$. Hence K is an r-AP graph with $C(K) = \{z\}$. □

In [9] a question was posed whether there exist r-AP graphs of order $n < 4r + 1$ for $r \geq 4$. A positive answer to this problem was given in [10] by demonstrating that for any $r \geq 1$ there exists an r-AP graph of order $4r - 1$. We reprove here this answer with a significantly simpler construction.

Proposition 2.2. *For any integer $r \geq 1$ there exists an r-AP graph of order $4r-1$.*

Proof. For $r = 1$ the path on three vertices is such a graph. Hence assume in the rest of the proof that $r \geq 2$. Let G_r be the graph constructed as follows. Start with a cycle C of length $4r-2$ and label its vertices consecutively with $v_1, v_2, \ldots, v_{4r-2}$. Add a new vertex x and finalize the construction by adding the edges xv_1, xv_{2r-1}, and $v_r v_{3r-1}$. The construction is illustrated in Fig. 1 on the graph $r = 6$.

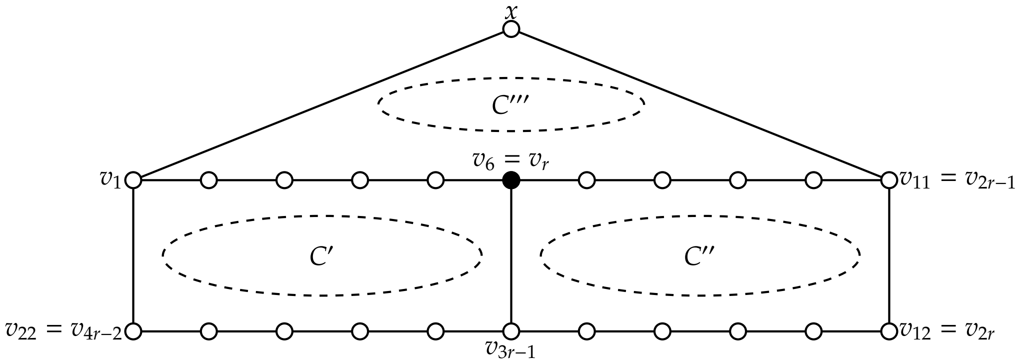

Figure 1: The graph G_6

It is straightforward to check that G_r is an r-AP graph with $C(G_r) = \{v_r\}$. To verify this, define the cycles:

$$\begin{aligned} C' &= v_1 - v_2 - \cdots - v_r - v_{3r-1} - v_{3r} - \cdots v_{4r-2} - v_1 \,, \\ C'' &= v_r - v_{r+1} - \cdots - v_{2r-1} - v_{2r} - v_{2r+1} - \cdots v_{3r-1} - v_r \,, \\ C''' &= v_1 - v_2 - \cdots - v_{2r-2} - v_{2r-1} - x - v_1 \,, \end{aligned}$$

(cf. Fig. 1 again) and note that they are all isometric cycles of length $2r$. Since v_r lies in $C' \cap C'' \cap C'''$ and these three cycles cover G_r we already get that $\mathrm{ecc}(v_r) = r$. To compute the other eccentricity it is useful to observe that also the cycle

$$x - v_{2r-1} - v_{2r} - v_{2r+1} - \cdots - v_{4r-3} - v_{4r-2} - v_1 - x$$

is isometric and that it is of length $2r+2$. So we are left with considering the distances between the vertices from C' and the vertices from C''. It is straightforward to verify that $d_{G_r}(v_i, v_{3r-1-i}) = r+1$ holds for $i = 1, \ldots, r-1$ and that $d_{G_r}(v_i, w) \leq r$ for any other vertex. Hence

$$\mathrm{ecc}(v_i) = \mathrm{ecc}(v_{3r-1-i}) = r+1$$

holds for $i = 1, \ldots, r-1$. By symmetry, the same conclusion holds also for the remaining vertices to be considered. □

Note that the graph G_2 constructed in Proposition 2.2 is the vertex-deleted 3-cube Q_3^-.

3. Embeddings into *r*-AP graphs

If G is a graph and r a positive integer, then the r-AP index $\mathrm{AP}_r(G)$ of G is

$$\mathrm{AP}_r(G) = \min\{|V(H)| - |V(G)| : H \text{ is } r\text{-AP graph}, G \text{ induced in } H\}\,.$$

Clearly, $\mathrm{AP}_r(G) = 0$ if and only if G is an r-AP graph. Moreover, if a graph G does not contain a unique universal vertex (equivalently $\mathrm{AP}_1(G) > 0$), then adding a new vertex and joining it to all vertices of G

yields an 1-AP graph. Consequently $\mathrm{AP}_1(G) \leq 1$ holds for every graph G. For $r \geq 2$ it was proved in [9] that if G is an arbitrary graph on at least two vertices, then

$$\mathrm{AP}_2(G) \leq 5\,, \tag{4}$$

where equality holds if and only if G is a complete graph. It was further shown that for every $r \geq 2$ and every graph G we have $\mathrm{AP}_r(G) \leq 4r+1$. This result was improved in [10] by proving that if G is an arbitrary graph, then

$$\mathrm{AP}_r(G) \leq 4r-2\,. \tag{5}$$

Based on (5) it was asked in [10, Problem 4.1] whether for $r \geq 3$ there exists a graph G with $AP_r(G) = 4r-2$. In the following theorem we answer this problem in negative.

Theorem 3.1. *If $r \geq 2$ and $n \geq 2$ then $AP_r(K_n) \leq 4r-3$. Moreover, if G is not complete and has at least three vertices, then $AP_r(G) \leq 4r-4$.*

Proof. In our construction we will essentially use the graphs G_r, $r \geq 2$, constructed in the proof of Proposition 2.2 (cf. Fig. 1). Let G be a graph and distinguish the following cases.

Case 1: $G = K_n$, $n \geq 2$.
In this case let $H_{r,n}$ be the graph obtained from G_r and (a disjoint copy of) K_n by identifying two vertices of K_n with v_{3r-1} and v_{3r}, respectively. Note that $H_{r,2} = G_r$. The graph $H_{6,5}$ is drawn in Fig. 2.

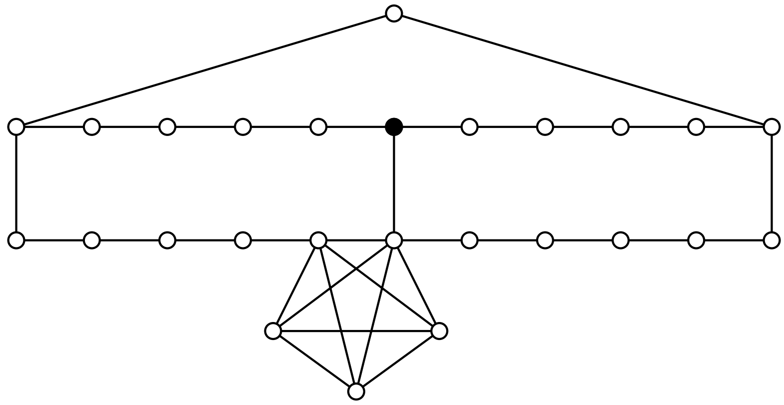

Figure 2: The graph $H_{6,5}$

We claim that $H_{r,n}$ is an r-AP graph. Clearly, the vertices of the complete subgraph K_n of $H_{r,n}$ do not decrease the eccentricities of the vertices from the subgraph G_r of $H_{r,n}$. Moreover, since each vertex of this subgraph is at distance at most r from at least one of the vertices v_{3r-1} and v_{3r} (for instance, $d_{H_{r,n}}(x, v_{3r}) = r$), we have $\mathrm{ecc}_{H_{r,n}}(u) = \mathrm{ecc}_{G_r}(u)$ holds for any vertex u from the subgraph G_r of $H_{r,n}$. Finally, if $v \in V(K_n) \setminus V(G_r)$, then $d_{H_{r,n}}(v, x) = r+1$, so that $\mathrm{ecc}_{H_{r,n}}(v) = r+1$. Hence $H_{r,n}$ is an r-AP graph, where $C(H_{r,n}) = \{v_r\}$. Since $|V(G_r)| = 4r-1$ it follows that $AP_r(K_n) \leq 4r-3$ holds for any $r \geq 2$ and $n \geq 2$.

Case 2: G contains an induced P_3.
Let u, v, and w be the vertices of G that induce P_3, where $uw \notin E(G)$. Let $X(r, G)$ be the graph obtained from G_r and G by identifying the path $u - v - w$ of G with the path $v_{3r} - v_{3r-1} - v_{3r-2}$ and joining v_r by an edge to every vertex of $V(G) \setminus \{u, v, w\}$. The graph $X(r, G)$ is schematically shown in Fig. 3.

The eccentricities of the vertices of G_r do not change in $G_r \circ G(uvw)$. Moreover, if $z \in V(G) \setminus \{u, v, w\}$, then $\mathrm{ecc}_{X(r,G)}(z) = \mathrm{ecc}_{G_r}(v_r) + 1 = r+1$. It follows that Hence $X(r, G)$ is an r-AP graph and since $|V(G_r)| = 4r-1$ we conclude that $AP_r(G) \leq (4r-1) - 3 = 4r-4$.

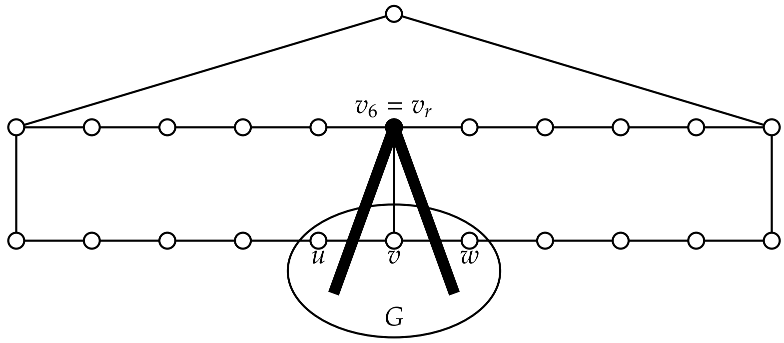

Figure 3: The graph $X(r, G)$

The remaining case to consider is:

Case 3: G is a disjoint union of complete graphs.
Let G be the disjoint union of $K_{n_1}, \ldots, K_{n_k}$, where $n_1 \leq \cdots \leq n_k$. Then we distinguish the following subcases.

Suppose first that $n_k = 1$, that is, G is edge-less. Since $|V(G)| \geq 3$, identify three vertices of G with three independent vertices of G_r pairwise different from v_r, and connect all the other isolated vertices of G with v_r. In this way G is embedded into an r-AP graph and hence $AP_r(G) \leq 4r - 4$.

If $n_k = 2$ and $n_{k-1} = 1$ (or if $n_k = 2$ and $n_{k-1} = 2$), then identify the K_2 and one independent vertex (or two K_2's, respectively) of G with a corresponding induced subgraph of G_r and connect all the other vertices of G with v_r to reach the same conclusion.

Finally let $n_k \geq 3$. In this subcase construct first the graph H_{r,n_1} as in Case 1. Further, if $n_{k-1} \geq 2$, then identify one vertex of the component $K_{n_{k-1}}$ with v, and connect all the vertices of the other $k - 2$ components with v. Otherwise we have $n_{k-1} = 1$, in which case identify one isolated vertex with a vertex of G_r independent from the vertices v_r, v_{3r-1}, and v_{3r}, and connect all the other isolated vertices with v_r.

The proof is completed by observing that in all the cases we have constructed r-AP graphs and consequently $AP_r(G) \leq 4r - 4$. □

4. Concluding remarks

Note that the inequality (4) and its equality case imply that Theorem 3.1 is best possible for $r = 2$. Hence we pose:

Problem 4.1. *Is Theorem 3.1 best possible for $r \geq 3$? More precisely:*

(i) Is it true that $AP_r(K_n) = 4r - 3$ for $r \geq 3$ and $n \geq 2$?
(ii) Let $r \geq 3$. Does there exist a non complete graph X_r such that $AP_r(X_r) = 4r - 4$?

The constructions from Theorem 3.1 in many cases yield graphs AP graphs with cut vertices. It would be interesting to see if there exist related embeddings into 2-connected AP graphs.

References

[1] J. Akiyama, K. Ando, D. Avis, Eccentric graphs, Discrete Math. 56 (1985) 1–6.
[2] K. Balakrishnan, B. Brešar, M. Changat, S. Klavžar, I. Peterin, A.R. Subhamathi, Almost self-centered median and chordal graphs, Taiwanese J. Math. 16 (2012) 1911–1922.
[3] F. Buckley, Self-centered graphs, Ann. New York Acad. Sci. 576 (1989) 71–78.
[4] G. Chartrand, W. Gu, M. Schultz, S. J. Winters, Eccentric graphs, Networks 34 (1999) 115–121.
[5] D. Coudert, G. Ducoffe, Data center interconnection networks are not hyperbolic, Theoret. Comput. Sci. 639 (2016) 72–90.
[6] A. M. Hinz, D. Parisse, The average eccentricity of Sierpiński graphs, Graphs Combin. 28 (2012) 671–686.
[7] K. M. Kathiresan, G. Marimuthu, Further results on radial graphs, Discuss. Math. Graph Theory 30 (2010) 75–83.

[8] S. Klavžar, K. P. Narayankar, H. B. Walikar, Almost self-centered graphs, Acta Math. Sin. (Engl. Ser.) 27 (2011) 2343–2350.
[9] S. Klavžar, K. P. Narayankar, H. B. Walikar, S. B. Lokesh, Almost peripheral graphs, Taiwanese J. Math. 18 (2014) 463-471.
[10] S. Klavžar, H. Liu, P. Singh, K. Xu, Constructing almost peripheral and almost self-centered graphs revisited, Taiwanese J. Math. 21 (2017) 705–717.
[11] M. Liang, J. Liu, Proofs of conjectures on the Randić index and average eccentricity, Discrete Appl. Math. 202 (2016) 188–193.
[12] A.K. Nagar, S. Sriram, On eccentric connectivity index of eccentric graph of regular dendrimer, Math. Comput. Sci. 10 (2016) 229–237.
[13] J. Sedlar, Remoteness, proximity and few other distance invariants in graphs, Filomat 27 (2013) 1425–1435.
[14] H. Smith, L. Székely, H. Wang, Eccentricity sums in trees, Discrete Appl. Math. 207 (2016) 120–131.
[15] Z. Stanić, Some notes on minimal self-centered graphs, AKCE Int. J. Graphs Comb. 7 (2010) 97–102.
[16] Y. Tang, B. Zhou, Ordering unicyclic graphs with large average eccentricities, Filomat 28 (2014) 207–210.
[17] K. Xu, K. Ch. Das, A. D. Maden, On a novel eccentricity-based invarient of a graph, Acta Math. Sin. (Engl. Ser.) 32 (2016) 1477–1493.