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Fuzzy stabilizers in BL-algebras

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Abstract. The primary goal of this paper is to develop fuzzy stabilizer theory in BL-algebras. Two types of fuzzy stabilizers are introduced and their related properties are given. Also, the relationships between fuzzy stabilizers and several fuzzy filters are discussed. Finally, by means of fuzzy stabilizers, it is proven that the collection of all fuzzy filters in BL-algebras forms a residuated lattice. These results will provide a solid algebraic foundation for the consequence connectives in fuzzy logic.

1. Introduction

BL-algebras [6] are the corresponding algebraic structures for Hájek basic logic. The interval $[0,1]$ with the structure induced by a continuous t-norm is an important example for a BL-algebra. It is well known that MV-algebras [5] are one of most important subclasses of BL-algebras. Moreover, MV-algebras and lattice implication algebras [14], bounded commutative BCK-algebras [12] are categorically equivalent, respectively.

In BL-algebras, the focus is deductive systems also called filters. From the viewpoint of Logic, diverse filters correspond to diverse collections of provable formulas. So far, the filter theory in BL algebras has been extensively researched and related important results have been gained [7, 13]. Especially, Turunnen [13] investigated some properties of (prime) filters of BL-algebras. Inspired by this, Haveshki [7] systematically studied filter theory in BL-algebras including the relations of various kinds of filters and their characterizations. At present, many authors studied a variety of fuzzy filters of BL-algebras [9–11, 13, 15, 16]. For instance, Liu and Li [9, 10] introduced the concepts of fuzzy Boolean (positive implicative) filters of BL-algebras and some characterizations of them were derived.

The notion of stabilizers is from analytic theory, which is helpful for studying structures and properties of algebraic systems. Haveshki [8] first introduced the notion of stabilizers in BL-algebras and investigated some basic properties of them. Also, they discuss the relations between stabilizers and filters in BL-algebras. Inspired by this, Borzooei [4] introduced some new types of stabilizers and determined the relations among stabilizers in BL-algebras, they also showed that fantastic filters and (semi) normal filters are equivalent via stabilizers. Based on the above, we introduce and study two types of fuzzy stabilizers in BL-algebras.

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2. Preliminaries

In the present section, we review some contents that will be used below.

Definition 2.1. [6] A BL-algebra is a structure $(Q, \odot, \to, \vee, \wedge, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ with the below axioms: for all $x, y, z \in Q$,

(1) $(Q, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(Q, \odot, 1)$ is an abelian monoid;
(3) $y \leq x \to z$ iff $y \odot x \leq z$;
(4) $1 = (x \to y) \vee (y \to x)$;
(5) $y \wedge x = y \odot (y \to x)$.

Throughout this paper, a BL-algebra $(Q, \odot, \to, \vee, \wedge, 0, 1)$ will be written by Q, unless otherwise stated. For any $y \in Q$, denote $y^0 = 1, y^n = y \odot y^{n-1}$, where $n \geq 1$.

Proposition 2.2. [6] In a BL-algebra Q, for any $x, y, z \in Q$,

(1) $y \leq x$ iff $y \to x = 1$;
(2) $y \leq x$ implies $y \odot z \leq x \odot z, x \to z \leq y \to z, z \to y \leq z \to x$;
(3) $y \leq x \to y, 1 \to y = y, y \odot x \leq y \wedge x$;
(4) $y \to x = y \vee x \to x, y \to x = y \to y \wedge x$;
(5) $y \to x \wedge z = (y \to x) \wedge (y \to z)$;
(6) $y \vee x \to z = (y \to z) \wedge (x \to z), y \wedge x \to z = (y \to z) \vee (x \to z)$;
(7) $y \vee x = ((y \to x) \to x) \wedge ((x \to y) \to y)$.

Definition 2.3. [6] For any $x, y \in Q$, a BL-algebra Q is called a (an)

(1) Gödel algebra *provided that* $y = y \odot y$;
(2) MV-algebra *provided that* $(y \to x) \to x = (x \to y) \to y$.

Definition 2.4. [9, 10] A fuzzy set λ of Q is said to be a fuzzy filter provided that for all $x, y \in Q$,

(1) $\lambda(1) \geq \lambda(y)$;
(2) $\lambda(y \odot x) \geq \lambda(y) \wedge \lambda(x)$;
(3) $y \leq x$ implies $\lambda(y) \leq \lambda(x)$.

Set $x, y \in Q$. A fuzzy set λ of Q is a fuzzy filter iff $\lambda(1) \geq \lambda(x), \lambda(y) \geq \lambda(x) \wedge \lambda(x \to y)$. Denote by the collection of all fuzzy filters in Q $\mathcal{F}(Q)$ and define the fuzzy filter generated by λ as $\langle\lambda\rangle = \bigcap_{\nu \in \mathcal{F}(Q), \lambda \subseteq \nu} \nu$. Furthermore $\langle\lambda\rangle(x) = \vee\{\wedge\lambda(a_k) | b_1, \cdots b_n \in Q, x \geq b_1 \odot b_2 \cdots b_n\}$. In $\mathcal{F}(Q)$, define $\lambda_1 \leq \lambda_2$ iff $\lambda_1 \subseteq \lambda_2, \lambda_1 \vee \lambda_2 = \langle\lambda_1 \cup \lambda_2\rangle, \lambda_1 \wedge \lambda_2 = \lambda_1 \cap \lambda_2$. Then $(\mathcal{F}(Q), \wedge, \vee, \emptyset, Q)$ is a complete distributive lattice [10, 17].

Definition 2.5. [9, 10] Set $x, y, z \in Q$. A fuzzy filter λ of Q is called a

(1) fuzzy Boolean filter *provided that* $\lambda((y \to x) \to y) \leq \lambda(y)$;
(2) *fuzzy prime filter* provided that $\lambda(y \vee x) = \lambda(y) \vee \lambda(x)$;
(3) *fuzzy fantastic filter* provided that $\lambda(y \to x) \leq \lambda(((x \to y) \to y) \to x))$.

Definition 2.6. [17] A fuzzy equivalent relation R on Q is said to be a *fuzzy congruence* provided that for any $w, t, u, v \in Q$, $R(w, t) \wedge R(u, v) \leq R(w \Xi u, t \Xi v)$, where $\Xi \in \{\to, \wedge, \odot, \vee\}$.

Proposition 2.7. [17] Given a fuzzy congruence R on Q, $R(y, x) = R(1, y \leftrightarrow x)$ whence $y \Leftrightarrow x = (y \to x) \wedge (x \to y), x, y \in Q$.

Proposition 2.8. [17] (1) Given a fuzzy congruence R on Q, $R(1, \cdot)$ is a fuzzy filter of Q.
(2) Given a fuzzy filter λ of Q with $\lambda(1) = 1$, $R(y, x) = \lambda(y \leftrightarrow x)$ where $y, x \in Q$, is a fuzzy congruence, which is called the fuzzy congruence generated by λ.

3. Fuzzy stabilizers in BL-algebras

Definition 3.1. Given a fuzzy set λ of Q and a fuzzy congruence R on Q, the fuzzy left and right stabilizer of λ w.r.t. R are defined as follows: for $x \in Q$,

$$X^l_R(\lambda)(x) = \bigwedge_{y\in Q}\{\lambda(y) \to R(y \to x, x)\},$$
$$X^r_R(\lambda)(x) = \bigwedge_{y\in Q}\{\lambda(y) \to R(x \to y, y)\},$$

where $\to$ is the residuum implication w.r.t. a continuous t-norm.

Notation 3.2. From Definition 3.1, one can see that if S is a classic subset of Q and R is the identity relation Id on Q. Then

$$X^l_{Id}(S) = \{c \in Q | c \to x = x, \textit{for all } x \in S\},$$
$$X^r_{Id}(S) = \{c \in Q | x \to c = c, \textit{for all } x \in S\},$$

are left and right stabilizers of X defined in [8], respectively.

Example 3.3. Set $Q = \{0, x, y, z, u, 1\}$ with $0 \le u \le x \le 1, 0 \le z \le y \le 1$. Consider $\odot, \to$ on Q below:

$\odot$	0	x	y	z	u	1
0	0	0	0	0	0	0
x	0	x	u	0	u	x
y	0	u	z	z	0	y
z	0	0	z	z	0	z
u	0	u	0	0	u	u
1	0	x	y	c	u	1

$\to$	0	x	y	z	u	1
0	1	1	1	1	1	1
x	z	1	y	z	y	1
y	u	x	1	y	x	1
z	x	x	1	1	x	1
u	y	1	1	y	1	1
1	0	x	y	z	u	1

Then $(Q, \odot, \to, \wedge, \vee, 0, 1)$ is a BL-algebra. The fuzzy set λ is defined by $\lambda(1) = 1, \lambda(y) = \lambda(z) = 0.7, \lambda(0) = \lambda(x) = \lambda(u) = 0.3$ and the fuzzy congruence R is generated by λ. Therefore one can compute that $X^l_R(\lambda)(y) = X^l_R(\lambda)(1) = 0.8, X^l_R(\lambda)(0) = X^l_R(\lambda)(x) == X^l_R(\lambda)(z) = X^l_R(\lambda)(u) = 0.5$; $X^r_R(\lambda)(y) = X^r_R(\lambda)(z) = X^r_R(\lambda)(1) = 0.9, X^r_R(\lambda)(0) = X^r_R(\lambda)(x) = X^r_R(\lambda)(u) = 0.4$.

Proposition 3.4. Given a fuzzy congruence R on Q and fuzzy sets μ_1, μ_2 of Q,

(1) $X^l_R(\chi_1) = X^r_R(\chi_1) = Q$;
(2) $X^l_R(\chi_0) = R(1, \cdot)$;
(3) If $\mu_1 \subseteq \mu_2$, then $X^l_R(\mu_2) \subseteq X^l_R(\mu_1)$, $X^r_R(\mu_2) \subseteq X^r_R(\mu_1)$;
(4) $X^l_R(\mu_1 \cup \mu_1) = X^l_R(\mu_1) \cap X^l_R(\mu_2)$, $X^r_R(\mu_1 \cup \mu_2) = X^r_R(\mu_1) \cap X^r_R(\mu_2)$;
(5) $X^l_R(\mu_1 \cap \mu_2) = X^l_R(\mu_1) \cup X^l_R(\mu_2)$, $X^r_R(\mu_1 \cap \mu_2) = X^r_R(\mu_1) \cup X^r_R(\mu_2)$.

Proof. (1) From $X^l_R(\chi_1)(x) = \bigwedge_{z\in L}\{\chi_1(z) \to R(z \to x, x)\} = 1 \to R(1 \to x, x) = 1$, we have $X^l_R(\chi_1) = L$. Moreover, by $X^r_R(\chi_1)(x) = \bigwedge_{z\in L}\{\chi_1(z) \to R(x \to z, z)\} = 1 \to R(1, 1) = 1$. Thus $X^r_R(\chi_1) = L$.

(2) $X^l_R(\chi_0) = \bigwedge_{z\in L}\{\chi_0(z) \to R(z \to x, x)\} = 1 \to R(1, x)$.

(3) If $\mu_1 \subseteq \mu_2$, that is, for any $z \in L$, we have $\mu_1(z) \le \mu_2(z)$. From Proposition 2.2, we have $X^l_R(\mu_2) = \bigwedge_{z\in L}\{\mu_2(z) \to R(z \to x, x)\} \le \bigwedge_{z\in L}\{\mu_1(z) \to R(z \to x, x)\} = X^l_R(\mu_1)$ and $X^r_R(\mu_2) = \bigwedge_{z\in L}\{\mu_2(z) \to R(x \to z, z)\} \le \bigwedge_{z\in L}\{\mu_1(z) \to R(x \to z, z)\} = X^l_R(\mu_1)$. Therefore, if $\mu_1 \subseteq \mu_2$, then $X^l_R(\mu_2) \subseteq X^l_R(\mu_1)$, $X^r_R(\mu_2) \subseteq X^r_R(\mu_1)$.

(4) From Proposition 2.2, we have $X^l_R(\mu_1 \cup \mu_2) = \bigwedge_{z\in L}\{\mu_1(z) \vee \mu_2(z) \to R(z \to x, x)\} = \bigwedge_{z\in L}\{\mu_1(z) \to R(z \to x, x)\} \cap \bigwedge_{z\in L}\{\mu_2(z) \to R(z \to x, x)\} = X^l_R(\mu_1) \cap X^l_R(\mu_2)$. Moreover, one can prove that $X^r_R(\mu_1 \cup \mu_2) = \bigwedge_{z\in L}\{\mu_1(z) \vee \mu_2(z) \to R(x \to z, z)\} = \bigwedge_{z\in L}\{\mu_1(z) \to R(x \to z, z)\} \cap \bigwedge_{z\in L}\{\mu_2(z) \to R(x \to z, z)\} = X^r_R(\mu_1) \cap X^r_R(\mu_2)$.

(5) Similar to (4).

□

Proposition 3.5. Given a fuzzy congruence R on Q and a fuzzy set μ of Q, $X_R^r(\mu)$ is a fuzzy filter of Q.

Proof. (1) $X_R^r(\mu_1)(1) = \bigwedge_{y\in Q}\{\mu(y) \to R(y, 1 \to y)\} = \bigwedge_{y\in Q}\{\mu(y) \to R(y, y)\} = 1 \geq X_R^r(\mu_1)(x)$.

(2) Let $x \leq y$. It follows from Proposition 2.2,2.7,2.8 that $X_R^r(\mu)(x) = \bigwedge_{z\in Q}\{\mu(z) \to R(x \to z, z)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (x \to z) \Leftrightarrow z)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (x \to z) \to z)\} \leq \bigwedge_{z\in Q}\{\mu(z) \to R(1, (y \to z) \to z)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (y \to z) \Leftrightarrow z)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(z, y \to z) = X_R^r(\mu)(y)$.

(3) For $x, y \in Q$, $X_R^r(\mu)(x \odot y) = \bigwedge_{z\in Q}\{\mu(z) \to R(z, (x \odot y) \to z)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(((x \odot y) \to z) \Leftrightarrow z, 1)\} = \bigwedge_{z\in Q}\{\mu(z) \to R((x \to (y \to z)) \Leftrightarrow z, 1)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(((x \odot y) \to z) \to z, 1)\} \geq \bigwedge_{z\in Q}\{\mu(z) \to R(((x \to z) \to z) \odot ((y \to z) \to z), 1) \geq \bigwedge_{z\in Q}\{\mu(z) \to R((x \to z) \to z, 1) \wedge R((y \to z) \to z, 1)\} \geq \bigwedge_{z\in Q}\{\mu(z) \to R((x \to z) \to z, 1)\} \wedge \bigwedge_{z\in Q}\{\mu(z) \to R((y \to z) \to z, 1)\} = \bigwedge_{z\in Q}\{\mu(z) \to R((x \to z, z)\} \wedge \bigwedge_{z\in Q}\{\mu(z) \to R((y \to z, z)\} = X_R^r(\mu)(x) \wedge X_R^r(\mu)(y)$.
□

The below example reveals that $X_R^l(\mu)$ is not a fuzzy filter of Q in general.

Example 3.6. Considering the Example 3.2, easy to verify that $X_R^l(\mu)$ is not a fuzzy filter of Q by $X_R^l(\mu)(y \odot y) = X_R^l(\mu)(z) = 0.5 \leq 0.8 = X_R^l(\mu)(y) \wedge X_R^l(\mu)(y)$.

Proposition 3.7. Given an MV-algebra and a fuzzy set μ of Q, $X_R^r(\mu)$ and $X_R^l(\mu)$ are fuzzy filters of Q.

Proof. (1) $X_R^l(\mu)(1) = \bigwedge_{y\in Q}\{\mu(y) \to R(1, y \to 1)\} = 1 \geq X_R^l(\mu_1)(x)$.

(2) Let $x \leq y$. Then from Proposition 2.2 $X_R^l(\mu)(x) = \bigwedge_{z\in Q}\{\mu(z) \to R(z \to x, x)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (z \to x) \Leftrightarrow x)\} = \bigwedge_{z\in Q}\{\mu(z) \to R((z \to x) \to x, 1)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, z \vee x)\} \leq \bigwedge_{z\in Q}\{\mu(z) \to R(1, z \vee y)\} = \bigwedge_{z\in L}\{\mu(z) \to R(1, (z \to y) \to y)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (z \to y) \Leftrightarrow y)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(z \to y, y)\} = X_R^l(\mu)(y)$.

(3) For any $x, y \in Q$, $X_R^l(\mu)(x \odot y) = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (z \to (x \odot y)) \Leftrightarrow (x \odot y))\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (z \to (x \odot y)) \to (x \odot y))\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, z \vee (x \odot y))\} = \bigwedge_{z\in Q}\{\mu(z) \to R(1, (z \vee x) \odot (z \vee y))\} \geq \bigwedge_{z\in Q}\{\mu(z) \to R(1, z \vee x)\} \wedge \bigwedge_{z\in Q}\{\mu(z) \to R(1, z \vee y)\} = \bigwedge_{z\in Q}\{\mu(z) \to R(z \to x, x)\} \wedge \bigwedge_{z\in G}\{\mu(z) \to R(y, z \to y)\} = X_R^l(\mu)(x) \wedge X_R^l(\mu)(y)$.
□

Definition 3.8. Given two fuzzy sets λ_1, λ_2 of Q, define fuzzy stabilizer of λ_1 w.r.t. λ_2 by

$$X(\lambda_1, \lambda_2)(x) = \bigwedge_{y\in Q}[\lambda_1(y) \to \lambda_2((x \to y) \to y)],$$

where $\to$ is the residuum implication w.r.t. a continuous t-norm.

Example 3.9. Considering the BL-algebra from Example 3.3 and fuzzy sets λ_1, λ_2 of Q where $\lambda_1(0) = \lambda_1(x) = \lambda_1(u) = 0.4, \lambda_1(y) = \lambda_1(z) = 0.8$; $\lambda_2(0) = \lambda_2(x) = \lambda_2(u) = 0.3, \lambda_2(y) = \mu_2(z) = \lambda_2(1) = 0.9$. Easy to calculate that $X(\lambda_1, \lambda_2)(0) = X(\lambda_1, \lambda_2)(x) = X(\lambda_1, \lambda_2)(y) = 0.9, X(\lambda_1, \lambda_2)(y) = X(\lambda_1, \lambda_2)(z) = X(\lambda_1, \lambda_2)(1) = 1$.

In what follows, we discuss the relation between these fuzzy stabilizers and other types of fuzzy filters in Q.

Proposition 3.10. Let $\mu_1, \mu_2, \lambda_1, \lambda_2, \mu_{2i} (i \in I)$ be fuzzy sets and λ be a fuzzy filter of Q. We have:

(1) If $X(\mu_1, \mu_2)(x) = Q$, then $\mu_1 \subseteq \mu_2$;
(2) If $\mu_1 \subseteq \lambda$, then $X(\mu_1, \lambda)(x) = Q$;
(3) $\lambda \subseteq X(\mu_1, \lambda)$;
(4) If $\lambda(1) = 1$, then $X(\chi_1, \lambda) = Q$;
(5) If $\mu_1 \subseteq \lambda_1$ and $\mu_2 \subseteq \lambda_2$, then $X(\lambda_1, \mu_2) \subseteq X(\mu_1, \lambda_2)$;
(6) $X(\mu_1, \cap_{i\in\Lambda}\mu_{2i}) = \cap_{i\in\Lambda}(\mu_1, \mu_{2i})$;
(7) $X(\cup_{i\in\Lambda}\mu_{2i}, \mu_1) = \cap_{i\in\Lambda}(\mu_1, \mu_{2i})$.

Proof. (1) For all $x \in Q, X(\mu_1,\mu_2)(x) = \bigwedge_{y\in Q}[\mu_1(y) \to \mu_2((x \to y) \to y)] = 1$. This implies $\mu_1(x) \to \mu_2((x \to x) \to x) = 1$ and hence $\mu_1(x) \le \mu_2(x)$, that is, $\mu_1 \subseteq \mu_2$.

(2) Considering $X(\mu_1,\lambda)(x) = \bigwedge_{y\in Q}[\mu_1(y) \to \lambda((x \to y) \to y)] \ge \bigwedge_{y\in Q}[\mu_1((x \to y) \to y) \to \lambda((x \to y) \to y)] = 1$, hence $X(\mu_1,\lambda) = Q$.

(3) Since $\mu_1(z) \to \lambda((x \to z) \to z) \ge \lambda((x \to z) \to z) \ge \lambda(x)$, we have $X(\mu_1,\lambda)(x) = \bigwedge_{z\in Q}[\mu_1(z) \to \lambda((x \to z) \to z)] \ge \lambda(x)$. Therefore $\lambda \subseteq X(\mu_1,\lambda)$.

(4) $X(\chi_1,\lambda) = Q$ follows from $X(\chi_1,\lambda)(x) = \bigwedge_{z\in Q}[\chi_1(z) \to \lambda((x \to z) \to z)] = 1$ for all $x \in Q$.

(5) By $X(\lambda_1,\mu_2)(x) = \bigwedge_{z\in Q}[\lambda_1(z) \to \mu_2((x \to z) \to z)] \le \bigwedge_{z\in Q}[\lambda_1(z) \to \lambda((x \to z) \to z)]$ for any $x \in Q$, $X(\lambda_1,\mu_2) \subseteq X(\mu_1,\lambda_2)$.

(6) For $x \in Q$, $X(\mu_1, \cap_{i\in\Lambda}\mu_{2i})(x) = \bigwedge_{z\in Q}[\mu_1(z) \to \bigwedge_{i\in\Lambda}\mu_{2i}((x \to z) \to z)] = \bigwedge_{z\in Q}\bigwedge_{i\in\Lambda}[\mu_1(z) \to \mu_{2i}((x \to z) \to z)] = \bigwedge_{i\in\Lambda}\bigwedge_{z\in Q}[\mu_1(z) \to \mu_{2i}((x \to z) \to z)] = \cap_{i\in\Lambda}(\mu_1,\mu_{2i})(x)$.

(7) For $x \in Q$, $X(\cup_{i\in\Lambda}\mu_{2i},\mu_1)(x) = \bigwedge_{z\in Q}[\bigvee_{i\in\Lambda}\mu_{2i}(z) \to \mu_2((x \to z) \to z)] = \bigwedge_{z\in Q}\bigvee_{i\in\Lambda}[\mu_{2i}(z) \to \mu_2((x \to z) \to z)] = \mu_{2i}(z) \to \mu_2((x \to z) \to z)] = \cap_{i\in\Lambda}(\mu_1,\mu_{2i})(x)$.

□

Proposition 3.11. Let μ_1,μ_2 be fuzzy sets of Q.

(1) If μ_2 is a fuzzy filter of Q, then $X(\mu_1,\mu_2)$ is also a fuzzy filter of Q;
(2) If μ_2 is a fuzzy prime filter of Q, then $X(\mu_1,\mu_2)$ is also a fuzzy prime filter of Q;
(3) If μ_2 is a fuzzy fantastic filter of Q, then $X(\mu_1,\mu_2)$ is also a fuzzy fantastic filter of Q;
(4) If μ_2 is a fuzzy Boolean filter of Q, then $X(\mu_1,\mu_2)$ is also a fuzzy Boolean filter of Q.

Proof. (1) Suppose that μ_2 is a fuzzy filter of Q. Hence $X(\mu_1,\mu_2)(1) = \bigwedge_{y\in Q}[\mu_1(y) \to \mu_2((1 \to y) \to y)] = \bigwedge_{y\in Q}[\mu_1(y) \to \mu_2(1)] \ge \bigwedge_{y\in Q}[\mu_1(y) \to \mu_2((x \to y) \to y)] = X(\mu_1,\mu_2)(x)$. If $x \le y$, we have $X(\mu_1,\mu_2)(x) = \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2((x \to z) \to z)] \le \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2((y \to z) \to z)] = X(\mu_1,\mu_2)(y)$. Furthermore, for any $x, y \in Q$, $X(\mu_1,\mu_2)(x \odot y) = \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2(((x \odot y) \to z) \to z)] \ge \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2(((x \to z) \to z)\odot((y \to z) \to z))] \ge \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2(((x \to z) \to z)\wedge\mu_2((y \to z) \to z))] \ge \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2(((x \to z) \to z)) \wedge (\mu_1(z) \to \mu_2((y \to z) \to z))] \ge \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2(((x \to z) \to z))] \wedge \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((y \to z) \to z))] = X(\mu_1,\mu_2)(x) \wedge X(\mu_1,\mu_2)(y)$. This proof is complete.

(2) Suppose that μ_2 is a fuzzy prime filter of Q. Hence $X(\mu_1,\mu_2)(x \vee y) = \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((x \vee y \to z) \to z)] = \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((x \to z) \to z) \vee ((y \to z) \to z)] = \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((x \to z) \to z) \vee \mu_2((y \to z) \to z)] = \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2((x \to z) \to z)] \vee \bigwedge_{z\in Q}[\mu_1(z) \to \mu_2((y \to z) \to z)] = X(\mu_1,\mu_2)(x) \vee X(\mu_1,\mu_2)(y)$. By (1) this proof is complete.

(3) Assume that μ_2 is a fuzzy fantastic filter of Q. Thus $X(\mu_1,\mu_2)((x \to y) = \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((x \to y) \to z) \to z))] \le \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2(((y \to x) \to x) \to y) \to z) \to z))] = X(\mu_1,\mu_2)(((y \to x) \to x) \to y))$. By (1) this proof is complete.

(4) Assume that μ_2 is a fuzzy Boolean filter of Q. Thus $X(\mu_1,\mu_2)((x \to (y \to x)) = \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2((x \to ((x \to y)) \to z) \to z))] \le \bigwedge_{z\in Q}[(\mu_1(z) \to \mu_2(x \to z) \to z)] = X(\mu_1,\mu_2)(x)$. By (1) this proof is complete.

□

Theorem 3.12. Given fuzzy filters μ_1,μ_2 of Q, the generated fuzzy filter of $\mu_1 \odot \mu_2$ is denoted by $\mu_1 \otimes \mu_2$, where $(\mu_2 \odot \mu_1)(x) = \mu_2(x) \odot \mu_1(x)$, for any $x \in Q$,

$$(\mu_1 \otimes \mu_2)(x) = \bigvee\{\bigwedge_{i=1,2,\cdots,n}(\mu_1(a_i) \odot \mu_2(a_i)|x \ge a_1 \odot \cdots \odot a_n\}.$$

Proof. Denote the right of the above equation by $\nu(x)$. Firstly, by $x \ge x \odot x$, $\nu(x) \ge (\mu_2(x) \odot \mu_1(x)) \wedge (\mu_2(x) \odot \mu_1(x)) = \mu_2(x) \odot \mu_1(x) = (\mu_1 \odot \mu_2)(x)$. Next, we prove that ν is a fuzzy filter. Obviously, for all $x \in L$, $\nu(1) \ge \nu(x)$. Let $x, y \in Q$. If there exist $a_1,\cdots,a_n,b_1,\cdots,b_m \in Q, x \ge a_1 \odot \cdots \odot a_n, x \to y \ge b_1 \odot \cdots b_m$, we have $y \ge x\odot(x \to y) \ge a_1\odot\cdots\odot a_n\odot b_1\odot\cdots\odot b_m$. Hence $\nu(y) \ge \bigwedge_{i=1,\cdots,n}(\mu_1(a_i)\odot\mu_2(a_i)) \bigwedge \bigwedge_{i=1,\cdots,m}(\mu_1(b_i)\odot\mu_2(b_i))$. On the other hand, $\nu(x) \wedge \nu(x \to y) = \bigvee\{\bigwedge_{i=1,\cdots,n}(\mu_1(s_i) \odot \mu_2(s_i))|x \ge s_1 \odot \cdots \odot s_n\} \bigwedge \bigvee\{\bigwedge_{i=1,\cdots,m}(\mu_1(t_i) \odot \mu_2(t_i))|x \to y \ge t_1 \odot \cdots \odot t_m\} = \bigvee\{\bigwedge_{i=1,\cdots,n}(\mu_1(s_i) \odot \mu_2(s_i)) \bigwedge \bigwedge_{i=1,\cdots,m}(\mu_1(t_i) \odot \mu_2(t_i))|x \ge s_1 \odot \cdots \odot s_n, x \to y \ge t_1 \odot \cdots \odot t_m\}$. This implies that ν is a fuzzy filter of Q. Finally, if λ is a fuzzy filter satisfying $\mu_1 \odot \mu_2 \le \lambda$, we get $\nu(x) = \bigvee\{\bigwedge_{i=1,\cdots,n}(\mu_1(a_i) \odot \mu_2(a_i))|x \ge a_1 \odot \cdots \odot a_n\} \le \bigvee\{\bigwedge_{i=1,\cdots,n}\lambda(a_i)|x \ge a_1 \odot \cdots \odot a_n\} \le \lambda(x)$. Summarizing the above results the proof is complete. □

Theorem 3.13. Let $\lambda_1, \lambda_2, \lambda_3$ be fuzzy filters of Q. Then $\lambda_1 \otimes \lambda_2 \subseteq \lambda_3$ if and only if $\lambda_1 \subseteq X(\lambda_2, \lambda_3)$.

Proof. Set $x \in Q$. If $\lambda_1 \otimes \lambda_2 \subseteq \lambda_3$, then $\lambda_1(x) \odot \lambda_2(x) \leq (\lambda_1 \otimes \lambda_2)(x) \leq \lambda_3(x)$. Hence $X(\lambda_2, \lambda_3)(x) = \bigwedge_{y \in Q}[\lambda_2(y) \to \lambda_3((x \to y) \to y)] \geq \bigwedge_{y \in Q}[\lambda_2((x \to y) \to y) \to \lambda_3((x \to y) \to y)] \geq \bigwedge_{y \in Q}[\lambda_2((x \to y) \to y) \to \lambda_1((x \to y) \to y) \odot \lambda_2((x \to y) \to y)] \geq \lambda_1((x \to y) \to y) \geq \lambda_1(x)$. This means that $\lambda_1 \subseteq X(\lambda_2, \lambda_3)$. Conversely, if $\lambda_1 \subseteq X(\lambda_2, \lambda_3)$, then $\lambda_1(x) \leq X(\lambda_2, \lambda_3)(x) = \bigwedge_{y \in Q}[\lambda_2(y) \to \lambda_3((x \to y) \to y)]$. Hence $\lambda_1(x) \leq \lambda_2(y) \to \lambda_3((x \to x) \to x) = \lambda_2(x) \to \lambda_3(x)$. So $\lambda_1(x) \otimes \lambda_2(x) \leq \lambda_3(x)$ and thus $\lambda_1 \otimes \lambda_2(x) = \bigvee\{\bigwedge_{i=1,2,\cdots,m}(\lambda_1(a_i) \odot \lambda_2(a_i)) | x \geq a_i \odot \cdots \odot a_n\} \leq \bigvee\{\bigwedge_{i=1,2,\cdots,m} \lambda_3(a_i) | x \geq a_i \odot \cdots \odot a_n\} = \bigvee\{\lambda_3(a_1 \odot \cdots \odot a_m) | x \geq a_i \odot \cdots \odot a_n\} \leq \lambda_3(x)$. This shows that $\lambda_1 \otimes \lambda_2 \subseteq \lambda_3$. □

We have the following theorem whence $\to$ in $X(\mu_1, \mu_2)$ is the Gödel residuum implication.

Theorem 3.14. Let $\lambda_1, \lambda_2 \in \mathcal{F}(Q)$. Then $\lambda_1 \wedge \lambda_2 = \lambda_1 \wedge X(\lambda_1, \lambda_2)$.

Proof. Obviously, $\lambda_1 \wedge \lambda_2 \leq \lambda_1 \wedge X(\lambda_1, \lambda_2)$. Now, we prove that $\lambda_1 \wedge X(\lambda_1, \lambda_2) \leq \lambda_1 \wedge \lambda_2$. Indeed, for any $x \in Q$, $[\lambda_1 \wedge X(\lambda_1, \lambda_2)](x) = \lambda_1(x) \wedge X(\lambda_1, \lambda_2)(x) = \lambda_1(x) \wedge \bigwedge_{z \in L}(\lambda_1(y) \to \lambda_2((y \to x) \to x)) \leq \lambda_1(x) \wedge (\lambda_1(x) \to \lambda_2(x)) = \lambda_1(x) \wedge \lambda_2(x) = (\lambda_1 \wedge \lambda_2)(x)$. Therefore $\lambda_1 \wedge \lambda_2 = \lambda_1 \wedge X(\lambda_1, \lambda_2)$. □

Theorem 3.15. $(\mathcal{F}(Q), \wedge, \vee, \otimes, X(\lambda_1, \lambda_2), \emptyset, L)$ is a complete residuated lattice.

Proof. It follows from Theorem 3.13. □

4. Conclusions

Inspired by the previous studies about fuzzy filters and stabilizers of BL-algebras, we introduce two classes of fuzzy stabilizers and investigated their related properties in BL-algebras. Also, we discuss the relation between these fuzzy stabilizers and other classes of fuzzy filters in BL-algebras. Finally, using the properties of the fuzzy stabilizers, we deduce that the collection of all fuzzy filters constitutes a residuated lattice. These results will provide a solid algebraic foundation for the consequence connectives in fuzzy logic.

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